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ON NEW MOMENT ESTIMATION OF PARAMETERS OF THE GENERALIZED GAMMA DISTRIBUTION USING IT'S CHARACTERIZATION

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Abstract. In this paper, the three estimators for three parameters of the generalized gamma distribution are proposed by using its characterization, and shown to be more convenient and more efficient than the maximum likelihood estimator for small samples. Furthermore, the estimators for the parameter of its special distribution such as gamma and Weibull distributions which are the most important in the reliability field and survival analysis are obtained.

1. Introduction

In early work with the generalized gamma distribution there was difficult in developing inference procedures with maximum likelihood estimation and large-sample method based on normal approximation to their distributions. (e.g. Stacy and Mihram, 1965., Parr and Webster, 1965; Harter 1967; Hager and Bain, 1970), but work by Prentice (1974) clarified matters a great deal and its inference can now be fairly easily handled (e.g. Farewell and Prentice, 1977; Lawless, 1980). Much of the difficulties with the model arise because the generalized gamma distribution with very different sets of parameter values look alike. Therefore Prentice (1974) considered the distribution in a reparameterized form that reduces this effect and makes properties of the distribution much more transparent.

Although Prentice (1974) have presented a procedure to obtain the three parameters of the generalized gamma distribution, his procedure still quit complicated. In this paper, we propose a simple procedure to obtain the three estimators by using its characterization and moment estimation approach. Note that Hwang and Hu (1999)

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have obtained more general characterizations with the independence of sample coefficient of variation V_n with sample mean $\overline{X}_n$ as one of its special cases when random samples are drawn from the generalized gamma distribution.

In this paper, their characterization is used to derive the expectation and the variance of V_n^2 in Section 2, and then the new estimators for the three parameters of generalized gamma distribution are proposed. Furthermore, we compare by simulation the new estimators with the maximum likelihood estimator in terms of mean absolute bias and mean square error for Weibull distribution in Section 3.

2. New Moment Estimator of Parameters of the Generalized Gamma Distribution

For deriving new moment estimators of three parameters of the generalized gamma distribution, we need the following theorem obtained by using the similar approach of Hwang and Hu (Theorem of 1999 and Cor. 4.1 of 2000).

Theorem 2.1. *Let $n \geq 3$ and let $X_1, X_2, \cdots, X_n$ be n positive i.i.d. random variables having a probability density function f(x). Then the independence of the sample mean $\bar{X}_n$ and the sample coefficient of variation $V_n = S_n/\bar{X}_n$ is equivalent to that $f(x)$ is a generalized gamma density where S_n is the sample standard deviation.*

The next theorem is easy to prove and used to derive the expectation and the variance of $V_n^2 = \left(S_n/\bar{X}_n\right)^2$, where $\bar{X}_n$ and S_n are respectively the sample mean and the sample standard deviation.

Theorem 2.2. *Let $n \geq 3$ and let $X_1, X_2, \cdots, X_n$ be n i.i.d. random samples drawn from a population having a generalized gamma density*

$$g(x; \lambda, \beta, k) = \frac{\lambda\beta}{\Gamma(k)} (\lambda x)^{k\beta - 1} \exp\left[-(\lambda x)^{\beta}\right], \quad x > 0, \lambda > 0, \beta > 0, k > 0.$$

Then

$$E\left(X^m\right) = \frac{\Gamma\left(k + \frac{m}{\beta}\right)}{\lambda^m \Gamma\left(k\right)} \quad m = 1, 2, 3, ... \tag{2.1}$$

$$E\left(\bar{X}_n\right) = \frac{\Gamma\left(k + \frac{1}{\beta}\right)}{\lambda \Gamma\left(k\right)}, \tag{2.2}$$

$$E\left(\bar{X}_n^2\right) = \frac{\Gamma\left(k\right)\Gamma\left(k + \frac{2}{\beta}\right) + (n-1)\,\Gamma^2\left(k + \frac{1}{\beta}\right)}{n\lambda^2\Gamma^2\left(k\right)}, \tag{2.3}$$

and

$$(2.4) \qquad E\left(S_n^2\right) = \frac{\Gamma(k)\,\Gamma\left(k+\frac{2}{\beta}\right) - \Gamma^2\left(k+\frac{1}{\beta}\right)}{\lambda^2\Gamma^2(k)},$$

where $\bar{X}_n$ and S_n^2 are respectively their sample mean and sample variance.

Note that the Theorem 2.2 gives the same results obtained by Hwang and Huang (Theorem 2.2 of 2002) when the generalized gamma distribution reduced to gamma distribution $(\beta = 1)$. For proposing the estimators $\hat{\lambda}$, $\hat{k}$ and $\hat{\beta}$ of λ, $k$and β based on moment estimation approach, we need the following two theorems.

Theorem 2.3. *Let $n \geq 3$ and let $X_1, X_2, \cdots, X_n$ be n i.i.d. random samples drawn from a population having a generalized gamma density*

$$g(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma(k)}(\lambda x)^{k\beta-1}\exp\left[-(\lambda x)^{\beta}\right], x>0, \lambda>0, \beta>0, k>0,$$

Then

$$(2.5) \qquad E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{n\cdot\left[\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta}) - \Gamma^2(k+\frac{1}{\beta})\right]}{\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta}) + (n-1)\cdot\Gamma^2(k+\frac{1}{\beta})},$$

where $\bar{X}_n$ and S_n^2 are respectively their sample mean and sample variance.

Proof. By Theorem 2.1, we have

$$E\left(S_n^2\right) = E\left(\frac{S_n^2}{\bar{X}_n^2}\cdot\bar{X}_n^2\right) = E\left(\frac{S_n^2}{\bar{X}_n^2}\right)\cdot E\left(\bar{X}_n^2\right)$$

and hence

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{E\left(S_n^2\right)}{E\left(\bar{X}_n^2\right)}$$

Applying Theorem 2.2 to the above identity yields that

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{n\cdot\left[\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta}) - \Gamma^2(k+\frac{1}{\beta})\right]}{\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta}) + (n-1)\cdot\Gamma^2(k+\frac{1}{\beta})}$$

Thus Theorem 2.3 is established. ■

Note that Theorem 2.3 give the same result obtained by Hwang and Huang (Theorem 2.3 of 2002) when generalized gamma distribution is reduced to gamma distribution, i.e., $\beta = 1$.

Note that $E\left(\frac{S_n^2}{\bar{X}_n^2}\right) \to \frac{\Gamma(k)\cdot\Gamma\left(k+\frac{2}{\beta}\right)}{\Gamma^2\left(k+\frac{1}{\beta}\right)} - 1$ as $n \to \infty$ and that this limit is the square of the population coefficient of variation. Thus $\frac{S_n^2}{\bar{X}_n^2}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation.

Theorem 2.4. *Let $n \geq 3$ and let $X_1, X_2, \cdots, X_n$ be n i.i.d. random samples drawn from a population having a generalized gamma density*

$$g(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma(k)}(\lambda x)^{k\beta-1}\exp\left[-(\lambda x)^\beta\right], \;\; x>0, \lambda>0, \beta>0, k>0.$$

Then

$$E\left(X^{\beta m}\right) = \frac{d^m}{dt^m}\left[\frac{\lambda^{k\beta}}{\left(\lambda^\beta - t\right)^k}\right]_{t=0}$$

Proof. Making a transformation X^β, we have

$$E\left(e^{tX^\beta}\right) = \frac{\lambda^{k\beta}}{\left(\lambda^\beta - t\right)^k}, \;\; t>0$$

By Taylor's expansion for e^{tX^β}, we have

$$E\left(e^{tX^\beta}\right) = E\left(\sum_{m=0}^{\infty}\frac{t^m X^{\beta m}}{m!}\right), t>0$$

Differentiating the both sides of the above equation m times with respect to t and evaluated at $t=0$, we have

$$E\left(X^{\beta m}\right) = \frac{d^m}{dt^m}\sum_{m=0}^{\infty}E\left(\frac{t^m X^{\beta m}}{m!}\right)\bigg|_{t=0} = \frac{d^m}{dt^m}\left[\frac{\lambda^{k\beta}}{\left(\lambda^\beta - t\right)^k}\right]_{t=0}$$

and Theorem 2.4 is established. ■

By Theorem 2.4 for $m=1$, we have $E\left(X^\beta\right) = \frac{k}{\lambda^\beta}$ and for a set of n i.i.d. random samples $X_1, X_2, \cdots, X_n$, the following equality holds:

$$E\left(\frac{X_1^\beta + X_2^\beta + ... + X_n^\beta}{n}\right) = \frac{k}{\lambda^\beta}. \tag{2.6}$$

Based on Theorems 2.2, 2.3 and (2.6), we set, by using moment estimation approach, three equations for finding three estimators $\left(\hat{\beta}, \hat{\lambda}, \hat{k}\ say\right)$ of parameters β, λ, k respectively as follows:

$$\bar{X}_n = \frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\lambda\cdot\Gamma(k)}, \tag{2.7}$$

$$\frac{S_n^2}{n\bar{X}_n^2} = \frac{\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta}) - \Gamma^2(k+\frac{1}{\beta})}{\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta}) + (n-1)\cdot\Gamma^2(k+\frac{1}{\beta})}, \tag{2.8}$$

$$\sum_{i=1}^{n} X_i^{\beta} = \frac{nk}{\lambda^{\beta}}. \tag{2.9}$$

Simplifying (2.8) further, we have

$$c\Gamma^2\left(k+\frac{1}{\beta}\right) = \Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k), \tag{2.10}$$

where $c = \frac{n\bar{X}_n^2+(n-1)S_n^2}{n\bar{X}_n^2-S_n^2}$. Thus the solutions of β, λ, k obtained by solving the three equations (2.7), (2.9) and (2.10) simultaneously are proposed for their estimators.

When the generalized gamma distribution reduces to the gamma distribution $(\beta = 1)$, we have

$$\hat{\lambda} = \frac{\hat{k}}{\bar{X}_n}, \;\; \hat{k} = \frac{\bar{X}_n^2}{S_n^2} - \frac{1}{n}, \tag{2.11}$$

which are the same as proposed by Hwang and Huang (2002), while to the Weibull distribution $(k = 1)$ the proposed two estimators can be obtained by solving the following two equations which are obtained by simplifying (2.7) and (2.9) for $k = 1$.

$$\lambda\beta\bar{X}_n = \Gamma\left(\frac{1}{\beta}\right), \tag{2.12}$$

$$c\Gamma^2\left(\frac{1}{\beta}\right) = 2\beta\Gamma\left(\frac{2}{\beta}\right). \tag{2.13}$$

and the estimator $\frac{1}{\bar{X}_n}$ of λ is obtained by (2.7) or (2.9) but not (2.10) for exponential distribution $(k = \beta = 1)$ which is as same as maximum likelihood estimator.

The variances of sample variance and sample coefficient of variation can be derived as the following theorem which procedure of proof is as similar as Theorem 2.3.

Theorem 2.5. *Let $n \geq 3$ and let $X_1, X_2, \cdots, X_n$ be n i.i.d. random samples drawn from a population having a generalized gamma density*

$$g(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma(k)}(\lambda x)^{k\beta-1}\exp\left[-(\lambda x)^{\beta}\right], \;\; x>0, \lambda>0, \beta>0, k>0$$

Then

$$Var\left(S_n^2\right) = E\left(S_n^4\right) - \left[\frac{c_0c_2 - c_1^2}{\lambda^2 c_0^2}\right]^2, \tag{2.14}$$

and

$$Var\left(V_n^2\right) = \frac{E\left(S_n^4\right)}{E\left(\bar{X}_n^4\right)} - \left[\frac{n\left(c_0c_2 - c_1^2\right)}{c_0c_2 + (n-1)\,c_1^2}\right]^2. \tag{2.15}$$

where

$$c_i = \Gamma\left(k + \frac{i}{\beta}\right), \quad i = 0, 1, 2, 3, 4. \tag{2.16}$$

$$\begin{aligned} E\left(\bar{X}_n^4\right) = &\frac{1}{n^3\lambda^4 c_0^4}\left\{c_0^3c_4 + 4\,(n-1)\,c_0^2c_1c_3\right. \\ &+3\,(n-1)\,c_0^2c_2^2 \\ &\left.+6\,(n-1)\,(n-2)\,c_0c_1c_2^2 + (n-1)\,(n-2)\,(n-3)c_1^4\right\}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} E\left(S_n^4\right) = &\frac{n}{(n-1)^2\,\lambda^4 c_0^2}\left\{c_0c_4 + (n-1)\,c_2^2 - \frac{1}{n^2c_0^2}\right. \\ &\left[\frac{(2n-1)\,c_0c_2 - (n-1)\,c_1^2}{c_0c_2 + (n-1)\,c_1^2}\right] \\ &\cdot\left(c_0^3c_4 + 4\,(n-1)\,c_0^2c_1c_3 + 3\,(n-1)\,c_0^2c_2^2\right. \\ &\left.\left.+6\,(n-1)\,(n-2)\,c_0c_1c_2^2 + (n-1)\,(n-2)\,(n-3)\,c_1^4\right)\right\}. \end{aligned} \tag{2.18}$$

When the generalized gamma distribution is reduced to the gamma distribution $(\beta = 1)$, (2.14) and (2.15) can be simplified respectively as following:

$$Var\left(S_n^2\right) = \frac{k}{\lambda^4}\left[\frac{2k}{n-1} + \frac{6}{n}\right], \tag{2.19}$$

and

$$Var\left(V_n^2\right) = \frac{2k\,(k+1)}{(n-1)\left(k+\frac{1}{n}\right)^2\left(k+\frac{2}{n}\right)\left(k+\frac{3}{n}\right)}, \tag{2.20}$$

which are the same as the Theorem 2.4 of Hwang and Huang (2002); there is some error in the equation (2.3) of Hwang and Huang (2002).

Furthermore, while the generalized gamma distribution is reduced to the Weibull distribution $(k = 1)$, (2.14) and (2.15) can be simplified respectively as following:

$$Var\left(S_n^2\right) = E\left(S_n^4\right) - \left[\frac{d_2 - d_1^2}{\lambda^2}\right]^2, \tag{2.21}$$

and

$$Var\left(V_n^2\right) = \frac{E\left(S_n^4\right)}{E\left(\bar{X}_n^4\right)} - \left[\frac{n\left(d_2 - d_1^2\right)}{d_2 + (n-1)\,d_1^2}\right]^2. \tag{2.22}$$

where $d_i = \frac{i}{\beta}\Gamma\left(\frac{i}{\beta}\right)$, $i = 1, 2, 3, 4$.

$$\begin{aligned} E\left(\bar{X}_n^4\right) = \frac{1}{n^3\lambda^4}\{&d_4 + 4\,(n-1)\,d_1 d_3 \\ &+3\,(n-1)\,d_2^2 + 6\,(n-1)\,(n-2)\,d_1 d_2^2 \\ &+(n-1)\,(n-2)\,(n-3)\,d_1^4\}, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} E\left(S_n^4\right) = \frac{n}{(n-1)^2\lambda^4}\{&d_4 + (n-1)\,d_2^2 \\ &-\frac{1}{n^2}\left[\frac{(2n-1)\,d_2 - (n-1)\,d_1^2}{d_2 + (n-1)\,d_1^2}\right] \\ &\cdot\left(d_4 + 4\,(n-1)\,d_1 d_3 + 3\,(n-1)\,d_2^2\right. \\ &\left.+6\,(n-1)\,(n-2)\,d_1 d_2^2. + (n-1)\,(n-2)\,(n-3)\,d_1^4\right)\} \end{aligned} \tag{2.24}$$

Theorem 2.5 implies that both $Var(S_n^2)$ and $Var(V_n^2)$ tend to zero as $n \to \infty$. Thus S_n^2 and V_n^2 are respectively consistent estimators of $\frac{\Gamma(k)\Gamma(k+\frac{2}{\beta})-\Gamma^2(k+\frac{1}{\beta})}{\lambda^2 p^2(k)}$ and $\frac{n\cdot\left[\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta})-\Gamma^2(k+\frac{1}{\beta})\right]}{\Gamma(k)\cdot\Gamma(k+\frac{2}{\beta})+(n-1)\cdot\Gamma^2(k+\frac{1}{\beta})}$ for large samples. After some computations, we find $ar(V_n^2)Var(S_n^2)$ when $n\left[\lambda^2\Gamma^2(k) - \Gamma^2(k+\frac{1}{\beta})\right] \le \Gamma(k)\Gamma(k+\frac{2}{\beta}) - \Gamma^2(k+\frac{1}{\beta})$.

Furthermore, the fact that $Var(V_n^2) \to 0$ as $n \to \infty$ also confirms the reason: why V_n can always be considered approximately as constant for large samples, and it can be used in checking experiment results and in estimating the standard deviation.

3. The Comparison with Maximum Likelihood Estimator

Since the comparison of our estimator $(\hat{k}_c, \hat{\lambda}_c)$ with maximum likelihood estimator $(\hat{k}_l, \hat{\lambda}_l)$ have been done by Hwang and Huang (2002) for the gamma distribution $(\beta = 1)$, thus the comparison of our estimators $(\hat{\beta}_c, \hat{\lambda}_c)$ with maximum likelihood estimators $(\hat{\beta}_l, \hat{\lambda}_l)$ would be done for Weibull distribution ($k = 1$) in this paper. In order to compare relative performances of the estimators mentioned above, the simulation procedures used in this paper is proposed by Greenwood and Durand (1960) which improved Thom (1958). Note that $(\hat{\beta}_l, \hat{\lambda}_l)$ are more difficult to compute than $(\hat{\beta}_c, \hat{\lambda}_c)$; the former even can't be found for some cases.

The scale parameter values $\lambda = 0.5, 1.0, 1.5$ and 2.0 were studied in the simulation study corresponding to $\beta = 0.5, 0.95, 1.0, 1.5,$ and 2.0. For each of these (β, λ)-values, 1000 random sample, each of size 10, were generated from a Weibull distribution. For each simulated sample, the MLE's and our estimates were obtained and compared their mean absolute bias and mean square errors respectively. The following results are the worst case in all simulations.

Tables 1-5 give the mean absolute bias and mean square error estimates error estimates for MLE and our estimates based on the undiscarded samples until 1000 samples are completed. The mean absolute bias and the mean square errors of $\hat{\beta}_c$ and $\hat{\lambda}_c$ estimators are always smaller than those of $\hat{\beta}$ and $\hat{\lambda}_l$ estimators. The numbers in the parentheses of Tables 1-5 are the bias of $\hat{\beta}_c$ and $\hat{\lambda}_c$ less than $\hat{\beta}_l$ and $\hat{\lambda}_l$ respectively within 1000 biases; although the numbers in the parentheses for various (β, λ,) less than 500, but their mean absolute biases always smaller; this means that absolute biases induced by $(\hat{\beta}_c, \hat{\lambda}_c)$ always smaller than by $(\hat{\beta}_l, \hat{\lambda}_l)$.

Table 1. Mean absolute bias and mean square error for Weibull parameter estimator using 1000 random samples of size 10 from Weibull distribution with β = 0.5

	λ	$\lvert\hat{\beta}_l - \beta\rvert$	$\lvert\hat{\beta}_c - \beta\rvert$		$\lvert\hat{\lambda}_l - \lambda\rvert$	$\lvert\hat{\lambda}_c - \lambda\rvert$	
Bias	0.5	0.2417	0.1101	(173)	0.2254	0.2326	(213)
	1.0	0.1565	0.1099	(172)	1.2964	0.4665	(206)
	1.5	0.7677	0.1123	(198)	0.6949	0.6876	(229)
	2.0	1.3592	0.1074	(170)	0.6985	0.8683	(193)
MSE	0.5	4.2774	0.0206		2.0404	0.1317	
	1.0	0.4792	0.0204		828.59	0.4596	
	1.5	125.536	0.0210		6.7129	0.9911	
	2.0	1303.79	0.0197		10.851	1.7305	

Table 2. Mean absolute bias and mean square error for Weibull parameter estimator using 1000 random samples of size 10 from Weibull distribution with β = 1.0

	λ	$\lvert\hat{\beta}_l - \beta\rvert$	$\lvert\hat{\beta}_c - \beta\rvert$		$\lvert\hat{\lambda}_l - \lambda\rvert$	$\lvert\hat{\lambda}_c - \lambda\rvert$	
Bias	0.5	0.4452	0.1272	(433)	0.1365	0.1220	(462)
	1.0	0.4753	0.1188	(475)	0.2840	0.2389	(473)
	1.5	0.6541	0.1249	(464)	0.4385	0.3809	(462)
	2.0	0.5648	0.1164	(448)	0.5795	0.4889	(466)
MSE	0.5	5.0403	0.0257		0.0750	0.0398	
	1.0	4.1884	0.0230		0.2934	0.1363	
	1.5	76.1254	0.0256		0.6847	0.3671	
	2.0	15.6411	0.0222		1.2717	0.6052	

Table 3. Mean absolute bias and mean square error for Weibull parameter estimator using 1000 random samples of size 10 from Weibull distribution with β = 1.5

	λ	$\lvert\hat{\beta}_l - \beta\rvert$	$\lvert\hat{\beta}_c - \beta\rvert$		$\lvert\hat{\lambda}_l - \lambda\rvert$	$\lvert\hat{\lambda}_c - \lambda\rvert$	
Bias	0.5	0.9957	0.1264	(490)	0.1359	0.1170	(450)
	1.0	0.9957	0.1264	(490)	0.1359	0.1170	(450)
	1.5	0.5694	0.1240	(442)	0.3986	0.3571	(451)
	2.0	0.5615	0.1257	(480)	0.5155	0.4510	(464)
MSE	0.5	221.608	0.0266		0.0690	0.0347	
	1.0	30.5121	0.0252		0.2447	0.1405	
	1.5	13.4106	0.0253		0.5511	0.3190	
	2.0	10.5556	0.0257		0.9287	0.4844	

Table 4. Mean absolute bias and mean square error for Weibull parameter estimator using 1000 random samples of size 10 from Weibull distribution with β = 2.0

	λ	$\lvert\hat{\beta}_l - \beta\rvert$	$\lvert\hat{\beta}_c - \beta\rvert$		$\lvert\hat{\lambda}_l - \lambda\rvert$	$\lvert\hat{\lambda}_c - \lambda\rvert$	
Bias	0.5	0.7320	0.1365	(491)	0.0740	0.0662	(422)
	1.0	0.5322	0.1390	(521)	0.1639	0.1447	(451)
	1.5	0.5484	0.1401	(533)	0.2498	0.2136	(431)
	2.0	0.5173	0.1383	(465)	0.2857	0.2550	(435)
MSE	0.5	92.1199	0.0318		0.0186	0.0109	
	1.0	4.4506	0.0320		0.0933	0.0513	
	1.5	5.7316	0.0322		0.2088	0.1100	
	2.0	10.6006	0.0317		0.2724	0.1642	

Table 5. Mean absolute bias and mean square error for Weibull parameter estimator using 1000 random samples of size 10 from Weibull distribution with β = 2.0

	λ	$\lvert\hat{\beta}_l - \beta\rvert$	$\lvert\hat{\beta}_c - \beta\rvert$		$\lvert\hat{\lambda}_l - \lambda\rvert$	$\lvert\hat{\lambda}_c - \lambda\rvert$	
Bias	0.5	0.3050	0.1216	(508)	0.0414	0.0372	(364)
	1.0	0.3652	0.1294	(511)	0.0898	0.0792	(363)
	1.5	0.3109	0.1228	(524)	0.1305	0.1161	(367)
	2.0	0.3987	0.1236	(543)	0.1953	0.1629	(384)
MSE	0.5	0.7342	0.0284		0.0066	0.0040	
	1.0	0.9782	0.0297		0.0306	0.0174	
	1.5	0.7629	0.0273		0.0693	0.0391	
	2.0	1.1217	0.0280		0.1477	0.0756	

Until now, we have done more than 100,000 times simulation for λ = 0.5, 1.0, 1.5, 2.0 and β = 0.5, 0.95, 1.0, 1.5, 2.0, when n = 5(5)25, 26(1)30, and obtained the following conclusions: (1) $(\hat{\beta}_c, \hat{\lambda}_c)$ is better than $(\hat{\beta}_l, \hat{\lambda}_l)$ for $n \leqslant 25$, and the smaller n the better $(\hat{\beta}_c, \hat{\lambda}_c)$; (2) $(\hat{\beta}_l, \hat{\lambda}_l)$ is better than $(\hat{\beta}_c, \hat{\lambda}_c)$ for $n > 25$, and the larger n the better $(\hat{\beta}_l, \hat{\lambda}_l)$.

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