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NONSMOOTH MULTIOBJECTIVE FRACTIONAL PROGRAMMING WITH GENERALIZED INVEXITY

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Abstract. In this paper, we consider nonsmooth multiobjective fractional programming problems involving locally Lipschitz functions. We introduce the property of generalized invexity for fractional function. We present necessary optimality conditions, sufficient optimality conditions and duality relations for nonsmooth multiobjective fractional programming problems, which is for a weakly efficient solution under suitable generalized invexity assumptions.

1. INTRODUCTION

Recently there has been an increasing interest in studying generalized convexity for nonsmooth multiobjective programming problems involving locally Lipschitz functions. In [2], Jeyakumar defined ρ-invexity for nonsmooth function, and Liu [10, 11] used a parametric approach to obtain necessary and sufficient conditions and established duality theorems for a class of nonsmooth generalized fractional programming problems involving either nonsmooth pseudoinvex functions or nonsmooth (F, ρ)-convex functions. As a generalization of V-invex functions [3, 12], Kuk et al. [6] defined the concept of V-ρ-invexity for vector valued functions. Also, Kuk et al. [7] presented generalized Karush-Kuhn-Tucker necessary and sufficient optimality theorems and established some duality theorems for nonsmooth multiobjective fractional programs involving V-ρ-invex functions.

On the other hand, Khan and Hanson [4] have used the ratio invexity concept in fractional programming problem. Recently, Reddy and Mukherjee [14] applied a generalized ratio invexity concept for single objective fractional programming

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problems. After Liang et al. [8] introduce the concept of (F, α, ρ, d)-convexity due to Preda [13] and present optimality and duality results for a class of nonlinear fractional programming problems under generalized convexity and the properties of sublinear functional and they [9] obtained optimality conditions and duality theorems for multiobjective fractional programming under generalized convexity assumptions. Very recently, Kim and Kim [5] established optimality and duality for nonsmooth fractional programming under generalized ratio invexity assumptions in the scalar case.

In this paper, we present some results about the multiobjective fractional objective function based on ρ-invexity assumptions. By using ρ-invexity of fractional function, we obtain necessary and sufficient optimality conditions and duality theorems for nonsmooth multiobjective fractional programming problems.

Now, we consider the following nonsmooth multiobjective fractional programming problem,

$$\begin{array}{lll} \text{(NMFP)} & \text{Minimize} & (f_1(x)/g(x),\ \cdots\ , f_p(x)/g(x)) \\ & \text{subject to} & h(x) \leqq 0, \quad x \in X_0, \end{array}$$

where X_0 is an open set of $\mathbb{R}^n$, $f_i : X_0 \to \mathbb{R},\ g : X_0 \to \mathbb{R},\ h_j : X_0 \to \mathbb{R}$ and $l_k : X_0 \to \mathbb{R}$ are locally Lipschitz functions. We let $I(x) := \{i \mid h_i(x) = 0\}$ for any $x \in X_0$. We assume in the sequel that $f_i(x) \geqq 0$ and $g(x) > 0$ for all $x \in X_0$.

2. Definitions and Generalized Invexity of Fractional Function

The following conventions for vector in $\mathbb{R}^n$ will be used :

$$\begin{array}{l} x < y \iff x_i < y_i, \quad i = 1, 2, \cdots, n; \\ x \leqq y \iff x_i \leqq y_i, \quad i = 1, 2, \cdots, n; \\ x \leq y \iff x_i \leqq y_i, \quad i = 1, 2, \cdots, n \quad \text{but} \quad x \neq y; \\ x \nleq y \quad \text{is the negation of} \quad x \leq y. \end{array}$$

The real valued function f is said to be *locally Lpischitz* if for any $z \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of z such that, for each $x, y \in N$,

$$|f(x) - f(y)| \leqq K||x - y||,$$

where $|| \cdot ||$ denotes any norm in $\mathbb{R}^n$. The Clarke [1] generalized subgradient of f at x is denoted by

$$\partial f(x) = \{\xi : f^0(x; d) \geqq \xi d, \quad \forall d \in \mathbb{R}^n\}.$$

Definition 2.1. A vector valued function $f = (f_1, \cdots, f_p) : X_0 \to \mathbb{R}^p$ is said to be $\rho - invex$ at $u \in X_0$ with respect to functions η and $\theta : X_0 \times X_0 \to \mathbb{R}^n$ if there exists $\rho \in \mathbb{R}$ such that for any $x \in X_0$ and any $\xi_i \in \partial f_i(u)$, for $i = 1, 2, \cdots, p$,

$$f_i(x) - f_i(u) \geqq \xi_i \eta(x, u) + \rho \|\theta(x, u)\|^2, \quad \text{for all } \ i = 1, \cdots, p.$$

Remark

(i) When $p = 1$ and $\rho = 0$, the definition of ρ-invexity reduces to the notion of invexity in the sense of Jeyakumar [2].

(ii) When $p = 1$, the definition of ρ-invexity reduces to the notion of ρ-invexity for the scalar function in Kim and Kim [5].

Definition 2.2. A real valued function f is said to be *regular* at x if for all $d \in \mathbb{R}^n$ the one-sided directional derivative $f'(x; d)$ exists and $f'(x; d) = f^0(x; d)$.

Definition 2.3. A point $u \in X_0$ is said to be a *weakly efficient solution* of (NMFP) if there exist no $x \in X_0$ such that

$$\left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_p(x)}{g(x)}\right) < \left(\frac{f_1(u)}{g(u)}, \cdots, \frac{f_p(u)}{g(u)}\right)$$

Theorem 2.1. *If f and $G = (-g, \cdots, -g)$ are ρ-invex at x_0 with respect to η and θ, and f_i and $-g$ are regular at x_0, then the fractional function $\left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_p(x)}{g(x)}\right)$ is ρ-invex at x_0 with respect to $\bar{\eta}$ and $\bar{\theta}$, where $\bar{\eta}(x, x_0) = (g(x_0)/g(x))\eta(x, x_0)$, and $\bar{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x, x_0)$.*

Proof. Let $x \in X_0$. By the ρ-invexity of f and G, we have

$$\begin{aligned} & f_i(x)/g(x) - f_i(x_0)/g(x_0) \\ & \geqq (1/g(x))\xi_i \eta(x, x_0) + \rho \|(1/g(x))^{1/2}\theta(x, x_0)\|^2 \\ & \quad + (f_i(x_0)/(g(x)g(x_0))(-\zeta\eta(x, x_0) + \rho\|\theta(x, x_0)\|^2), \end{aligned}$$

for any $x \in X_0$, any $\xi_i \in \partial f_i(x_0)$ and any $\zeta \in \partial g(x_0)$. Since $f_i(x) \geqq 0$ and $g(x) > 0$,

$$\begin{aligned} & f_i(x)/g(x) - f_i(x_0)/g(x_0) \\ & \geqq (g(x_0)/g(x))(\xi_i/g(x_0))\eta(x, x_0) + (f_i(x_0)(-\zeta)/(g^2(x_0))\eta(x, x_0)) \\ & \quad + \rho\|(1/g(x))^{1/2}(1 + (f_i(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2. \end{aligned}$$

Since f_i and $-g$ are regular at x_0, we obtain, for any $\delta_i \in \partial(f_i(x_0)/g(x_0))$,

$$(f_i(x)/g(x)) - (f_i(x_0)/g(x_0))$$
$$\geqq (g(x_0)/g(x))\delta_i\eta(x,x_0) + \rho\|(1/g(x))^{1/2}(1+(f_i(x_0)/g(x_0)))^{1/2}\theta(x,x_0)\|^2.$$

Considering that $1 + f_i(x_0)/g(x_0) \geqq 1$, we have

$$\frac{f_i(x)}{g(x)} - \frac{f_i(x_0)}{g(x_0)} \geqq (\frac{g(x_0)}{g(x)})\delta_i\eta(x,x_0) + \rho\|(\frac{1}{g(x)})^{1/2}\theta(x,x_0)\|^2.$$

Therefore, $(f_1(x)/g(x), \cdots, f_p(x)/g(x))$ is ρ-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, where

$$\bar{\eta}(x,x_0) = (g(x_0)/g(x))\eta(x,x_0), \ \ \bar{\theta}(x,x_0) = (1/g(x))^{1/2}\theta(x,x_0).$$ ■

3. Optimality Conditions

In this section, we present Fritz John and Karush-Kuhn-Tucker necessary conditions and establish Karush-Kuhn-Tucker sufficient conditions for a weakly efficient solution of the multiobjective nonsmooth fractional programming problem (NMFP).

By Theorem 6.1.1 in [1], we can obtain the following Fritz John necessary conditions.

Theorem 3.1. (Fritz John Necessary Conditions). *If $x_0 \in X_0$ is a weakly efficient solution of (NMFP), then there exist $\lambda_i \geqq 0, i = 1, 2, \cdots, p$ and $\mu_j \geqq 0, j = 1, 2, \cdots, m$, such that*

$$0 \in \sum_{i=1}^{p} \lambda_i \partial(f_i(x_0)/g(x_0)) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),$$

equivalently, there exist $a_i \in \partial(f_i(x_0)/g(x_0)), i = 1, \cdots, p$ and $b_j \in \partial h_j(x_0), j = 1, \cdots, m$ such that

$$0 = \sum_{i=1}^{p} \lambda_i a_i + \sum_{j=1}^{m} \mu_j b_j, \tag{1}$$

and

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$
$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,$$
$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0.$$

If $(\lambda_1, \cdots, \lambda_p) = 0$ in (1), then $0 = \sum_{j=1}^{m} \mu_j b_j$ and hence under the following condition (2), which is a kind of constraint qualifications for (NMFP), we can obtain the following Karush-Kuhn-Tucker necessary condtions.

Theorem 3.2. (Karush-Kuhn-Tucker Necessary Conditions). *Assume that for* $b_j \in \partial h_j(x_0), j = 1, \cdots, m$,

(2) *there exists* $x^* \in X_0$ *such that* $\langle b_j, x^* \rangle < 0,\ j \in I(x_0) = \{i | h_i(x_0) = 0\}$.

If $x_0 \in X_0$ *is a weakly efficient solution of (NMFP), then there exist* $\lambda_i \geqq 0,\ i = 1, 2, \cdots, p$ *and* $\mu_j \geqq 0, j = 1, 2, \cdots, m$, *such that*

$$\text{(KTC)} \qquad 0 \in \sum_{i=1}^{p} \lambda_i \partial(f_i(x_0)/g(x_0)) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,\ (\lambda_1, \cdots, \lambda_p) \neq 0.$$

Remark. If $h_j,\ j = 1, \cdots, m$ are convex functions, the well-known Slater condition implies (2)

Theorem 3.3. (Karush-Kuhn-Tucker Sufficient Conditions). *Let* (x_0, λ, μ) *satisfy the conditions of (KTC). Assume that* f *and* G *are* ρ*-invex at* x_0 *with respect to* η *and* θ *and* f_i *and* $-g$ *are regular functions at* x_0*, and* h *is* ρ'*-invex at* x_0 *with respect to* $\bar{\eta}$ *and* $\bar{\theta}$ *with* $\rho \sum_{i=1}^{p} \lambda_i + \rho' \sum_{j=1}^{m} \mu_j \geqq 0$*. Then* x_0 *is a weakly efficient solution of* $(NMFP)$.

Proof. Suppose that x_0 is not a weakly efficient solution. Then there exists $x \in X_0$ such that

$$\frac{f_i(x)}{g(x)} < \frac{f_i(x_0)}{g(x_0)}, \text{ for } i = 1, \cdots, p.$$

Since f and G are ρ-invex at x_0 and f_i and $-g$ are regular, we have

$$a_i \bar{\eta}(x, x_0) + \rho ||\bar{\theta}(x, x_0)||^2 < 0, \text{ for some } a_i \in \partial \left(\frac{f_i(x_0)}{g(x_0)} \right).$$

From $\lambda \geq 0$, we obtain

$$\sum_{i=1}^{p} \lambda_i a_i \bar{\eta}(x, x_0) + \rho \sum_{i=1}^{p} \lambda_i ||\bar{\theta}(x, x_0)||^2 < 0.$$

By using Karush-Kuhn-Tucker conditions and $\rho \sum_{i=1}^{p} \lambda_i + \rho' \sum_{j=1}^{m} \mu_j \geqq 0$, we have

(3) $$\sum_{j=1}^{m} \mu_j b_j \bar{\eta}(x, x_0) + \rho' \sum_{j=1}^{m} \mu_j ||\bar{\theta}(x, x_0)||^2 > 0, \text{ for some } b_j \in \partial h_j(x_0).$$

Since h is ρ'-invex at x_0 with respect to $\bar{\eta}$ and $\bar{\theta}$,

$$\sum_{j=1}^{m} \mu_j h_j(x) - \sum_{j=1}^{m} \mu_j h_j(x_0) \geqq \sum_{j=1}^{m} \mu_j b_j \bar{\eta}(x, x_0) + \rho' \sum_{j=1}^{m} \mu_j ||\bar{\theta}(x, x_0)||^2.$$

Hence

$$\sum_{j=1}^{m} \mu_j b_j \bar{\eta}(x, x_0) + \rho' \sum_{j=1}^{m} \mu_j ||\bar{\theta}(x, x_0)||^2 \leqq 0.$$

This inequality contradicts (3). Therefore, x_0 is a weakly efficient solution. ■

4. Duality Theorems

In this section, we formulate the dual models and establish the duality theorems for a weakly efficient solution of the multiobjective nonsmooth fractional programming problem (NMFP).

We consider Mond-Weir dual problem to (NMFP):

(NMFD)$_M$ Maximize $(f_1(u)/g(u), \cdots, f_p(u)/g(u))$

subject to

$$0 \in \sum_{i=1}^{p} \lambda_i \partial (f_i(u)/g(u)) + \sum_{j=1}^{m} \mu_j \partial h_j(u),$$

$$\sum_{j=1}^{m} \mu_j h_j(u) \geqq 0,$$

$$(\lambda, \mu) \geqq 0,$$

$$\lambda^T e = 1, \ \ e = (1, \cdots, 1) \in \mathbb{R}^p$$

Theorem 4.1. (Weak Duality). *Let x be feasible for (NMFP) and (u, λ, μ) feasible for (NMFD)$_M$. Assume that f and G are ρ-invex at u with respect to η and θ and f_i and $-g$ are regular functions at u, and h is ρ'-invex at u with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \rho' \sum_{j=1}^{m} \mu_j \geqq 0$.*

Then the following holds:

$$\left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_p(x)}{g(x)} \right) \nless \left(\frac{f_1(u)}{g(u)}, \cdots, \frac{f_p(u)}{g(u)} \right).$$

Proof. Suppose that

$$\frac{f_i(x)}{g(x)} < \frac{f_i(u)}{g(u)}, \ \text{for} \ \ i = 1, \cdots, p.$$

Since f and G are ρ-invex at x_0 and f_i and $-g$ are regular, we have

$$a_i\bar{\eta}(x,u)+\rho||\bar{\theta}(x,u)||^2<0, \text{ for some } a_i\in\partial\left(\frac{f_i(u)}{g(u)}\right).$$

From $\lambda \geq 0$, we obtain

$$\sum_{i=1}^{p}\lambda_i a_i\bar{\eta}(x,u)+\rho\sum_{i=1}^{p}\lambda_i||\bar{\theta}(x,u)||^2<0.$$

Since (u,λ,μ) is feasible for (NMFD)$_M$ and $\rho+\rho'\sum_{j=1}^{m}\mu_j \geqq 0$, we have

$$\text{(4)} \qquad \sum_{j=1}^{m}\mu_j b_j\bar{\eta}(x,u)+\rho'\sum_{j=1}^{m}\mu_j||\bar{\theta}(x,u)||^2>0, \text{ for some } b_j\in\partial h_j(u).$$

By the ρ'-invexity of h with respect to $\bar{\eta}$ and $\bar{\theta}$, for some $b_j\in\partial h_j(u)$,

$$\sum_{j=1}^{m}\mu_j h_j(x)-\sum_{j=1}^{m}\mu_j h_j(u)\geqq\sum_{j=1}^{m}\mu_j b_j\bar{\eta}(x,u)+\rho'\sum_{j=1}^{m}\mu_j||\bar{\theta}(x,u)||^2.$$

Since x is a feasible solution of (NMFP), we obtain

$$\sum_{j=1}^{m}\mu_j b_j\bar{\eta}(x,u)+e'\sum_{j=1}^{m}\mu_j||\bar{\theta}(x,u)||^2\leqq 0.$$

This inequality contradicts (4). Hence the weak duality theorem holds. ■

Theorem 4.2. (Strong Duality). *Let $\bar{x}$ is a weakly efficient solution of (NMFP). Assume that there exists $x^*\in X_0$ such that $\langle b_j,x^*\rangle<0,\ j\in I(\bar{x})$ and $b_j\in\partial h_j(x_0), j=1,\cdots,m$. Then $(\bar{x},\bar{\lambda},\bar{\mu})$ is feasible for $($NMFD$)\ _M$. Moreover, if f,G and h satisfy the conditions of Theorem 4.1, then $(\bar{x},\ \bar{\lambda},\bar{\mu})$ is a weakly efficient solution of $($NMFD$)\ _M$.*

Proof. From the Karush-Kuhn-Tucker necessary condition, there exist $\bar{\lambda}_i\geqq 0,\ i=1,\cdots,p$ and $\bar{\mu}_j\geqq 0, j=1,\cdots,m$ such that

$$0\in\sum_{i=1}^{p}\bar{\lambda}_i\partial(f_i(\bar{x})/g(\bar{x}))+\sum_{j=1}^{m}\bar{\mu}_j\partial h_j(\bar{x}),$$

$$\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})=0,$$

$$(\bar{\lambda}_1,\cdots,\bar{\lambda}_p)\neq 0.$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (NMFD)$_M$. So by Theorem 4.1, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (NMFD)$_M$. ■

We propose the following Wolfe dual problem to (NMFP).

$$\text{(5)}\quad \begin{aligned} &\text{(NMFD)}_W \quad \text{Maximize} && \Big(f_1(u)/g(u) + \sum_{j=1}^{m} \mu_j h_j(u), \cdots, \\ &&& f_p(u)/g(u) + \sum_{j=1}^{m} \mu_j h_j(u)\Big) \\ &\qquad\qquad \text{subject to} && 0 \in \sum_{i=1}^{p} \lambda_i \partial (f_i(u)/g(u)) + \sum_{j=1}^{m} \mu_j \partial h_j(u), \\ &&& (\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0, \\ &&& \lambda^T e = 1, e = (1, \cdots, 1) \in \mathbb{R}^p \end{aligned}$$

Theorem 4.3. (Weak Duality). *Let x be feasible for (NMFP) and (u, λ, μ) feasible for (NMFD)$_W$. Assume that f and G are ρ-invex at u with respect to η and θ and f_i and $-g$ are regular functions at u, and h is ρ'-invex at u with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \rho' \sum_{j=1}^{m} \mu_j \geqq 0$.*

Then the following holds:

$$\left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_p(x)}{g(x)} \right) \not< \left(\frac{f_1(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u), \cdots, \frac{f_p(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u) \right).$$

Proof. Suppose that

$$\frac{f_i(x)}{g(x)} < \frac{f_i(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u), \ \text{ for } \ i = 1, \cdots, p.$$

Since x is feasible for (NMFP),

$$\frac{f_i(x)}{g(x)} + \sum_{j=1}^{m} \mu_j h_j(x) < \frac{f_i(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u), \ \text{ for } \ i = 1, \cdots, p.$$

By the ρ-invexity of f and G and ρ'-invexity of h, we have

$$a_i \bar{\eta}(x, u) + \rho ||\bar{\theta}(x, u)||^2 + \sum_{j=1}^{m} \mu_j b_j \bar{\eta}(x, u) + \rho' \sum_{j=1}^{m} \mu_j ||\bar{\theta}(x, u)||^2 < 0,$$

for some $a_i \in \partial\left(\frac{f_i(u)}{g(u)}\right)$ and some $b_j \in \partial h_j(u)$. From $\lambda \geq 0$ and $\rho+\rho'\sum_{j=1}^m \mu_j \geqq 0$, we obtain

$$\Big(\sum_{i=1}^p \lambda_i a_i + \sum_{j=1}^m \mu_j b_j\Big)\bar{\eta}(x,u) < 0.$$

This contradicts (5). Hence the weak duality theorem holds. ■

Theorem 4.4. (Strong Duality). *Let $\bar{x}$ is a weakly efficient solution of (NMFP). Assume that there exists $x^* \in X_0$ such that $\langle b_j, x^*\rangle < 0,\ j \in I(\bar{x})$ and $b_j \in \partial h_j(x_0), j=1,\cdots,m$. Then $(\bar{x},\bar{\lambda},\bar{\mu})$ is feasible for (NMFD)$_W$. Moreover, if f, G and h satisfy the conditions of Theorem 4.3, then $(\bar{x},\ \bar{\lambda},\bar{\mu})$ is a weakly efficient solution of (NMFD)$_W$.*

Proof. From the Karush-Kuhn-Tucker necessary condition, there exist $\bar{\lambda}_i \geqq 0,\ i=1,\cdots,p$ and $\bar{\mu}_j \geqq 0, j=1,\cdots,m$ such that

$$\begin{aligned}
&0 \in \sum_{i=1}^p \bar{\lambda}_i \partial(f_i(\bar{x})/g(\bar{x})) + \sum_{j=1}^m \bar{\mu}_j \partial h_j(\bar{x}),\\
&\sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}) = 0,\\
&(\bar{\lambda}_1,\cdots,\bar{\lambda}_p) \neq 0.
\end{aligned}$$

Thus $(\bar{x},\bar{\lambda},\bar{\mu})$ is feasible for (NMFD)$_W$. So by Theorem 4.3, $(\bar{x},\bar{\lambda},\bar{\mu})$ is a weakly efficient solution of (NMFD)$_W$. ■

We formulate the following general dual problem to primal problem (NMFP).

$$\begin{aligned}
\text{(NMFD)}_G\quad \text{Maximize}\quad &\Big(f_1(u)/g(u) + \sum_{j\in I_0}\mu_j h_j(u),\cdots,\\
&f_p(u)/g(u) + \sum_{j\in I_0}\mu_j h_j(u)\Big)\\
\text{subject to}\quad &0 \in \sum_{i=1}^p \lambda_i \partial(f_i(u)/g(u)) + \sum_{j=1}^m \mu_j \partial h_j(u),\\
&\sum_{j\in I_\alpha}\mu_j h_j(u) \geqq 0, \quad \alpha=1,\cdots,r,\\
&(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_m) \geqq 0,\\
&\lambda^T e = 1,\ \ e=(1,\cdots,1)\in\mathbb{R}^p
\end{aligned} \tag{6}$$

where $I_\alpha \subset M = \{1, \cdots, m\},\ \alpha = 0, 1, \cdots, r$ with $\cup_{\alpha=0}^r I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$.

Theorem 4.5. (Weak Duality). *Let x be feasible for (NMFP) and (u, λ, μ) feasible for (NMFD)$_G$. Assume that f and G are ρ-invex at u with respect to η and θ and f_i and $-g$ are regular fuctions at u and h is ρ'-invex at u, $j \in J_\alpha(\alpha = 0, 1, \cdots, r)$ with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \rho' \sum_{j=1}^m \mu_j \geqq 0$.*

Then the following holds:

$$\left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_p(x)}{g(x)}\right)$$
$$\not< \left(\frac{f_1(u)}{g(u)} + \sum_{j\in I_0} \mu_j h_j(u), \cdots, \frac{f_p(u)}{g(u)} + \sum_{j\in I_0} \mu_j h_j(u)\right).$$

Proof. Suppose that

$$\frac{f_i(x)}{g(x)} < \frac{f_i(u)}{g(u)} + \sum_{j\in I_0} \mu_j h_j(u), \text{ for } i = 1, \cdots, p.$$

Since x is feasible for (NMFP) and $\sum_{j\in I_\alpha} \mu_j h_j(u) \geqq 0$,

$$\frac{f_i(x)}{g(x)} + \sum_{j\in I_0} \mu_j h_j(x) + \sum_{j\in I_\alpha} \mu_j h_j(x) < \frac{f_i(u)}{g(u)}$$
$$+ \sum_{j\in I_0} \mu_j h_j(u) + \sum_{j\in I_\alpha} \mu_j h_j(u), \text{ for } i = 1, \cdots, p.$$

Hence

$$\frac{f_i(x)}{g(x)} + \sum_{j=1}^m \mu_j h_j(x) < \frac{f_i(u)}{g(u)} + \sum_{j=1}^m \mu_j h_j(u), \text{ for } i = 1, \cdots, p.$$

By the ρ-invexity of f and G and ρ'-invexity of h, we have

$$a_i \bar{\eta}(x, u) + \rho||\bar{\theta}(x, u)||^2 + \sum_{j=1}^m \mu_j b_j \bar{\eta}(x, u) + \rho' \sum_{j=1}^m \mu_j ||\bar{\theta}(x, u)||^2 < 0,$$

for some $a_i \in \partial\left(\frac{f_i(u)}{g(u)}\right)$ and some $b_j \in \partial h_j(u)$. From $\lambda \geq 0$ and $\rho + \rho' \sum_{j=1}^m \mu_j \geqq 0$, we obtain

$$\Big(\sum_{i=1}^p \lambda_i a_i + \sum_{j=1}^m \mu_j b_j\Big)\bar{\eta}(x, u) < 0.$$

This contradicts (6). Hence the proof is completed. ■

Theorem 4.6. (Strong Duality). *Let $\bar{x}$ is a weakly efficient solution of (NMFP). Assume that there exists $x^* \in S$ such that $\langle b_j, x^* \rangle < 0,\ j \in I(\bar{x})$ and $b_j \in \partial h_j(x_0), j = 1, \cdots, m$. Then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (NMFD)$_G$. Moreover, if f, G and h satisfy the conditions of Theorem 4.5, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (NMFD)$_G$.*

Proof. From the Karush-Kuhn-Tucker necessary condition, there exist $\bar{\lambda}_i \geqq 0,\ i = 1, \cdots, p$ and $\bar{\mu}_j \geqq 0, j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i \partial(f_i(\bar{x})/g(\bar{x})) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0,$$

$$(\bar{\lambda}_1, \cdots, \bar{\lambda}_p) \neq 0.$$

Since $\sum_{j \in I_0} \bar{\mu}_j h_j(\bar{x}) + \sum_{j \in I_\alpha} \bar{\mu}_j h_j(\bar{x}) = 0$, we have $\sum_{j \in I_\alpha} \bar{\mu}_j h_j(\bar{x}) \geqq -\sum_{j \in I_0} \bar{\mu}_j h_j(\bar{x}) \geqq 0$. Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (NMFD)$_G$. So by Theorem 4.5, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (NMFD)$_G$. ■

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