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ON CONVERGENCE OF A RECURSIVE SEQUENCE $x_{n+1} = f(x_{n-1}, x_n)$

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Abstract. C. H. Gibbons, M. R. S. Kulenovic and G. Ladas [1] have posed the following problem: Is there a solution of the difference equation:

$$x_{n+1} = \frac{\beta x_{n-1}}{\beta + x_n}, \quad x_{-1}, x_0 > 0, \beta > 0 \quad (n = 0, 1, 2, \dots)$$

such that $\lim_{n\to\infty} x_n = 0$? S. Stevic [2] gives an affirmative answer to this open problem and generalize this result to the equation of the form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)}, \quad x_{-1}, x_0 > 0 \quad (n = 0, 1, 2, \dots)$$

by using his ingenious device. In this note, we generalize the result of Stevic to the equation of the form:

$$x_{n+1} = f(x_{n-1}, x_n), \quad x_{-1}, x_0 > 0 \quad (n = 0, 1, 2, \dots).$$

However our proof is simple and short.

1. Introduction and Main Result

Recently S. Stevic [2] has proved the following result which gives an affirmative answer to the open problem on the convergency of a recursive sequence posed in [1]:

Theorem A. *Let g be a C^1-function on $[0, \infty)$ such that $g(0) = 1$ and $g'(x) > 0$ for all $x \in [0, \infty)$. Then for any $a > 0$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}}{g(x_n)}$ with $x_{-1} = a$ such that $x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.*

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In this note, we generalize his result. To do this we consider the convergency of the following nonlinear recursive sequence:

(1) $$x_{n+1} = f(x_{n-1}, x_n), \quad x_{-1}, x_0 > 0 \quad (n = 0, 1, 2, \dots),$$

where $f\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ is a continuous function which satisfies the following conditions:

(a) $f(x,y) \le x$ for each $x, y > 0$;

(b) If $f(y, f(x,y)) \le f(x,y)$, then $x \ge y$.

Let $a = x_{-1}$, $b = x_0$ and $x_n = x_n(a,b) \quad (n = 1, 2, \dots)$. Then $\{x_n(a,b)\}$ denotes the solution of Equation (1) with initial conditions $x_{-1} = a$ and $x_0 = b$. Also we can regard x_n as a continuous function : $(0,\infty) \times (0,\infty) \to (0,\infty)$ with variable (a,b). By (a), we see that the sequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are decreasing and hence there exist $p, q \ge 0$ such that $\lim_{n\to\infty} x_{2n} = p$ and $\lim_{n\to\infty} x_{2n-1} = q$. Therefore the sequence defined by the Equation (1) converges if and only if $p = q$ and hence the following problem is naturally posed:

(2) Is there $(a,b) \in (0,\infty) \times (0,\infty)$ such that $p(a,b) = q(a,b)$?

To solve the above problem, let $\varepsilon > 0$ and set

$$\begin{aligned} A_f(\varepsilon) &= \{a \in [\varepsilon,\infty) : b < f(a,b) \text{ for some } b \ge \varepsilon\}, \\ B_f(\varepsilon) &= \{b \in [\varepsilon,\infty) : b < f(a,b) \text{ for some } a \ge \varepsilon\}, \\ C_f(b;\varepsilon) &= \{a \in [\varepsilon,\infty) : b \ge f(a,b)\} \quad (b > 0). \end{aligned}$$

Furthermore set

$$A_f = \bigcup_{\varepsilon > 0} A_f(\varepsilon) \text{ and } B_f = \bigcup_{\varepsilon > 0} B_f(\varepsilon).$$

Then our main result is the following assertion which gives an affirmative answer to the problem (2) under some condition.

Theorem 1.

(i) *Suppose that A_f is non-empty and a is in A_f. Then there exists a solution $\{x_n\}$ of the Equation (1) such that $a = x_{-1} \ge x_0 \ge x_1 \ge x_2 \ge \cdots > 0$.*

(ii) *Suppose that B_f is non-empty and b is in B_f such that $C_f(b;\varepsilon)$ is a bounded set in $[\varepsilon,\infty)$ for each $\varepsilon \in (0,b)$. Then there exists a solution $\{x_n\}$ of the Equation (1) such that $x_{-1} \ge b = x_0 \ge x_1 \ge x_2 \ge \cdots > 0$.*

2. Proof of the Main Result

Let $\varepsilon > 0$. Choose $a \in A_f(\varepsilon)$ and $b \in B_f(\varepsilon)$ with $b > \varepsilon$. For each $n \geq -1$, set

$$A_n(b;\varepsilon) = \{u \in [\varepsilon, \infty) : x_n(u,b) \geq x_{n+1}(u,b)\}$$

and

$$B_n(a;\varepsilon) = \{v \in [\varepsilon, \infty) : x_n(a,v) \geq x_{n+1}(a,v)\}.$$

Then both $A_n(b;\varepsilon)$ and $B_n(a;\varepsilon)$ are closed sets in $[\varepsilon, \infty)$. Note that

$$A_{n+2}(b;\varepsilon) \subseteq A_n(b;\varepsilon) \text{ and } B_{n+2}(a;\varepsilon) \subseteq B_n(a;\varepsilon). \tag{3}$$

Indeed, if $u \in A_{n+2}(b;\varepsilon)$, then

$$\begin{aligned} f(x_n(u,b), x_{n+1}(u,b)) &= x_{n+2}(u,b) \geq x_{n+3}(u,b) \\ &= f(x_{n+1}(u,b), f(x_n(u,b), x_{n+1}(u,b))). \end{aligned}$$

By (b), we have $x_n(u,b) \geq x_{n+1}(u,b)$ and so $u \in A_n(b;\varepsilon)$. Consequently, $A_{n+2}(b;\varepsilon) \subseteq A_n(b;\varepsilon)$. Similarly for $B_{n+2}(a;\varepsilon) \subseteq B_n(a;\varepsilon)$. Now set

$$X_n(b;\varepsilon) = A_n(b;\varepsilon) \cap A_{n+1}(b;\varepsilon) \text{ and } Y_n(a;\varepsilon) = B_n(a;\varepsilon) \cap B_{n+1}(a;\varepsilon).$$

Then both $X_n(b;\varepsilon)$ and $Y_n(a;\varepsilon)$ are closed sets in $[\varepsilon, \infty)$ such that

$$X_{-1}(b;\varepsilon) \supseteq X_1(b;\varepsilon) \supseteq X_3(b;\varepsilon) \supseteq \ldots$$

and

$$Y_{-1}(a;\varepsilon) \supseteq Y_1(a;\varepsilon) \supseteq Y_3(a;\varepsilon) \supseteq \ldots$$

by (3). We assert that $X_{2n+1}(b;\varepsilon) \neq \emptyset$ and $Y_{2n+1}(a;\varepsilon) \neq \emptyset$. Indeed, suppose $X_{2n+1}(b;\varepsilon) = \emptyset$. Then $A_{2n+1}(b;\varepsilon)^{\mathrm{c}} \cup A_{2n+2}(b;\varepsilon)^{\mathrm{c}} = [\varepsilon, \infty)$. Also $A_{2n+1}(b;\varepsilon)^{\mathrm{c}} \cap A_{2n+2}(b;\varepsilon)^{\mathrm{c}} = \emptyset$. Suppose to the contrary that there is a $u \in [\varepsilon, \infty)$ such that $x_{2n+1}(u,b) < x_{2n+2}(u,b) < x_{2n+3}(u,b)$. This contradicts the fact that the sequence $\{x_{2k-1}\}$ is decreasing. Note that $A_{-1}(b;\varepsilon)^{\mathrm{c}} = \{u \in [\varepsilon, \infty) : u < b\} \neq \emptyset$ because $b > \varepsilon$ and that $A_0(b;\varepsilon)^{\mathrm{c}} = \{u \in [\varepsilon, \infty) : b < f(u,b)\} \neq \emptyset$ because $b \in B_f(\varepsilon)$. By (3), $A_{-1}(b;\varepsilon)^{\mathrm{c}} \subseteq A_{2n+1}(b;\varepsilon)^{\mathrm{c}}$ and $A_0(b;\varepsilon)^{\mathrm{c}} \subseteq A_{2n+2}(b;\varepsilon)^{\mathrm{c}}$ and so both $A_{2n+1}(b;\varepsilon)^{\mathrm{c}}$ and $A_{2n+2}(b;\varepsilon)^{\mathrm{c}}$ are non-empty disjoint open sets in $[\varepsilon, \infty)$. Then we arrive at a contradiction since $[\varepsilon, \infty)$ is connected. Consequently, we have $X_{2n+1}(b;\varepsilon) \neq \emptyset$. Also since $B_{-1}(a;\varepsilon)^{\mathrm{c}} = \{v \in [\varepsilon, \infty) : a < v\} \neq \emptyset$ and $B_0(a;\varepsilon)^{\mathrm{c}} = \{v \in [\varepsilon, \infty) : v < f(a,v)\} \neq \emptyset$ because $a \in A_f(\varepsilon)$, it follows from a similar argument that $Y_{2n+1}(a;\varepsilon) \neq \emptyset$.

Proof of (*i*). Let $a \in A_f$. Then there is an $\varepsilon_0 > 0$ such that $a \in A_f(\varepsilon_0)$. Since $Y_{-1}(a;\varepsilon_0) \subseteq B_{-1}(a;\varepsilon_0) = \{v \in [\varepsilon_0, \infty) : a \geq v\}$, it follows that $Y_{-1}(a;\varepsilon_0)$ is a bounded set in $[\varepsilon_0, \infty)$. Therefore by the above argument, we see that $\{Y_{-1}(a;\varepsilon_0), Y_1(a;\varepsilon_0), Y_3(a;\varepsilon_0), \ldots\}$ is a decreasing sequence of non-empty compact sets in $[\varepsilon_0, \infty)$. Then there exists an element v_0 of $\bigcap_{n=-1}^{\infty} Y_{2n+1}(a;\varepsilon_0)$ by the Heine-Borel covering theorem. Hence we have that

$$a = x_{-1}(a, v_0) \geq x_0(a, v_0) \geq x_1(a, v_0) \geq x_2(a, v_0) \geq \cdots > 0,$$

and then the assertion (i) holds.

Proof of (*ii*). Let $b \in B_f$ be such that $C_f(b;\varepsilon)$ is a bounded set in $[\varepsilon, \infty)$ for each $\varepsilon \in (0, b)$. Then there is an $\varepsilon_1 > 0$ such that $b \in B_f(\varepsilon_1)$. Note that $B_f(\varepsilon_1) \subseteq B_f(\varepsilon_1/2)$. Then $b \in B_f(\varepsilon_1/2)$ and $b > \frac{\varepsilon_1}{2}$. Since $X_{-1}(b;\varepsilon_1/2) \subseteq A_0(b;\varepsilon_1/2) = C_f(b;\varepsilon_1/2)$, it follows that $X_{-1}(b;\varepsilon_1/2)$ is a bounded set in $\left[\frac{\varepsilon_1}{2}, \infty\right)$. Therefore by the above argument, we see that $\{X_{-1}(b;\varepsilon_1/2), X_1(b;\varepsilon_1/2), X_3(b;\varepsilon_1/2), \ldots\}$ is a decreasing sequence of non-empty compact sets in $\left[\frac{\varepsilon_1}{2}, \infty\right)$. Then there exists an element u_0 of $\bigcap_{n=-1}^{\infty} X_{2n+1}(b;\varepsilon_1/2)$ by the Heine-Borel covering theorem. Hence we have that

$$x_{-1}(u_0, b) \geq b = x_0(u_0, b) \geq x_1(u_0, b) \geq x_2(u_0, b) \geq \cdots > 0,$$

and then the assertion (ii) holds. ∎

3. APPLICATION

Let $g\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ be a continuous function which satisfies the following conditions

(c) $g(x, \cdot)$ is an increasing function for any fixed $x > 0$;

(d) $\dfrac{g(y,x) - g(x,y)}{x - y} \geq 0$ for each $x, y > 0$ with $x \neq y$.

Set $f(x,y) = \dfrac{x}{1 + g(x,y)}$ for each $x, y > 0$. Then f is a continuous function of $(0,\infty)\times(0,\infty)$ into $(0,\infty)$ which satisfies the condition (a). Also f satisfies the condition (b). In fact, let $x, y > 0$ with $x \neq y$ and suppose $f(y, f(x,y)) \leq f(x,y)$. By (c), we have

$$\begin{aligned}
\frac{x}{1+g(x,y)} &= f(x,y) \geq f(y, f(x,y)) \\
&= \frac{y}{1 + g\left(y, \dfrac{x}{1+g(x,y)}\right)} \geq \frac{y}{1+g(y,x)}
\end{aligned}$$

and hence

$$(x-y)\left(1+g(y,x)+y\,\frac{g(y,x)-g(x,y)}{x-y}\right)\geq 0.$$

It follows from (d) that $x-y\geq 0$ and so f satisfies the condition (b). Moreover since

$$A_f(\varepsilon)=\{a\in[\varepsilon,\infty):b(1+g(a,b))<a \text{ for some } b\geq\varepsilon\}$$

for each $\varepsilon>0$, it follows from (c) that $A_f=(0,\infty)$. Therefore we have from Theorem 1 that for any $a>0$, there exists a solution $\{x_n\}$ of Equation (1) such that $a=x_{-1}\geq x_0\geq x_1\geq x_2\geq\cdots>0$. Set $\alpha=\lim_{n\to\infty}x_n$. If $\alpha\neq 0$, then $\alpha=\dfrac{\alpha}{1+g(\alpha,\alpha)}$ and so $\alpha g(\alpha,\alpha)=0$, hence we arrive at a contradiction since $g(\alpha,\alpha)>0$. Therefore we have that $\lim_{n\to\infty}x_n=0$. Moreover if $g(x,\cdot)$ is strictly increasing for any fixed $x>0$, then we have $a=x_{-1}>x_0>x_1>x_2>\cdots>0$. In fact, suppose that there exists an $N\geq -1$ such that $x_N=x_{N+1}$. Then we have

$$\frac{x_N}{1+g(x_N,x_{N+2})}=x_{N+3}\leq x_{N+2}=\frac{x_N}{1+g(x_N,x_{N+1})}$$

and hence $g(x_N,x_{N+1})\leq g(x_N,x_{N+2})$. Therefore $x_{N+1}\leq x_{N+2}$ and so $x_{N+1}=x_{N+2}$ whenever $g(x,\cdot)$ is strictly increasing for any fixed $x>0$. By repeating this argument, we have that $x_N=x_{N+1}=x_{N+2}=x_{N+3}=\ldots$ and so $\lim_{n\to\infty}x_n=x_N>0$, a contradiction. Therefore we have the following:

Theorem 2. *Let* $g\colon(0,\infty)\times(0,\infty)\to(0,\infty)$ *be a continuous function which satisfies the conditions* (c) *and* (d). *Then for any* $a>0$*, there exists a solution* $\{x_n\}$ *of* $x_{n+1}=\dfrac{x_{n-1}}{1+g(x_{n-1},x_n)}$ *such that* $a=x_{-1}\geq x_0\geq x_1\geq x_2\geq\cdots>0$ *and* $\lim_{n\to\infty}x_n=0$.

In particular if $g(x,\cdot)$ *is a strictly increasing function for any fixed* $x>0$*, then the above solution* $\{x_n\}$ *is strictly decreasing.*

Let $h\colon(0,\infty)\to(0,\infty)$ be a continuous increasing function and set

$$f(x,y)=\frac{x}{1+h(y)}\quad(x,y>0).$$

Note that $g(x,y)=h(y)\quad(x,y>0)$ satisfies the conditions (c) and (d). Note also that $A_f=B_f=(0,\infty)$ and $C_f(b;\varepsilon)=\{u\geq\varepsilon:u\leq b(1+h(b))\}$, hence bounded, for each pair (b,ε) with $0<\varepsilon<b$. Then by Theorems 1 and 2, we have the following

Corollary 3. *Let* $h\colon(0,\infty)\to(0,\infty)$ *be a continuous increasing function. Then*

(i) *For any* $a > 0$, *there exists a solution of the equation* $x_{n+1} = \dfrac{x_{n-1}}{1+h(x_n)}$ *such that* $a = x_{-1} \geq x_0 \geq x_1 \geq x_2 \geq \cdots > 0$ *and* $\lim_{n\to\infty} x_n = 0$.

In particular if h *is strictly increasing, then the above solution* $\{x_n\}$ *is strictly decreasing.*

(ii) *For any* $b > 0$, *there exists a solution of the equation* $x_{n+1} = \dfrac{x_{n-1}}{1+h(x_n)}$ *such that* $x_{-1} \geq b = x_0 \geq x_1 \geq x_2 \geq \cdots > 0$ *and* $\lim_{n\to\infty} x_n = 0$.

In particular if h *is strictly increasing, then the above solution* $\{x_n\}$ *is strictly decreasing.*

Remark. We note that Theorem A follows immediately from Corollary 3-(i): In fact take h to be a C^1-function such that $h(0) = 0$ and $h'(x) > 0$ for all $x \in [0, \infty)$.

4. Other Typical Examples

In this section, we give other typical examples of Theorem 2.

1. Let $f(x, y) = \dfrac{x}{1+x+y}$. Then $A_f = (0, \infty)$, $B_f = (0, 1)$ and $C_f = \left[\varepsilon, \dfrac{b(1+b)}{1-b}\right]$ for each pair (b, ε) with $0 < \varepsilon < b \in B_f$. Then it follows from Theorems 1 and 2 that

   (i) For any $a > 0$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}}{1+x_{n-1}+x_n}$ such that $a = x_{-1} > x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.

   (ii) For any $b \in (0, 1)$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}}{1+x_{n-1}+x_n}$ such that $x_{-1} > b = x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.

2. Let $f(x, y) = \dfrac{x}{1+xy}$. Then $A_f = (0, \infty)$, $B_f = (0, 1)$ and $C_f(b; \varepsilon) = \left[\varepsilon, \dfrac{b}{1-b^2}\right]$ for each pair (b, ε) with $0 < \varepsilon < b \in B_f$. Then it follows from Theorems 1 and 2 that

   (i) For any $a > 0$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}}{1+x_{n-1}x_n}$ such that $a = x_{-1} > x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.

(ii) For any $b \in (0,1)$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}}{1+x_{n-1}x_n}$ such that $x_{-1} > b = x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.

3. Let $f(x,y) = \dfrac{x^2}{x+y}$. Then $A_f = B_f = (0,\infty)$ and $C_f(b;\varepsilon) = \left[\varepsilon, \dfrac{\sqrt{5}+1}{2}b\right]$ for each pair (b,ε) with $0 < \varepsilon < b \in B_f$. Then it follows from Theorems 1 and 2 that

(i) For any $a > 0$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}^2}{x_{n-1}+x_n}$ such that $a = x_{-1} > x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.

(ii) For any $b > 0$, there exists a solution of the equation $x_{n+1} = \dfrac{x_{n-1}^2}{x_{n-1}+x_n}$ such that $x_{-1} > b = x_0 > x_1 > x_2 > \cdots > 0$ and $\lim_{n\to\infty} x_n = 0$.

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