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SPACES OF CESÀRO DIFFERENCE SEQUENCES OF ORDER r DEFINED BY A MODULUS FUNCTION IN A LOCALLY CONVEX SPACE

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Abstract. The idea of difference sequence spaces was introduced by Kizmaz [12] and was generalized by Et and Colak [6]. In this paper we introduce and examine some properties of the sequence spaces $[V,\lambda,f,p]_0(\Delta_v^r,q)$, $[V,\lambda,f,p]_1(\Delta_v^r,q)$, $[V,\lambda,f,p]_\infty(\Delta_v^r,q)$, $S_\lambda(\Delta_v^r,q)$ and give some inclusion relations on these spaces. We also show that the space $S_\lambda(\Delta_v^r,q)$ may be represented as a $[V,\lambda,f,p]_1(\Delta_v^r,q)$ space. Furthermore, we compute Köthe-Toeplitz duals of the spaces of generalized Cesàro difference sequences spaces.

1. Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|=\sup|x_k|$, where $k\in\mathbb{N}=\{1,2,\ldots\}$, the set of positive integers.

Throughout this paper, let $\lambda=(\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$. The generalized de laVallée-Pousin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k,\ \text{where } I_n=[n-\lambda_n+1,n]\ \text{for } n=1,2,....$$

A sequence $x=(x_k)$ is said to be $(V,\lambda)-$summable to a number L [14] if $t_n(x)\to L$ as $n\to\infty$. If $\lambda_n=n$, then $(V,\lambda)-$summability and strongly (V,λ)-summability are reduced to $(C,1)$-summability and strongly $(C,1)$-summability, respectively.

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The difference sequence spaces was first introduced by Kizmaz [12] and then the concept was generalized by Et and Colak [6]. Later on, difference sequence spaces have been discussed in [1, 2, 4, 5, 7, 10, 17].

Ruckle [20] used the idea of a modulus function to construct some spaces of complex sequences. Maddox [16] investigated and discussed some properties of the three sequence spaces defined using a modulus function f. Recently, Malkowsky and Savaş $[18]$ defined some $\lambda-$sequence spaces by using a modulus function.

The main object of this paper is to study some sequence spaces which arise from the notation of generalized de la Vallée-Pousin mean, the generalized difference operator Δ_v^r and the concept of a modulus function.

We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,

(ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Since $|f(x)-f(y)|\leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. A modulus may be unbounded or bounded.

Let $X,Y\subset w$. Then we shall write

$$M(X,Y)=\bigcap_{x\in X}x^{-1}*Y=\{a\in w: ax\in Y \quad \text{for all } x\in X\}\,[22].$$

The set $X^\alpha=M(X,\ell_1)$ is called Köthe-Toeplitz dual or the $\alpha-$dual of X. If $X\subset Y$, then $Y^\alpha\subset X^\alpha$. It is clear that $X^\alpha\subset(X^\alpha)^\alpha=X^{\alpha\alpha}$. If $X=X^{\alpha\alpha}$, then X is called an $\alpha-$space. In particular, an $\alpha-$space is called a Köthe space or a perfect sequence space.

Let X be a sequence space. Then X is called:

(i) *Solid* (or *normal*), if $(\alpha_k x_k)\in X$ for all sequences (α_k) of scalars with $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$, whenever $(x_k)\in X$,

(ii) *Symmetric*, if $(x_k)\in X$ implies $(x_{\pi(k)})\in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$,

(iii) *Sequence algebra* if $x.y\in X$, whenever $x,y\in X$.

It is well known that if X is perfect, then X is normal [11].

The following inequality will be used in the sequel.

$$|a_k+b_k|^{p_k}\leq G\{|a_k|^{p_k}+|b_k|^{p_k}\}, \tag{1}$$

where $a_k,b_k\in\mathbb{C}$, $0<p_k\leq\sup_k p_k=H$, $G=\max\left(1,2^{H-1}\right)$ [15].

2. Main Results

In this section we prove some results involving the sequence spaces $[V,\lambda,f,p]_0$ (Δ_v^r,q), $[V,\lambda,f,p]_1$ (Δ_v^r,q) and $[V,\lambda,f,p]_\infty$ (Δ_v^r,q).

Definition 1. Let f be a modulus function, X be a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous seminorms q and $p=(p_k)$ be a sequence of strictly positive real numbers. By $w(X)$ we shall denote the space of all sequences defined over X. Let $v=(v_k)$ be any fixed sequence of nonzero complex numbers. Now we define the following sequence spaces:

$$[V,\lambda,f,p]_1(\Delta_v^r,q) = \left\{ x\in w(X): \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(q\left(\Delta_v^r x_k - L\right)\right)\right]^{p_k}=0, \right.$$

$$\left. \text{for some } L\right\},$$

$$[V,\lambda,f,p]_0(\Delta_v^r,q) = \left\{ x\in w(X): \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(q\left(\Delta_v^r x_k\right)\right)\right]^{p_k}=0 \right\},$$

$$[V,\lambda,f,p]_\infty(\Delta_v^r,q) = \left\{ x\in w(X): \sup_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(q\left(\Delta_v^r x_k\right)\right)\right]^{p_k}<\infty \right\},$$

where $r\in\mathbb{N}$, $\Delta_v^0 x=(v_k x_k)$, $\Delta_v x=(v_k x_k - v_{k+1}x_{k+1})$, $\Delta_v^r x=(\Delta_v^{r-1}x_k - \Delta_v^{r-1}x_{k+1})$ and so $\Delta_v^r x_k=\sum_{i=0}^{r}(-1)^i\binom{r}{i}v_{k+i}x_{k+i}$.

The above sequence spaces contain some unbounded sequences for $r\geq 1$. For example, let $X=\mathbb{C}$, $f(x)=x$, $q(x)=|x|$, $\lambda_n=n$ for all $n\in\mathbb{N}$, $v=(1,1,1,...)$ and $p_k=1$ for all $k\in\mathbb{N}$, then $(k^r)\in[V,\lambda,f,p]_\infty(\Delta_v^r,q)$ but $(k^r)\notin\ell_\infty$.

If $x\in[V,\lambda,f,p]_1(\Delta_v^r,q)$, then we will write that $x_k\to L\,[V,\lambda,f,p]_1(\Delta_v^r,q)$ and L will be called λ_{qv}^r- difference limit of x with respect to the modulus f.

Throughout the paper Z will denote any one of the notation 0, 1, or ∞.

In the case $f(x)=x$, $p_k=1$ for all $k\in\mathbb{N}$ and $p_k=1$ for all $k\in\mathbb{N}$, we shall write $[V,\lambda]_Z(\Delta_v^r,q)$ and $[V,\lambda,f]_Z(\Delta_v^r,q)$ instead of $[V,\lambda,f,p]_Z(\Delta_v^r,q)$, respectively.

Also in the special case $q(x)=|x|$, $\lambda_n=n$ for all $n\in\mathbb{N}$ and $X=\mathbb{C}$ we shall write $[C,1]_Z(\Delta_v^r)$ instead of $[V,\lambda]_Z(\Delta_v^r,q)$.

The proofs of the following theorems are obtained by using the well-known standard techniques, therefore we give them without proofs.

Theorem 2.1. *Let the sequence* (p_k) *be bounded. Then the spaces* $[V,\lambda,f,p]_Z$ (Δ_v^r,q) *are linear spaces.*

Theorem 2.2. *Let f be a modulus function, then* $[V,\lambda,f,p]_0\left(\Delta_v^r,q\right)\subset[V,\lambda,f,p]_1\left(\Delta_v^r,q\right)\subset[V,\lambda,f,p]_\infty\left(\Delta_v^r,q\right)$ *and the inclusions are strict.*

Theorem 2.3. $[V,\lambda,p]_0\left(\Delta_v^r,q\right)$ *is a paranormed (need not total paranorm) space with*

$$g_\Delta(x)=\sup_n\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[\left(q\left(\Delta_v^r x_k\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

where $M=\max(1,\sup p_k)$.

Theorem 2.4. *If* $r\geq 1$, *then the inclusion* $[V,\lambda,f,p]_Z\left(\Delta_v^{r-1},q\right)\subset[V,\lambda,f,p]_Z\left(\Delta_v^r,q\right)$ *is strict. In general* $[V,\lambda,f,p]_Z\left(\Delta_v^i,q\right)\subset[V,\lambda,f,p]_Z\left(\Delta_v^r,q\right)$ *for all* $i=1,2,\ldots,r-1$ *and the inclusions are strict.*

Proof. Proof follows from the inequality

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(q\left(\Delta_v^r x_k\right)\right)\right]^{p_k} \leq\ & \frac{G}{\lambda_n}\sum_{k\in I_n}\left[f\left(q\left(\Delta_v^{r-1}x_k\right)\right)\right]^{p_k}\\ &+\frac{G}{\lambda_n}\sum_{k\in I_n}\left[f\left(q\left(\Delta_v^{r-1}x_{k+1}\right)\right)\right]^{p_k}.\end{aligned}$$

To show the inclusion is strict, let $X=\mathbb{C}$, $q(x)=|x|$, $v=(1,1,1,...)$, $p_k=1$ for all $k\in\mathbb{N}$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then the sequence $x=(k^r)$, for example, belongs to $[V,\lambda,f,p]_\infty\left(\Delta_v^r,q\right)$, but does not belong to $[V,\lambda,f,p]_\infty\left(\Delta_v^{r-1},q\right)$ for $f(x)=x$.

The proof of the following result is a routine work.

Proposition 2.5. $[V,\lambda,f,p]_1\left(\Delta_v^{r-1},q\right)\subset[V,\lambda,f,p]_0\left(\Delta_v^r,q\right)$.

Theorem 2.6. *Let f, f_1, f_2 be modulus functions. For any two sequences $p=(p_k)$ and $t=(t_k)$ of strictly positive real numbers and any two seminorms q_1, q_2 we have*

(i) $[V,\lambda,f_1,p]_Z\left(\Delta_v^r,q\right)\subset[V,\lambda,f\circ f_1,p]_Z\left(\Delta_v^r,q\right)$,

(ii) $[V,\lambda,f_1,p]_Z\left(\Delta_v^r,q\right)\cap[V,\lambda,f_2,p]_Z\left(\Delta_v^r,q\right)\subset[V,\lambda,f_1+f_2,p]_Z\left(\Delta_v^r,q\right)$,

(iii) $[V,\lambda,f,p]_Z\left(\Delta_v^r,q_1\right)\cap[V,\lambda,f,p]_Z\left(\Delta_v^r,q_2\right)\subset[V,\lambda,f,p]_Z\left(\Delta_v^r,q_1+q_2\right)$,

(iv) *If* $\limsup\frac{f_1(x)}{f_2(x)}<\infty$, *then* $[V,\lambda,f_2,p]_Z\left(\Delta_v^r,q\right)\subset[V,\lambda,f_1,p]_Z\left(\Delta_v^r,q\right)$,

(v) *If q_1 is stronger than q_2 then* $[V,\lambda,f,p]_Z\left(\Delta_v^r,q_1\right)\subset[V,\lambda,f,p]_Z\left(\Delta_v^r,q_2\right)$,

(vi) *If q_1 is equivalent to q_2 then* $[V,\lambda,f,p]_Z\left(\Delta_v^r,q_1\right)=[V,\lambda,f,p]_Z\left(\Delta_v^r,q_2\right)$,

(vii) $[V,\lambda,f,p]_Z\left(\Delta_v^r,q_1\right)\cap[V,\lambda,f,t]_Z\left(\Delta_v^r,q_2\right)\neq\emptyset$.

Proof. (i) We shall only prove (i) for $Z=0$ and the other cases can be proved by using the similar arguments. Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $f(t)<\varepsilon$ for $0\leq t\leq\delta$. Write $y_k=f_1\left(q\left(\Delta_v^r x_k\right)\right)$ and consider

$$\sum_{k\in I_n}[f(y_k)]^{p_k}=\sum_1[f(y_k)]^{p_k}+\sum_2[f(y_k)]^{p_k}$$

where the first summation is over $y_k\leq\delta$ and second summation is over $y_k>\delta$. Since f is continuous, we have

$$\sum_1[f(y_k)]^{p_k}<\lambda_n\max\left(\varepsilon^{\inf p_k},\varepsilon^H\right). \tag{2}$$

By the definition of f we have for $y_k>\delta$,

$$f(y_k)<2f(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{\lambda_n}\sum_2[f(y_k)]^{p_k}\leq\max\left(1,\left(2f(1)\delta^{-1}\right)^H\right)\frac{1}{\lambda_n}\sum_2[y_k]^{p_k}. \tag{3}$$

From (2) and (3), we obtain $[V,\lambda,f_1,p]_0\left(\Delta_v^r,q\right)\subset[V,\lambda,f\circ f_1,p]_0\left(\Delta_v^r,q\right)$.

The following result is a consequence of Theorem 2.6 (i).

Corollary 2.7. *Let f be a modulus function. Then $[V,\lambda,p]_Z\left(\Delta_v^r,q\right)\subset[V,\lambda,f,p]_Z\left(\Delta_v^r,q\right)$.*

Theorem 2.8. *Let (X,q) be a complete seminormed space. Then the sequence spaces $[V,\lambda]_Z\left(\Delta_v^r,q\right)$ are complete, seminormed by*

$$g_\Delta(x)=\sum_{i=1}^r q\left(v_i x_i\right)+\sup\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\left(q\left(\Delta_v^r x_k\right)\right)\Big)$$

where $v_k\neq 0$ for each $k\in\mathbb{N}$ and X is any sequence space.

Proof. We shall prove only that $[V,\lambda]_\infty\left(\Delta_v^r,q\right)$ is complete with respect to the above seminorm. The others can be proved by the same way. Let (x^s) be a Cauchy sequence in $[V,\lambda]_\infty\left(\Delta_v^r,q\right)$, where $x^s=(x_i^s)_{i=1}^\infty$. Then we have

$$g_\Delta(x^s-x^t)\to 0,\ \text{as } s,t\to\infty.$$

Let $\varepsilon > 0$ be given, then there exists a positive integer n_0 such that $g_\Delta(x^s - x^t) < \varepsilon$ for all $s, t > n_0$. So we have

$$\sum_{i=1}^{r} q\left(x_i^s - x_i^t\right) + \sup\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left(q(\Delta_v^r\left(x_i^s - x_i^t\right))\right)\right) < \varepsilon, \text{ for all } s, t > n_0.$$

Hence (x_i^s) , (for $i \le r$) and $(\Delta_v^r(x_k^s))$ for all $k \in \mathbb{N}$, are Cauchy sequences in X. Since X is complete, these sequences are convergent in X. Suppose that $x_i^s \to x_i$, (for $i \le r$) and $\Delta_v^r(x_k^s) \to y_k$ as $s \to \infty$, for each $k \in \mathbb{N}$. Then we can find a sequence (x_k) such that $y_k = \Delta_v^r x_k$ for each $k \in \mathbb{N}$. These $x_k\,'s$ can be written as

$$x_k = v_k^{-1}\sum_{i=1}^{k-r}(-1)^r\binom{k-i-1}{r-1}y_i = v_k^{-1}\sum_{i=1}^{k}(-1)^r\binom{k+r-i-1}{r-1}y_{i-r},$$

for sufficiently large k, for instance $k > r$, where $y_{1-r} = y_{2-r} = \ldots = y_0 = 0$. Thus $(\Delta_v^r(x_k^s)) = \left(\left(\Delta_v^r\left(x_k^1\right)\right), \left(\Delta_v^r\left(x_k^2\right)\right), \ldots\right)$ converges to $\Delta_v^r x_k$, for each $k \in \mathbb{N}$ in X. Hence $g_\Delta(x^s - x) \to 0$ as $s \to \infty$. Since $(x^s - x)$, $(x^s) \in [V,\lambda]_\infty(\Delta^r, q)$ and the space $[V,\lambda]_\infty(\Delta_v^r, q)$ is a linear space we have $x = x^s - (x^s - x) \in [V,\lambda]_\infty(\Delta_v^r, q)$. Hence $[V,\lambda]_\infty(\Delta_v^r, q)$ is complete.

Theorem 2.9. *Let* $0 < p_k \le t_k$ *and* $\left(\frac{t_k}{p_k}\right)$ *be bounded. Then*$[V,\lambda,f,t]_Z(\Delta_v^r, q) \subset [V,\lambda,f,p]_Z(\Delta_v^r, q)$.

Proof. We prove it for $Z = 0$ and the other cases will follow on applying similar techniques. Let $x \in [V,\lambda,f,t]_0(\Delta_v^r, q)$. Write $w_k = [f(q(\Delta_v^r x_k))]^{t_k}$ and $\mu_k = \frac{p_k}{t_k}$, so that $0 < \mu < \mu_k \le 1$ for each k.

We define the sequences (u_k) and (s_k) as follows:

Let $u_k = w_k$ and $s_k = 0$ if $w_k \ge 1$, and let $u_k = 0$ and $s_k = w_k$ if $w_k < 1$. Then it is clear that for all $k \in \mathbb{N}$, we have $w_k = u_k + s_k$, $w_k^{\mu_k} = u_k^{\mu_k} + s_k^{\mu_k}$. Now it follows that $u_k^{\mu_k} \le u_k \le w_k$ and $s_k^{\mu_k} \le s_k^{\mu}$. Therefore

$$\lambda_n^{-1}\sum_{k\in I_n} w_k^{\mu_k} \le \lambda_n^{-1}\sum_{k\in I_n} w_k + \left(\lambda_n^{-1}\sum_{k\in I_n} s_k\right)^{\mu}.$$

Hence $x \in [V,\lambda,f,p]_0(\Delta_v^r, q)$.

3. Statistical Convergence

The notion of statistical convergence was introduced by Fast [8]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from sequence space point of view and linked with summability theory by Fridy [9], Connor [3], Salát [21], Mursaleen [19], Işik [10], Kolk [13], Malkowsky and Savaş [18] and many others. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the characteristic function of E.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$.

In this section we introduce λ_{qv}^r- statistically convergent sequences and give some inclusion relations between $S_\lambda(\Delta_v^r, q)$ and $[V, \lambda, f, p]_1(\Delta_v^r, q)$.

Definition 2. A sequence $x = (x_k)$ is said to be λ_{qv}^r- statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : q(\Delta_v^r x_k - L) \geq \varepsilon\}| = 0,$$

where the vertical bars denote the cardinality of the enclosed set. In this case we write $S_\lambda(\Delta_v^r, q) - \lim x = L$ or $x_k \to LS_\lambda(\Delta_v^r, q)$.

In the case $\lambda_n = n$ and $L = 0$ we shall write $S(\Delta_v^r, q)$ and $S_{\lambda_0}(\Delta_v^r, q)$ instead of $S_\lambda(\Delta_v^r, q)$, respectively.

The proofs of the following two theorems are easily obtained by using the same techniques of Mursaleen [19, Theorem 2.1 and Theorem 3.1], therefore we give them without proofs.

Theorem 3.1. *Let $\lambda = (\lambda_n)$ be the same as in Section 1, then*

(i) If $x_k \to L[V,\lambda]_1(\Delta_v^r, q)$, *then* $x_k \to LS_\lambda(\Delta_v^r, q)$,

(ii) If $x \in \ell_\infty(\Delta_v^r, q)$ *and* $x_k \to LS_\lambda(\Delta_v^r, q)$, *then* $x_k \to L[V,\lambda]_1(\Delta_v^r, q)$,

(iii) $S_\lambda(\Delta_v^r, q) \cap \ell_\infty(\Delta_v^r, q) = [V,\lambda]_1(\Delta_v^r, q) \cap \ell_\infty(\Delta_v^r, q)$,

where $\ell_\infty(\Delta_v^r, q) = \{x \in w(X) : \sup_k q(\Delta_v^r x_k) < \infty\}$.

Remark 1. In fact the set $[V,\lambda]_1(\Delta_v^r, q)$ is a proper subset of $S_\lambda(\Delta_v^r, q)$.

Theorem 3.2. *If* $\liminf \frac{\lambda_n}{n} > 0$, *then* $S(\Delta_v^r, q) \subseteq S_\lambda(\Delta_v^r, q)$.

Theorem 3.3. *Let f be a modulus function and* $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. *Then* $[V, \lambda, f, p]_1(\Delta_v^r, q) \subset S_\lambda(\Delta_v^r, q)$.

Proof. Let $x \in [V, \lambda, f, p]_1 (\Delta_v^r, q)$ and $\varepsilon > 0$ be given. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(q(\Delta_v^r x_k - L))]^{p_k} \geq \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ q(\Delta_v^r x_k - L) \geq \varepsilon}} [f(q(\Delta_v^r x_k - L))]^{p_k}$$
$$\geq \frac{1}{\lambda_n} |\{k \in I_n : q(\Delta_v^r x_k - L) \geq \varepsilon\}| \min\left([f(\varepsilon)]^h, [f(\varepsilon)]^H\right).$$

Hence $x \in S_\lambda (\Delta_v^r, q)$.

Theorem 3.4. *If f is bounded then $S_\lambda (\Delta_v^r, q) \subset [V, \lambda, f, p]_1 (\Delta_v^r, q)$.*

Proof. Suppose that f is bounded and let $\varepsilon > 0$ be given. Since f is bounded there exists an integer K such that $f(x) < K$, for all $x \geq 0$. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(q(\Delta_v^r x_k - L))]^{p_k} = \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ q(\Delta_v^r x_k - L) \geq \varepsilon}} [f(q(\Delta_v^r x_k - L))]^{p_k}$$
$$+ \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ q(\Delta_v^r x_k - L) < \varepsilon}} [f(q(\Delta_v^r x_k - L))]^{p_k}$$
$$\leq \max\left(K^h, K^H\right) \frac{1}{\lambda_n} |\{k \in I_n : q(\Delta_v^r x_k - L) \geq \varepsilon\}|$$
$$+ \max\left(f(\varepsilon)^h,\ f(\varepsilon)^H\right).$$

Hence $x \in [V, \lambda, f, p]_1 (\Delta_v^r, q)$.

Theorem 3.5. *$S_\lambda (\Delta_v^r, q) = [V, \lambda, f, p]_1 (\Delta_v^r, q)$ if and only if f is bounded.*

Proof. Let f be bounded. By Theorem 3.3 and Theorem 3.4 we have $S_\lambda (\Delta_v^r, q) = [V, \lambda, f, p]_1 (\Delta_v^r, q)$.

Conversely suppose that f is unbounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, \ldots$. If we choose

$$\Delta_v^r x_i = \begin{cases} t_k, & i = k^2,\ i = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$\frac{1}{\lambda_n} |\{k \in I_n : |\Delta_v^r x_k| \geq \varepsilon\}| \leq \frac{\sqrt{\lambda_{n-1}}}{\lambda_n}$$

for all n and so $x \in S_\lambda\left(\Delta_v^r, q\right)$, but $x \notin [V, \lambda, f, p]_1\left(\Delta_v^r, q\right)$ for $X = \mathbb{C}$ and $p_k = 1$ for all $k \in \mathbb{N}$. This contradicts to $S_\lambda\left(\Delta_v^r, q\right) = [V, \lambda, f, p]\left(\Delta_v^r, q\right)$.

Theorem 3.6. *The sequence spaces* $[V, \lambda, f, p]_Z\left(\Delta_v^r, q\right)$, $S_\lambda\left(\Delta_v^r, q\right)$ *and* S_{λ_0} $\left(\Delta_v^r, q\right)$ *are not solid for* $r \geq 1$.

Proof. Let $X = \mathbb{C}$, $p_k = 1$ for all $k \in \mathbb{N}$, $f(x) = x$, $q(x) = |x|$, $v = (1, 1, ...)$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (k^r) \in [V, \lambda, f, p]_\infty\left(\Delta_v^r, q\right)$ but $(\alpha_k x_k) \notin [V, \lambda, f, p]_\infty\left(\Delta_v^r, q\right)$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $[V, \lambda, f, p]_\infty\left(\Delta_v^r, q\right)$ is not solid. The other cases can be proved on considering similar examples.

From the above theorem we may give the following corollary.

Corollary 3.7. *The sequence spaces* $[V, \lambda, f, p]_Z\left(\Delta_v^r, q\right)$ *are not perfect for* $r \geq 1$.

Remark 2. *If* $|v_k| \leq 1$ *for all* $k \in \mathbb{N}$*, then* $[V, \lambda]_0\left(\Delta_v^r, q\right)$ *and* $[V, \lambda]_\infty\left(\Delta_v^r, q\right)$ *are solid for* $r = 0$.

Theorem 3.8. *The sequence spaces* $[V, \lambda, f, p]_1\left(\Delta_v^r, q\right)$*,* $[V, \lambda, f, p]_\infty\left(\Delta_v^r, q\right)$, $S_\lambda\left(\Delta_v^r, q\right)$ *and* $S_{\lambda_0}\left(\Delta_v^r, q\right)$ *are not symmetric for* $r \geq 1$.

Proof. Under the restrictions on X, p, f, q, v and λ as given in the proof of Theorem 3.6, consider the sequence $x = (k^r)$, then $x \in [V, \lambda, f, p]_\infty\left(\Delta_v^r, q\right)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}.$$

Then $(y_k) \notin [V, \lambda, f, p]_\infty\left(\Delta_v^r, q\right)$. For the space $S_{\lambda_0}\left(\Delta_v^r, q\right)$, consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & \text{if } (2i-1)^2 \leq k < (2i)^2, \quad i = 1, 2, ... \\ 4, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in S_0(\Delta)$. Let (y_k) be the same as above, then $(y_k) \notin S_0(\Delta)$.

Remark 3. The space $[V, \lambda, f, p]_0\left(\Delta_v^r, q\right)$ is not symmetric for $r \geq 2$.

Theorem 3.9. *The sequence spaces* $[V, \lambda, f, p]_Z\left(\Delta_v^r, q\right)$*,* $S_\lambda\left(\Delta_v^r, q\right)$ *and* S_{λ_0} $\left(\Delta_v^r, q\right)$ *are not sequence algebras.*

Proof. Under the restrictions on X, p, f, q, v and λ as given in the proof of Theorem 3.6, consider the sequences $x = \left(k^{r-2}\right)$ and $y = \left(k^{r-2}\right)$, then $x, y \in [V, \lambda, f, p]_Z\left(\Delta_v^r, q\right)$, but $x.y \notin [V, \lambda, f, p]_Z\left(\Delta_v^r, q\right)$. The other cases can be proved on considering similar examples.

4. The Spaces of Cesàro Summable Generalized Difference Sequences

In this section we compute Köthe-Toeplitz duals of the spaces of Cesàro summable and strongly Cesàro summable difference sequences of order r.

We define the general sequence space $X(\Delta_v^r)$ as follows:

$$X(\Delta_v^r) = \{x \in w : (\Delta_v^r x) \in X\},$$

where $r \in \mathbb{N}$ and X is any sequence space.

It can be shown that the following inclusions are strict.

(i) $(C,1)_0(\Delta_v^r) \subset (C,1)(\Delta_v^r) \subset (C,1)_\infty(\Delta_v^r)$,

(ii) $(C,1)_0(\Delta_v^r) \subset (C,1)_0(\Delta_v^{r+1})$, $(C,1)(\Delta_v^r) \subset (C,1)(\Delta_v^{r+1})$ and $(C,1)_\infty$ $(\Delta_v^r) \subset (C,1)_\infty(\Delta_v^{r+1})$,

(iii) $c_0(\Delta_v^r) \subset (C,1)_0(\Delta_v^r)$, $c(\Delta_v^r) \subset (C,1)(\Delta_v^r)$ and $\ell_\infty(\Delta_v^r) \subset (C,1)_\infty(\Delta_v^r)$.

Theorem 4.1. *$(C,1)_0(\Delta_v^r)$, $(C,1)(\Delta_v^r)$ and $(C,1)_\infty(\Delta_v^r)$ are Banach spaces normed by*

$$\|x\|_\Delta = \sum_{i=1}^{r} |v_i x_i| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta_v^m x_k \right|,$$

and $[C,1]_0(\Delta_v^r)$, $[C,1]_1(\Delta_v^r)$ and $[C,1]_\infty(\Delta_v^r)$ are Banach spaces normed by

$$\|x\|_{\Delta'} = \sum_{i=1}^{r} |v_i x_i| + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta_v^r x_k| \right),$$

where $v_k \neq 0$ for each $k \in \mathbb{N}$.

Proof. Proof follows from Theorem 2.8.

Let us define the operator $D : X(\Delta_v^r) \to X(\Delta_v^r)$ by $Dx = (0,0,...,0,x_{m+1}, x_{m+2},...)$, where $x = (x_1,x_2,x_3...)$. It is trivial that D is a linear operator on $X(\Delta_v^r)$. Furthermore, the set

$$D[X(\Delta_v^r)] = DX(\Delta_v^r) = \{x = (x_k) : x \in X(\Delta_v^r),\ x_1 = x_2 = ... = x_m = 0\}$$

is a subspace of $X(\Delta_v^r)$. The operator $\Delta_v^r : DX(\Delta_v^r) \to X$ defined by $\Delta_v^r x = y = (\Delta_v^r x_k)$ is bijective (one to one and onto).

Let X stand for $(C,1)_0$, $(C,1)$, $(C,1)_\infty$, $[C,1]_0$, $[C,1]_1$ and $[C,1]_\infty$ and $r \in \mathbb{N}$. Now we will compute Köthe-Toeplitz duals of the sequence spaces $X(\Delta_v^r)$. For this we need the following lemma.

Lemma 4.2.

(i) $x \in D\,(C,1)_\infty\,(\Delta_v^r)$ *implies* $\sup_k k^{-r}\,|v_k x_k| < \infty$ *for all* $r \in \mathbb{N}$,

(ii) *There exist positive constants* K_1 *and* K_2 *such that* $K_1 k^r \leq \binom{r+k}{k} \leq K_2 k^r$, $k = 1, 2, \ldots$,

(iii) $\sum\limits_{j=0}^{k} \binom{r+j-1}{j} = \binom{k+r}{r} = \binom{r+k}{k}$.

Proof. Omitted.

Lemma 4.3. *Let* r *be a positive integer. Then*

$$\left(D\,(C,1)_\infty\,(\Delta_v^r)\right)^\alpha = U_1, \textit{where } U_1 = \left\{a \in w : \sum_{k=1}^{\infty} k^r \left|v_k^{-1} a_k\right| < \infty\right\}.$$

Proof. This can be proved by the same technique of Lemma 2.7 of Et and Colak [6].

Remark 4. Let r be a positive integer. Then $(D(C,1)_\infty(\Delta_v^r))^\alpha = ((C,1)_\infty\,(\Delta_v^r))^\alpha$.

Theorem 4.4. *Let* r *be a positive integer. Then*

(i) $((C,1)_0\,(\Delta_v^r))^\alpha = ((C,1)\,(\Delta_v^r))^\alpha = U_1$,

(ii) $((C,1)_0\,(\Delta_v^r))^{\alpha\alpha} = ((C,1)\,(\Delta_v^r))^{\alpha\alpha} = ((C,1)_\infty\,(\Delta_v^r))^{\alpha\alpha} = U_2$.

where $U_2 = \left\{a \in w : \sup_{k \geq 1} k^{-r}\,|a_k v_k| < \infty\right\}$.

Proof. (i) Since $((C,1)_\infty\,(\Delta_v^r))^\alpha = U_1$ and $(C,1)_0\,(\Delta_v^r) \subset (C,1)\,(\Delta_v^r) \subset (C,1)_\infty\,(\Delta_v^r)$, we have $U_1 \subset ((C,1)_\infty\,(\Delta_v^r))^\alpha \subset ((C,1)\,(\Delta_v^r))^\alpha \subset ((C,1)_o\,(\Delta_v^r))^\alpha$.

Conversely, let $a \notin U_1$. By Lemma 4.2 (ii), we can choose a sequence $(k\,(i))$ of integers, $0 = k\,(0) < k\,(1) < \ldots$, such that

$$\sum_{k=k(i)}^{k(i+1)-1} \left|v_k^{-1} a_k\right| \binom{r+k}{k} \geq i+1 \qquad (i = 0, 1, \ldots)\,.$$

Define a sequence $x = (x_k)$ as follows:

$$x_k \;=\; v_k^{-1}\left(\sum_{\ell=0}^{i-1} \frac{1}{\ell+1} \sum_{j=k(\ell)}^{k(\ell+1)-1} \binom{r+k-j-1}{k-j} + \frac{1}{i+1} \sum_{j=k(i)}^{k} \binom{r+k-j-1}{k-j}\right),$$

$$(k\,(i) \leq k \leq k\,(i+1)-1;\; i = 0, 1, \ldots)\,.$$

Then $x \in (C,1)_0 (\Delta_v^r)$. On the other hand we have

$$\sum_{k=k(i)}^{k(i+1)-1} |a_k x_k| \geq \sum_{k=k(i)}^{k(i+1)-1} \left|v_k^{-1} a_k\right| \frac{1}{i+1} \sum_{j=0}^{k} \binom{r+j-1}{j}$$
$$= \frac{1}{i+1} \sum_{k=k(i)}^{k(i+1)-1} \left|v_k^{-1} a_k\right| \binom{r+k}{k} \geq 1 \qquad (i = 0, 1, \ldots).$$

Thus $a \notin ((C,1)_0 (\Delta_v^r))^{\alpha}$, and hence $((C,1)_0 (\Delta_v^r))^{\alpha} \subset U_1$. This completes the proof.

(ii) Omitted.

Theorem 4.5. *Let* r *be a positive integer. Then*

(i) $([C,1]_0 (\Delta_v^r))^{\alpha} = ([C,1]_1 (\Delta_v^r))^{\alpha} = ([C,1]_{\infty} (\Delta_v^r))^{\alpha} = U_1$,

(ii) $([C,1]_0 (\Delta_v^r))^{\alpha\alpha} = ([C,1]_1 (\Delta_v^r))^{\alpha\alpha} = ([C,1]_{\infty} (\Delta_v^r))^{\alpha\alpha} = U_2$.

The proof is similar to that of Theorem 4.4.

Corollary 4.6. *The sequence spaces* $(C,1)_0 (\Delta_v^r)$, $(C,1) (\Delta_v^r)$, $(C,1)_{\infty} (\Delta_v^r)$, $[C,1]_0 (\Delta_v^r)$, $[C,1]_1 (\Delta_v^r)$ *and* $[C,1]_{\infty} (\Delta_v^r)$ *are not perfect.*

Theorem 4.7. *Let* r *be a positive integer. Then* $\ell_{\infty} (\Delta_v^r) \cap S (\Delta_v^r) \subset (C,1) (\Delta_v^r)$.

Proof. Omitted.

The converse of Therem 4.7 does not hold, for example the sequence $x = (0, -1, -1, -2, -2, -3, -3, ...)$ belongs to $(C,1) (\Delta_v)$ and does not belong to $S (\Delta_v)$ for $v = (1, 1, 1, ...)$.

The proof of the following theorem is a routine work, therefore we give it without proof.

Theorem 4.8. *Let* $u = (u_k)$ *and* $v = (v_k)$ *be any fixed sequences of nonzero complex numbers, then*

(i) *If* $\sup_k k^r \left|v_k^{-1} u_k\right| < \infty$ *, then* $(C,1)_{\infty} (\Delta_v^r) \subset (C,1)_{\infty} (\Delta_u^r)$ *and* $[C,1]_{\infty} (\Delta_v^r) \subset [C,1]_{\infty} (\Delta_u^r)$

(ii) *If* $k^r \left|v_k^{-1} u_k\right| \to L \, (k \to \infty)$ *, then* $(C,1) (\Delta_v^r) \subset (C,1) (\Delta_u^r)$ *and* $[C,1] (\Delta_v^r) \subset [C,1] (\Delta_u^r)$

(iii) *If* $k^r \left|v_k^{-1} u_k\right| \to 0 \, (k \to \infty)$ *, then* $(C,1)_0 (\Delta_v^r) \subset (C,1)_0 (\Delta_u^r)$ *and* $[C,1]_0 (\Delta_v^r) \subset [C,1]_0 (\Delta_u^r)$

If we take $u = (1, 1, 1, ...)$ and $v = (1, 1, 1, ...)$ in the last theorem, then we have the following results.

Corollary 4.9.

(i) *If* $\sup_k k^r \left|v_k^{-1}\right| < \infty$ *, then* $(C,1)_\infty (\Delta_v^r) \subset (C,1)_\infty (\Delta^r)$ *and* $[C,1]_\infty (\Delta_v^r) \subset [C,1]_\infty (\Delta^r)$,

(ii) *If* $k^r \left|v_k^{-1}\right| \to L \ (k \to \infty)$ *, then* $(C,1) (\Delta_v^r) \subset (C,1) (\Delta^r)$ *and* $[C,1] (\Delta_v^r) \subset [C,1] (\Delta^r)$,

(iii) *If* $k^r \left|v_k^{-1}\right| \to 0 \ (k \to \infty)$ *, then* $(C,1)_0 (\Delta_v^r) \subset (C,1)_0 (\Delta^r)$ *and* $[C,1]_0 (\Delta_v^r) \subset [C,1]_0 (\Delta^r)$.

Corollary 4.10.

(i) *If* $\sup_k k^r |v_k| < \infty$ *, then* $(C,1)_\infty (\Delta^r) \subset (C,1)_\infty (\Delta_v^r)$ *and* $[C,1]_\infty (\Delta^r) \subset [C,1]_\infty (\Delta_v^r)$,

(ii) *If* $k^r |v_k| \to L \ (k \to \infty)$ *, then* $(C,1) (\Delta^r) \subset (C,1) (\Delta_v^r)$ *and* $[C,1] (\Delta^r) \subset [C,1] (\Delta_v^r)$,

(iii) *If* $k^r |v_k| \to 0 \ (k \to \infty)$ *, then* $(C,1)_0 (\Delta^r) \subset (C,1)_0 (\Delta_v^r)$ *and* $[C,1]_0 (\Delta^r) \subset [C,1]_0 (\Delta_v^r)$.

5. Particular Cases

Firstly, we note that $X(\Delta^r)$ and $X(\Delta_v^r)$ overlap but one neither one contains the other, where X is any sequence space. For example if we choose $x = (k^r)$ and $v = (k)$, then $x \in [C,1]_\infty (\Delta^r)$, but $x \notin [C,1]_\infty (\Delta_v^r)$, conversely if we choose $x = \left(k^{r+1}\right)$ and $v = \left(k^{-1}\right)$, then $x \notin [C,1]_\infty (\Delta^r)$, but $x \in [C,1]_\infty (\Delta_v^r)$.

Definition 3. Let X be any sequence space and $v = (v_k)$ be any sequence of nonzero complex numbers. We say that the sequence space $X(\Delta^r)$ is $v-$invariant if $X(\Delta_v^r) = X(\Delta^r)$.

Now there is an open problem for researchers. It may be investigated the conditions of the sequence $v = (v_k)$ for which the equalities $[V, \lambda, f, p]_Z (\Delta_v^r, q) = [V, \lambda, f, p]_Z (\Delta^r, q)$ and $S_\lambda (\Delta_v^r, q) = S_\lambda (\Delta^r, q)$ hold.

If one considers the sequence spaces

(i) $[V, \lambda, f, p]_Z (\Delta^r, q)$ and $S_\lambda (\Delta^r, q)$ instead of $[V, \lambda, f, p]_Z (\Delta_v^r, q)$ and $S_\lambda (\Delta_v^r, q)$,

(ii) $[V, \lambda, f, p]_Z (\Delta_v^r)$ and $S_\lambda (\Delta_v^r)$ instead of $[V, \lambda, f, p]_Z (\Delta_v^r, q)$ and $S_\lambda (\Delta_v^r, q)$,

(iii) $[V, f, p]_Z (\Delta_v^r, q)$ and $S(\Delta_v^r, q)$ instead of $[V, \lambda, f, p]_Z (\Delta_v^r, q)$ and $S_\lambda (\Delta_v^r, q)$,

most of the results which have been proved in the previous sections will be true for these spaces as well.

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