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COLORING THE SQUARE OF AN OUTERPLANAR GRAPH

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Abstract. Let G be an outerplanar graph with maximum degree $\Delta(G) \geq 3$. We prove that the chromatic number $\chi(G^2)$ of the square of G is at most $\Delta(G)+2$. This confirms a conjecture of Wegner [8] for outerplanar graphs. The upper bound can be further reduced to the optimal value $\Delta(G)+1$ when $\Delta(G) \geq 7$.

1. Introduction

Only simple graphs are considered in this paper. For two vertices u and v of a graph $G(V, E)$, let $\mathrm{dist}_G(u, v)$ denote the distance between u and v in G, that is the length of a shortest path connecting them. The *square* G^2 of a graph G is the graph defined on the vertex set $V(G)$ such that u and v are adjacent in G^2 if and only if $1 \leq \mathrm{dist}_G(u, v) \leq 2$. A proper k-coloring is a mapping ϕ from $V(G)$ to the set $\{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever u and v are adjacent. Obviously, a k-coloring ϕ of G gives rise to a proper coloring of G^2 if and only if $\phi(u) \neq \phi(v)$ whenever $1 \leq \mathrm{dist}_G(u, v) \leq 2$. We call such a coloring ϕ a *square-k-coloring* of G. The chromatic number $\chi(G)$ is the least number k such that G admits a proper k-coloring. Let $\Delta(G)$ denote the maximum degree of a vertex of the graph G. It is evident that $\chi(G^2) \geq \Delta(G) + 1$ for any graph G. This lower bound is sharp. For instance, $\chi(T^2) = \Delta(T) + 1$ for every tree T with at least one edge. On the other hand, it is easy to see that $\chi(G^2) \leq \Delta^2(G) + 1$ for any graph G. This upper bound is also sharp. The 5-cycle and the Petersen graph are two examples.

Wegner [8] first investigated the chromatic number of the square of a planar graph. He proved that $\chi(G^2) \leq 8$ for every planar graph G with $\Delta(G) = 3$ and conjectured that the upper bound could be reduced to 7. Recently, Thomassen [6] has established Wegner's conjecture. Wegner [8] also proposed the following conjecture. The upper bounds are sharp if the conjecture is true.

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Conjecture 1. Let G be a planar graph. Then

$$\chi(G^2) \le \begin{cases} \Delta(G)+5 & if\ 4 \le \Delta(G) \le 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & if\ \Delta(G) \ge 8. \end{cases}$$

This conjecture remains open. The best upper bound as far as we know is $5\Delta(G)/3 + 78$ established by Molloy and Salavatipour [5]. This improves other recently obtained upper bounds: $\lfloor 9\Delta/5 \rfloor + 2$ for $\Delta(G) \ge 749$ ([1]), $\lceil 9\Delta/5 \rceil + 1$ for $\Delta(G) \ge 47$ ([2]), and $2\Delta(G) + 25$ ([4]). For planar graphs of large girth, better upper bounds for $\chi(G^2)$ are known. Wang and Lih [7] proved that if G is a planar graph with girth $g(G)$, then $\chi(G^2) \le \Delta(G) + 5$ when $g(G) \ge 7$, $\chi(G^2) \le \Delta(G) + 10$ when $g(G) \ge 6$, and $\chi(G^2) \le \Delta(G) + 16$ when $g(G) \ge 5$.

The focus of this paper is to study the chromatic number of the square of an outerplanar graph. A planar graph is said to be *outerplanar* if it has a plane embedding such that all vertices lie on the boundary of the unbounded face. An *outerplane* graph is a particular embedding of an outerplanar graph. Bodlaender et al. [3] showed that there are polynomial time algorithms for coloring the square of an outerplanar graph G and $\chi(G^2) \le \Delta(G) + 5$. We will reduce the upper bound to $\Delta(G) + 2$ when $\Delta(G) \ge 3$, and even to the optimal result $\chi(G^2) = \Delta(G) + 1$ when $\Delta(G) \ge 7$.

2. Special Vertices of Degree 2

A vertex of degree k is called a *k-vertex*. The degree of v in the graph G is denoted by $d_G(v)$. For a vertex v of a graph G, define $N_i^G(v) = \{u \in V(G) \mid \text{dist}_G(u,v) = i\}$ for $i \ge 1$ and define $\beta_G(v) = |N_1^G(v)| + |N_2^G(v)|$.

For an outerplane graph G, all faces are called *inner* faces, except that the unbounded one is called the *outer* face. The boundary edges of the outer face are called *outer* edges. All other edges are called *inner* edges. If G is 2-connected and $\Delta(G) \ge 3$, then an inner face f of G is called an *end* face if the boundary of f contains exactly one inner edge, i.e., the boundary of f contains exactly two vertices of degree 3 or more. The dual graph of G becomes a tree of order at least 2 when the vertex corresponding to the outer face is deleted. Thus there exist at least two leaves that determine two end faces of G.

Let $|G|$ denote the order of G. It is well-known that a 2-connected outerplane graph G has at least one 2-vertex if $|G| \ge 3$, and at least two nonadjacent 2-vertices if $|G| \ge 4$.

Let M_3 be the graph obtained from a path $x_1, x_2, \ldots, x_7$ of length 6, where $x_1 \ne x_7$, by adding the edges x_1x_3, x_3x_5, and x_5x_7. A graph G is said to contain the configuration M_3 if M_3 appears in G as a subgraph such that $d_G(x_i) = 2$

for $i = 2, 4, 6$ and $d_G(x_j) = 4$ for $j = 1, 3, 5, 7$. Define the function f so that $f(\Delta) = \Delta + 1$ when $3 \leq \Delta \leq 6$, and $f(\Delta) = \Delta$ when $\Delta \geq 7$.

Theorem 1. *Let G be a 2-connected outerplane graph with $|G| \geq 3$. Suppose that $\Delta(G) \geq 3$ and G does not contain the configuration M_3 when $\Delta(G) = 4$. Then there exists a vertex u of degree 2 such that $\beta_G(u) \leq f(\Delta)$.*

Proof. Let $\Delta = \Delta(G)$. If $\Delta = 3$, then it is easy to see that $\beta_G(u) \leq 4$ for some 2-vertex in the boundary of an end face.

So we may assume that $\Delta \geq 4$. The smallest 2-connected outerplane graph with maximum degree Δ consists of a vertex joined to every vertex of a path of length $\Delta - 1$. Evidently, this graph has a 2-vertex u such that $\beta_G(u) = \Delta \leq f(\Delta)$.

We now proceed by induction on $|G|$. Let C be the cycle consisting of all the outer edges.

Since $\Delta \geq 4$, there exists a subpath P of C of length at least 2 whose ends have degree at least 3 in G, but all of whose internal vertices have degree 2 in G. If the length of P is at least 3, then we may contract the second edge from one end of P to get a shorter path. The result then follows by induction. Therefore we assume that no two 2-vertices in C are adjacent.

Let an arbitrary 2-vertex u of G have neighbors v and w. If v and w are not adjacent and their degrees are both less than Δ, then we add a new edge vw. In so doing, we do not change the value of $\beta_G(u)$. We perform such additions wherever possible. So we may assume the following for G.

Convention. Let u be a 2-vertex with neighbors v and w such that $d_G(v) \geq d_G(w)$. If v and w are not adjacent, then $d_G(v) = \Delta$.

In the sequel, we always label the two neighbors v and w of a 2-vertex u in such a way that $d_G(v) \geq d_G(w)$. Let x_v and x_w denote the neighbors in $C - u$ of v and w, respectively. Let $P(u) : z_0, z_1, \ldots, z_t$ be the shortest subpath of C containing the path x_v, v, u, w, x_w and containing all the vertices in $N_1^G(v) \cup N_1^G(w)$. Then there exists an index i such that $z_{i-2} = x_v$, $z_{i-1} = v$, $z_i = u$, $z_{i+1} = w$, and $z_{i+2} = x_w$. Note that z_0 and z_t are in $N_1^G(v) \cup N_1^G(w)$. Now we choose a vertex u such that $P(u)$ has the minimum length.

Case 1. The neighbors of u are not adjacent.

Then by the Convention, $d_G(v) = \Delta$. Let H be the graph obtained from G by deleting u and adding the edge vw. We see that $|H| < |G|$, $\Delta(H) = \Delta(G)$, and H satisfies the assumptions of the theorem. By the induction hypothesis, there exists a vertex x such that $d_H(x) = 2$ and $\beta_H(x) \leq f(\Delta)$. Since $d_H(x) = 2$ and both $d_H(v)$ and $d_H(w)$ are at least 3, x is different from v and w. Obviously, at most one of v and w may be a neighbor of x. Suppose that both v and w are not neighbors of x. Then $\beta_G(x) = \beta_H(x)$ and x is what we are looking for.

Suppose that x is adjacent to v and z for some z different from w. If z is not a neighbor of w, then $N_2^G(x) = (N_2^H(x) \setminus \{w\}) \cup \{u\}$. Again, $\beta_G(x) = \beta_H(x)$. If z is a neighbor of w, then it implies that $d_G(v) = \Delta \leq 3$, contradicting our present assumption that $\Delta \geq 4$.

Suppose that x is adjacent to w and z for some z different from v. If z is not a neighbor of v, then $N_2^G(x) = (N_2^H(x) \setminus \{v\}) \cup \{u\}$. Again $\beta_G(x) = \beta_H(x)$. If z is a neighbor of v, then z and w must be adjacent. Now let the neighbor of z in $C - x$ be y, where $y \neq v$ as $d_G(v) = \Delta \geq 4$. Hence $d_G(z) \geq 4$. If y is adjacent to v, then $\beta_G(x) = 5 \leq f(\Delta)$, and we are done. Suppose that y is not a neighbor of v. If $d_G(v) = 4$, then $d_G(z) = 4$ since $d_G(v) = \Delta$. Again, it follows that $\beta_G(x) = 5 \leq f(\Delta)$, and we are done.

Suppose next $d_G(v) \geq 5$. Let $j \leq i - 2$ be the largest index such that z_j is a neighbor of v. If $j = 0$, then all the vertices at distance at most 2 from x are included in the path $z_{i-1}, z_i, \ldots, z_t$. It follows that $P(x)$ is strictly shorter than $P(u)$, a contradiction. Hence $j > 0$. Let k be the smallest index, $j < k \leq i - 2$, such that z_k is a 2-vertex. If $k < i - 2$, then all the vertices at distance at most 2 from z_k are included in the path $z_0, z_1, \ldots, z_{i-1}$. It follows that $P(z_k)$ is strictly shorter than $P(u)$, a contradiction. Now suppose that $k = i - 2$. In this case, z_{i-2} and z_j must be adjacent, i.e., $j = i - 3$. If z_{j-1} is also a neighbor of v, then $\beta_G(z_{i-2}) = d_G(v) \leq f(\Delta)$.

Suppose that z_{j-1} is not a neighbor of v. If there is at least one vertex z_p, $0 < p < j - 1$, that is adjacent to v, then there is some 2-vertex z_m, $p < m < j$, such that all the vertices at distance at most 2 from z_m are included in the path z_0, $z_1, \ldots, z_{i-1}$. Therefore, $P(z_m)$ is strictly shorter than $P(u)$, a contradiction. Now suppose that no such z_p exists. It implies that $d_G(v) = \Delta = 5$. Hence, $d_G(z) \leq 5$. It follows that $\beta_G(x) \leq 6 = f(\Delta)$, and we are done.

Case 2. The neighbors of u are adjacent.

If v is adjacent to x_w, then $\beta_G(u) = d_G(v) \leq \Delta \leq f(\Delta)$ and u is what we are looking for. Henceforth we assume that v and x_w are not adjacent.

Subcase 2.1. The number of indices j, $0 \leq j < i - 2$, such that z_j is adjacent to v is at least two.

Let $j < i - 2$ be the largest index such that z_j is a neighbor of v. Let k be the smallest index, $j < k \leq i - 2$, such that z_k is a 2-vertex. If $k < i - 2$ or z_t is not adjacent to v, then all the vertices at distance at most 2 from z_k are included in the path $z_0, z_1, \ldots, z_{i+1}$. It follows that $P(z_k)$ is strictly shorter than $P(u)$, a contradiction. Now suppose that $k = i - 2$ and z_t is a neighbor of v. In this case,

z_{i-2} and z_j must be adjacent, i.e., $j = i - 3$. If z_{j-1} is also a neighbor of v, then $\beta_G(z_{i-2}) = d_G(v) \le f(\Delta)$, and we are done.

Suppose that z_{j-1} is not a neighbor of v. If there is at least one vertex z_p, $0 < p < j - 1$, that is adjacent to v, then there is some 2-vertex z_m, $p < m < j$, such that all the vertices at distance at most 2 from z_m are included in the path z_0, z_1, $\dots$, z_{i-1}. Therefore, $P(z_m)$ is strictly shorter than $P(u)$, a contradiction.

Now suppose that no such z_p exists. If $d_G(v) \ge 7$, or $d_G(w) \ge 5$, or $d_G(w) = 4$ but w is not adjacent to z_t, then there is some z_s, $i + 2 < s < t$, that is a neighbor of v or w. It follows that there is some 2-vertex z_m, $i + 2 \le m < s$, such that all the vertices at distance at most 2 from z_m are included in the path z_{i-1}, z_i, $\dots$, z_t. Therefore, $P(z_m)$ is strictly shorter than $P(u)$, a contradiction. The remaining possibilities are such that $d_G(v) = 6$ and $d_G(w) = 3$, or $d_G(w) = 4$ and w is adjacent to z_t. We see that $\beta_G(u) = 7 \le f(\Delta)$ in both cases and u satisfies the theorem.

Subcase 2.2. The vertex z_0 is precisely x_v.

If z_t is a neighbor of w, then $\beta_G(u) \le 5 \le f(\Delta)$ since $3 \le d_G(w) \le d_G(v) \le 4$ in this case. Suppose that z_t is adjacent to v, but not to w. It is obvious that $d_G(v) \ge 4$. If $d_G(w) \ge 4$, then there is some z_j, $i + 2 < j < t$, that is adjacent to w. Thus there is some 2-vertex z_k, $i + 2 \le k < j$, such that all the vertices at distance at most 2 from z_k are included in the path z_{i-1}, z_i, $\dots$, z_t. It follows that $P(z_k)$ is strictly shorter than $P(u)$, a contradiction. So suppose $d_G(w) = 3$. If $d_G(v) = 4$, then $\beta_G(u) = 5 \le f(\Delta)$, hence u satisfies the theorem. If $d_G(v) \ge 5$, then there is some z_p, $i + 2 < p < t$, that is adjacent to v. Since v is not adjacent to x_w, there is some 2-vertex z_q, $i+2 \le q < p$, such that all the vertices at distance at most 2 from z_q are included in the path z_{i-1}, z_i, $\dots$, z_t. It follows that $P(z_q)$ is strictly shorter than $P(u)$, a contradiction.

Subcase 2.3. The vertex z_0 is different from x_v and is a neighbor of v such that no vertices among z_j, $0 < j < i - 2$, are adjacent to v.

First assume that z_t is not a neighbor of v. Thus $d_G(v) = 4$. If $d_G(w) = 3$, then $\beta_G(u) = 5 \le f(\Delta)$. If $d_G(w) = 4$, then $\beta_G(u) = 6 \le f(\Delta)$ when $\Delta \ge 5$. In both cases, u satisfies the theorem.

Now assume that $\Delta = 4$. If the degree of z_{i-2} is 4, or is 3 but z_{i-2} is not a neighbor of z_0, then some z_j, $0 < j < i - 2$, is a neighbor of z_{i-2}. Thus there is some 2-vertex z_k, $j < k < i - 2$, such that all the vertices at distance at most 2 from z_k are included in the path z_0, z_1, $\dots$, z_{i-1}. It follows that $P(z_k)$ is strictly shorter than $P(u)$, a contradiction.

Suppose that z_{i-2} is of degree 3 and adjacent to z_0. If z_{i-3} is of degree 2 and adjacent to z_0, then $\beta_G(z_{i-3}) = 4 < f(\Delta)$. Thus z_{i-3} satisfies the theorem. If z_{i-3} is of degree 2, but not adjacent to z_0, then all the vertices at distance at most

2 from z_{i-3} are included in the path $z_0, z_1, \ldots, z_{i-1}$. It follows that $P(z_{i-3})$ is strictly shorter than $P(u)$, a contradiction.

Suppose that the degree of z_{i-3} is at least 3. Then z_{i-3} cannot be a neighbor of z_0, for otherwise v would be a cut vertex. Then there is some 2-vertex z_k, $1 < k < i-3$, such that all the vertices at distance at most 2 from z_k are included in the path $z_0, z_1, \ldots, z_{i-2}$. It follows that $P(z_k)$ is strictly shorter than $P(u)$, a contradiction.

Consequently, the only possibility left for z_{i-2} is its degree is 2. If z_{i-2} is not adjacent to z_0, then all the vertices at distance at most 2 from z_{i-2} are included in the path $z_0, z_1, \ldots, z_{i+1}$. It follows that $P(z_{i-2})$ is strictly shorter than $P(u)$, a contradiction. If z_{i-2} is adjacent to z_0, i.e., $i = 3$, and $d_G(z_0) = 3$, then $\beta_G(z_1) = 5 = f(\Delta)$. Thus z_1 satisfies the theorem. So the last remaining possibility is that z_1 is a 2-vertex, z_1 is adjacent to z_0, and z_0 is a 4-vertex.

Now since $d_G(v) = d_G(w) = 4$, an argument similar to the above for z_{i-2} can be applied to z_5. We either obtain a desired 2-vertex or the degrees of z_5 and z_6 are 2 and 4, respectively. Note that the vertices $z_0, z_1, \ldots, z_6$ would induce a configuration M_3. However, that is ruled out by the assumptions of the theorem.

Finally, suppose that z_t is a neighbor of v. This implies that $d_G(v) \geq 5$. If $d_G(v) \geq 6$, or $d_G(w) \geq 5$, or if $d_G(v) = 5$, $d_G(w) = 4$, and z_t is not adjacent to w, then there is some z_j, $i+2 < j < t$, that is adjacent to v or w. It follows that there is some 2-vertex z_k, $i+2 \leq k < j$, such that all the vertices at distance at most 2 from z_k are included in the path $z_{i-1}, z_i, \ldots, z_t$. Therefore, $P(z_k)$ is strictly shorter than $P(u)$, a contradiction. If $d_G(v) = 5$, and $d_G(w) = 3$ or $d_G(w) = 4$ but z_t is adjacent to w, then $\beta_G(u) = 6 \leq f(\Delta)$. Thus u satisfies the theorem. ■

Let $C_5 + e$ be the graph obtained from a cycle of length 5 with two non-consecutive vertices joined. Then its maximum degree is 3 and $\beta_{C_5+e}(u) = 4$ for any vertex u.

For $n \geq 3$, let O_n denote the outerplane graph obtained by adding n edges $u_1u_2, u_2u_3, \ldots, u_nu_1$ inside a cycle $u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_1$ of length $2n$. We have $\Delta(O_3) = 4$ and $\beta_{O_3}(u) = 5$ for any vertex u. Let A be the graph obtained from O_4 by joining the vertices u_1 and u_3. Then $\Delta(A) = 5$ and $\beta_A(u) = 6$ for any 2-vertex u. Let B be the graph obtained from O_6 by adding the new triangle $u_1u_3u_5$. Then $\Delta(B) = 6$ and $\beta_B(u) = 7$ for any 2-vertex u. Therefore, the upper bound in Theorem 1 cannot be further reduced when $3 \leq \Delta \leq 6$.

3. Coloring the Square

Let G be a connected graph. It is straightforward to verify the following facts.

(1) If $\Delta(G) = 1$, then $\chi(G^2) = 2$.

(2) If $\Delta(G) = 2$ and G is a path, then $\chi(G^2) = 3 = \Delta(G) + 1$. If $\Delta(G) = 2$ and G is a cycle, then $3 \leq \chi(G^2) \leq 5$. Moreover, $\chi(G^2) = 3 = \Delta(G) + 1$ if and only if $|G| \equiv 0 \pmod 3$; $\chi(G^2) = 5 = \Delta(G) + 3$ if and only if $|G| = 5$.

Lemma 2. *Let x be a cut vertex of the graph G. Let G_i be the subgraph induced by $V_i \cup \{x\}$ for $i = 1, 2, \ldots, m$, where V_i's are the vertex sets of the components of $G - x$. Then $\chi(G^2) = \max_{1 \leq i \leq m}\{d_G(x) + 1, \chi(G_i^2)\}$.*

Proof. Let $k = \max_{1 \leq i \leq m}\{d_G(x) + 1, \chi(G_i^2)\}$. Since G_i is a subgraph of G, $\chi(G^2) \geq \chi(G_i^2)$ for every i, $1 \leq i \leq m$. Moreover, it is obvious that $\chi(G^2) \geq \Delta(G) + 1 \geq d_G(x) + 1$. It follows that $\chi(G^2) \geq k$. Conversely, let each G_i be colored with a square-$\chi(G_i^2)$-coloring. Then all the neighbors of x in G_i have different colors. By suitably renaming the colors, we can color x with the same color in every G_i and all the neighbors of x in G are colored differently. It follows that $k \geq \chi(G^2)$. ■

Theorem 3. *Let G be an outerplane graph with $\Delta(G) \geq 3$. Then $\chi(G^2) \leq \Delta(G) + 2$. Moreover, $\chi(G^2) = \Delta(G) + 1$ if $\Delta(G) \geq 7$.*

Proof. We proceed by induction on the order $|G|$. We may suppose the connectedness of G. If $|G| \leq 4$, the theorem holds trivially. Let $\Delta(G) \geq 3$ and $|G| \geq 5$.

Suppose that G is 2-connected. If $\Delta(G) \neq 4$, or $\Delta(G) = 4$ but G does not contain the configuration M_3, then there is a 2-vertex u of G such that $\beta_G(u) \leq \Delta(G) + 1$ by Theorem 1. Let v and w be the neighbors of u. If v and w are not adjacent, define H to be $G - u + vw$. If v and w are adjacent, define H to be $G - u$. Then $|H| < |G|$, $\Delta(H) = \Delta(G)$, and H is 2-connected. By the induction hypothesis, H has a square-$(\Delta(G) + 2)$-coloring. We can extend this coloring to G since the vertex u has at most $\Delta(G) + 1$ forbidden colors.

Now let $\Delta(G) = 4$ and G contains the configuration M_3. Let $y_1, y_2 \in N_1(x_1) \setminus \{x_2, x_3\}$ and $z_1, z_2 \in N_1(x_7) \setminus \{x_5, x_6\}$. If x_1 is adjacent to x_7, we stipulate that $y_2 = x_7$ and $z_2 = x_1$. Define the graph H to be $G - \{x_2, x_3, \ldots, x_6\}$ if x_1 is adjacent to x_7; to be $G - \{x_2, x_3, \ldots, x_6\} + x_1x_7$ otherwise. By the inductive hypothesis, H has a square-6-coloring ϕ with the color set $L = \{1, 2, \cdots, 6\}$. In order to extend ϕ into a square-6-coloring of G, we consider the following two cases.

Assume that x_1 is adjacent to x_7. Without loss of generality, we may let $\phi(y_1) = 1$, $\phi(x_1) = 2$, $\phi(x_7) = 3$, and $\phi(z_1) = a$. We first color x_4 with 1, x_5 with $b \in L \setminus \{1, 2, 3, a\}$, and x_2 and x_6 with $c \in L \setminus \{1, 2, 3, a, b\}$. Afterward, we assign a to x_3 when $a \neq 1$; we color x_3 with $d \in L \setminus \{1, 2, 3, b, c\}$ when $a = 1$.

Assume that x_1 is not adjacent to x_7. Since x_1 is adjacent to x_7 in H, $\phi(x_1) \notin \{\phi(z_1), \phi(z_2)\}$ and $\phi(x_7) \notin \{\phi(y_1), \phi(y_2)\}$. Suppose that $\phi(y_1) = 1$, $\phi(y_2) = 2$,

$\phi(x_1) = 3$, and $\phi(x_7) = 4$. First we color x_4 with 1, x_6 with 3, x_2 with 4, and x_3 with 5. If 2 or $6 \notin \{\phi(z_1), \phi(z_2)\}$, we further color x_5 with 2 or 6. If $\{\phi(z_1), \phi(z_2)\} = \{2, 6\}$, we recolor x_4 with 6 and then color x_5 with 1.

Next suppose that G has a cut vertex x. Let G_i, $1 \le i \le m$, be the subgraphs induced by the components of $G - x$ together with x. Then each G_i satisfies the assumptions of the theorem. If $\Delta(G_i) \ge 3$, then $\chi(G_i^2) \le \Delta(G_i) + 2 \le \Delta(G) + 2$ by the induction hypothesis. If $\Delta(G_i) \le 2$, then $\chi(G_i^2) \le 5 \le \Delta(G) + 2$ as noted at the beginning of this section. Thus $\chi(G^2) \le \Delta(G) + 2$ by Lemma 2.

The "moreover" part can also be proved by induction since the 2-vertex u could have been chosen so that $\beta_G(u) \le \Delta(G)$ by Theorem 1. ■

It is yet to be determined if any outerplanar graph G with $\Delta(G) = 5$ or 6 satisfies $\chi(G^2) = \Delta(G) + 2$. We would conjecture that none exists. If an outerplanar graph G with $\Delta(G) = 3$ contains a 5-cycle, then $\chi(G^2) = 5 = \Delta(G) + 2$. This example together with the following theorem shows that the upper bound $\Delta(G) + 2$ in Theorem 3 is tight for $\Delta(G) = 3$ or 4.

Theorem 4. *For any $n \ge 3$, $\chi(O_n^2) = 5$ except $\chi(O_3^2) = \chi(O_4^2) = \chi(O_7^2) = 6$.*

Proof. It is easy to see that $5 \le \chi(O_n^2) \le 6$ for every $n \ge 3$. Since O_3^2 is K_6 and O_4^2 contains K_6 as a subgraph, we have $\chi(O_3^2) = \chi(O_4^2) = 6$. We observe that every color class contains at most three vertices for a square-k-coloring of O_7. If a color class is of size 3, then it contains at least two vertices of degree 2. Since O_7 has seven vertices of degree 2, there are at most three color classes of size 3. This implies $k \ge 6$ and $\chi(O_7^2) = 6$.

Now assume $n \ge 5$ and $n \ne 7$. We are going to construct a square-5-coloring of O_n in every possible case.

If $n \equiv 0$ (mod 5), we color the sequence of vertices $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$ with the color sequence 1, 2, 3, 4, 5 repeatedly.

If $n \equiv 1$ (mod 5), we first color u_1 and u_4 with 1, u_2 and u_5 with 2, u_3 and u_6 with 3, v_1, v_3, v_5 with 4, and v_2, v_4, v_6 with 5. Then we color the sequence of vertices $u_7, v_7, u_8, v_8, \ldots, u_n, v_n$ with the color sequence 1, 4, 2, 3, 5 repeatedly.

If $n \equiv 2$ (mod 5) and $n \ge 12$, we first color u_1, u_4, u_7, u_{10} with 1, u_2, u_5, u_8, u_{11} with 2, u_3, u_6, u_9, u_{12} with 3, $v_1, v_3, v_5, v_7, v_9, v_{11}$ with 4, and $v_2, v_4, v_6, v_8, v_{10}, v_{12}$ with 5. Then we color the sequence of vertices u_{13}, v_{13}, u_{14}, v_{14}, $\ldots, u_n$, v_n with the color sequence 1, 4, 2, 3, 5 repeatedly.

If $n \equiv 3$ (mod 5), we first color v_1, v_3, v_5, v_7 with 1, u_1, u_4, v_6 with 2, u_2, v_4, u_7 with 3, v_2, u_5, u_8 with 4, and u_3, u_6, v_8 with 5. Then we color the sequence of vertices u_9, v_9, u_{10}, v_{10}, $\ldots, u_n$, v_n with the color sequence 2, 3, 1, 4, 5 repeatedly.

If $n \equiv 4$ (mod 5), we first color u_1, u_4, u_7 with 1, u_2, v_4, v_6, v_8 with 2, v_2, u_5, v_7, v_9 with 3, v_1, v_3, v_5, u_8 with 4, and u_3, u_6, u_9 with 5. Then color the sequence of vertices u_{10}, v_{10}, u_{11}, v_{11}, $\ldots, u_n$, v_n with the color sequence 1, 4, 2, 5, 3 repeatedly. ■

References

1. G. Agnarsson and M. M. Halldórsson, Coloring powers of planar graphs, *SIAM J. Discrete Math.* **16** (2003), 651-662.
2. O. V. Borodin, H. J. Broersma, A. Glebov, and J. van den Heuvel, Stars and bunches in planar graphs. part II: general planar graphs and colourings, CDAM Research Report, London School of Economics and Politics, 2002, a translated and adapted version of a paper that appeared in *Diskretn. Anal. Issled. Oper. Ser. 1.*, **8** (2001), 9-33 (in Russian).
3. H. L. Bodlaender, T. Kloks, R. B. Tan and J. van Leeuwen, λ-coloring of graphs, *STACS 2000* (*Lille*), *Lecture Notes in Comput. Sci. 1770,* Springer, Berlin, 2000, 395-406.
4. J. van den Heuvel and S. McGuinness, Coloring the square of a planar graph, *J. Graph Theory* **42** (2003), 110-124.
5. M. Molloy and M. R. Salavatipour, *A bound on the chromatic number of the square of a graph,* preprint, University of Toronto, 2002.
6. C. Thomassen, *Applications of Tutte cycles,* preprint, Technical University of Denmark, 2001.
7. Weifan Wang and Ko-Wei Lih , Labeling planar graphs with conditions on girth and distance two, *SIAM J. Discrete Math.*, **17** (2003), 264-275.
8. G. Wegner, *Graphs with given diameter and a coloring problem*, preprint, University of Dortmund, 1977.

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