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ON THE BANACH-STONE PROBLEM FOR L^p-SPACES

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Abstract. The Banach-Stone problem for L^p-spaces is to assert when a linear isometry between L^p-spaces is a weighted composition operator. We shall show that every σ-finite measure space with Sikorski's property solves the Banach-Stone probelm. In addition, we show that if X is a totally ordered and Dedekind complete, then every σ-finite μ-separable measure space $(X, \mathcal{B}, \mu)$ has Sikorski's property.

1. Introduction

Let X and Y be locally compact Hausdorff spaces. A classical Banach-Stone Theorem (see, e.g., Behrends [2, p. 138]) states that every isometry T from the Banach space $C_0(X)$ of continuous functions vanishing at infinity *onto* another $C_0(Y)$ is a weighted composition operator $Tf = h \cdot (f \circ \varphi)$, for all f in $C_0(X)$. In this paper, we call such a map T a *BS map*. We ask similar questions for operators between L^p-spaces. Given two measure spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$, is every linear isometry $T : L^p(X) \to L^q(Y)$ a BS map?

In [1] (see also [13, p. 415]), an affirmative answer is given for linear isometries $T : L^p[0,1] \to L^p[0,1]$ ($1 \le p < \infty$, $p \ne 2$). In [10], Lamperti showed that, for $(X, \mathcal{B}, \mu)$ a σ-finite measure space, every linear isometry $T : L^p(X) \to L^p(X)$ ($1 \le p < \infty$, $p \ne 2$) is given by a still simple form $T\chi_B = h \cdot \chi_{\Phi(B)}$ for all B in $\mathcal{B}$. We call such a map T a *Lamperti map*. Note that, in [10], it is not said if T is a BS map or not, and this is still unknown for the time being.

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be measure spaces and $1 \le p, q \le \infty$. A map $T : L^p(X) \to L^q(Y)$ is called *disjointness preserving* if $f \cdot g = 0$ a.e. $[\mu]$ implies $Tf \cdot Tg = 0$ a.e. $[\nu]$ for all f, g in $L^p(X)$. We shall see that every (surjective when $p = q = \infty$) linear isometry $T : L^p(X) \to L^q(Y)$ (either $1 \le p, q < \infty, p \ne$

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$2, q \neq 2$ or $p = q = \infty$, we will call such an pair (p, q) *accessible*) is disjointness preserving. Therefore, it suffices to study merely bounded disjointness preserving operators.

We shall prove that (i) every σ-finite measure space $(X, \mathcal{B}, \mu)$ solves the *Lamperti problem* for L^p-spaces, that is, for an arbitrary measure space $(Y, \mathcal{A}, \nu)$ and an accessible (p, q), every (surjective when $p = q = \infty$) bounded disjointness preserving operator $T : L^p(X) \to L^q(Y)$ is a Lamperti map; (ii) every σ-finite measure space $(X, \mathcal{B}, \mu)$ with Sikorski's property solves the *Banach-Stone problem* for L^p-spaces, that is, for an arbitrary measure space $(Y, \mathcal{A}, \nu)$ and an accessible (p, q), every (surjective when $p = q = \infty$) bounded disjointness preserving operator $T : L^p(X) \to L^q(Y)$ is a BS map. Note that we have included the case $p = q = \infty$ here.

In [11], Lessard used a topological approach with some technical lifting theorems to give the result: Every Lamperti map $T : L^p(X, \mathcal{B}, \mu) \to L^p(Y, \mathcal{A}, \nu)$ is a BS map, if μ is tight. A finite Baire measure μ on a topological space X is said to be *tight* if for every $\epsilon > 0$, there exists a compact set K in X such that $\mu^*(K) > \mu(X) - \epsilon$, where μ^* denotes the outer measure determined by μ.

We shall use an order theoretical approach to give a different sufficient condition (see Proposition 8) for a measure space $(X, \mathcal{B}, \mu)$ solving the *Banach-Stone problem* for L^p-spaces. We note that $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is, in general, not tight and thus Lessard's theorem does not apply. However, Proposition 8 below does help. And as well-known examples, $\mathbb{R}^n$ with Borel measure and Hilbert cube with an appropriate measure, satisfy our conditions and solve the Banach-Stone problem.

2. The Lamperti Problem

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be arbitrary measure spaces. We first show that every isometry $T : L^p(X) \to L^q(Y)$ is disjointness preserving for $1 \leq p, q < \infty$, $p, q \neq 2$. Indeed, it is easy to see that $\|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$ if and only if $f \cdot g = 0$ a.e. $[\mu]$ (ref. [13, p. 416]). Since T is an isometry, for all f, g in $L^p(X)$

$$\begin{aligned} f \cdot g = 0 \text{ a.e. } [\mu] &\Leftrightarrow \|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p) \\ &\Leftrightarrow \|Tf + Tg\|^q + \|Tf - Tg\|^q = 2(\|Tf\|^q + \|Tg\|^q) \\ &\Leftrightarrow Tf \cdot Tg = 0 \text{ a.e. } [\nu]. \end{aligned}$$

Hence T is disjointness preserving.

Recall that the function space $L^\infty(X)$ is a commutative C^*-algebra with identity, equipped with the natural algebraic structure and the natural involution. By the Gelfand-Naimark theorem (see, e.g., [4, p. 236], $L^\infty(X)$ is isometrically $*$-isomorphic to $C(\Sigma)$, where Σ is the maximal ideal space of $L^\infty(X)$. Note that Σ is compact.

Let Λ be the Gelfand transform from $L^\infty(X)$ onto $C(\Sigma)$. We write $\widehat{f}$ for $\Lambda(f)$ for simplicity of notations. For B in $\mathcal{B}$, since $\chi_B^2 = \chi_B$, we have $\widehat{\chi_B}^2 = \widehat{\chi_B}$. Then $\widehat{\chi_B}$ is the characteristic function of a closed and open subset U_B of Σ. Conversely, if U is a closed and open subset of Σ, then $\chi_U \in C(\Sigma)$ and $\chi_U = \widehat{f}$ for some f in $L^\infty(X)$. Moreover, $\widehat{f^2} = \widehat{f}$ and $f^2 = f$ in $L^\infty(X)$. It follows that $f = \chi_B$ for some B in $\mathcal{B}$. Consequently, we have

Lemma 1. *Every closed and open subset of Σ is of the form U_B for some B in $\mathcal{B}$.*

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be measure space, and Σ_1 (resp. Σ_2) the maximal ideal space of $L^\infty(X)$ (resp. $L^\infty(Y)$). For any given map $T : L^\infty(X) \to L^\infty(Y)$, define $\widehat{T} : C(\Sigma_1) \to C(\Sigma_2)$ by $\widehat{T}\widehat{f} = \widehat{Tf}$ for all f in $L^\infty(X)$. It is clear to get the following proposition.

Proposition 1. *T is a bounded linear operator if and only if $\widehat{T}$ is (and $\|T\| = \|\widehat{T}\|$); T is a linear isometry if and only if $\widehat{T}$ is; T is disjointness preserving if and only if $\widehat{T}$ is; and T is invertible if and only if $\widehat{T}$ is (in this case, $\widehat{T}^{-1} = \widehat{T^{-1}}$).*

Lemma 2. *Every surjective linear isometry $T : L^\infty(X) \to L^\infty(Y)$ is disjointness preserving.*

Proof. It follows Banach-Stone Theorem and the proposition above.

It is plain that there exists a linear isometry $T : L^\infty(X) \to L^\infty(Y)$ such that T is not disjointness preserving. (Consider, e.g., $T(x_1, x_2, \ldots,) = (\frac{x_1+x_2}{2}, x_1, x_2, \ldots)$ from ℓ^∞ into ℓ^∞.)

For the Lamperti problem, Lamperti's proof [10, p. 461] can be modified to prove the following theorem.

Theorem 2. *If $(X, \mathcal{B}, \mu)$ is a σ-finite measure space, $(Y, \mathcal{A}, \nu)$ an arbitrary measure space and $T : L^p(X) \to L^q(Y)$ ($1 \le p, q < \infty$ and $p, q \ne 2$) a bounded disjointness preserving linear operator, then T is a Lamperti map.*

It remains to prove the case $p = q = \infty$ for Lamperti problem. We need the following theorem.

Theorem 3. ([8]) *If X and Y are compact Hausdorff space and $T : C(X) \to C(Y)$ is a surjective disjointness preserving linear operator, then there exists a homeomorphism $\varphi : Y \to X$ and a function h in $C(Y)$ with $h(y) \ne 0$ for all y in Y such that $Tf = h \cdot (f \circ \varphi)$ for all f in $C(X)$.*

Theorem 4. *Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be measure spaces. If $T : L^\infty(X) \to L^\infty(Y)$ is a bounded surjective disjointness preserving linear operator, then there* *exist a proper regular set homomorphism* $\Phi : \mathcal{B} \to \mathcal{A}$ *with* $\Phi(X) = Y$ *and a function* h *in* $L^\infty(Y)$ *with* $h \neq 0$ *a.e.* $[\nu]$ *such that* $T\xi_B = h \cdot \xi_{\Phi(B)}$ *for all* B *in* $\mathcal{B}$*. In other words,* T *is a Lamperti map.*

Proof. Let Σ_1 (resp. Σ_2) be the maximal ideal space of $L^\infty(X)$ (resp. $L^\infty(Y)$). Let $\widehat{T} : C(\Sigma_1) \to C(\Sigma_2)$ be defined by $\widehat{T}\widehat{f} = \widehat{Tf}$ for all f in $L^\infty(X)$ (via the Gelfand transform Λ). By Proposition 1, $\widehat{T}$ is a bounded surjective disjointness preserving linear operator. By Theorem 3, there exist a homeomorphism $\varphi : \Sigma_2 \to \Sigma_1$ and a function h in $L^\infty(X)$ with $h \neq 0$ a.e. $[\nu]$ such that $\widehat{T}\widehat{f} = \widehat{h} \cdot (\widehat{f} \circ \varphi)$ for all f in $L^\infty(X)$. Let $\Phi : \mathcal{B} \to \mathcal{A}$ be defined, modulo null sets, by $\Phi(B) = A$ if $U_A = \varphi^{-1}(U_B)$ in the notations of Lemma 1. Φ is a proper regular set homomorphism.

It is easy to see that Φ preserves differences and finite unions, and $\nu(\Phi(B)) = 0$ if and only if $\mu(B) = 0$. By the homeomorphism of φ, Φ is surjective and $\Phi(X) = Y$.

It remains to show that Φ preserves countable union. Suppose that $\{B_n\}_n$ is a sequence of measurable sets in $\mathcal{B}$, we need to show that $\Phi(\bigcup_{n=1}^\infty B_n) = \bigcup_{n=1}^\infty \Phi(B_n)$, or equivalently, $\varphi^{-1}(U_{\cup B_n}) = \bigcup \varphi^{-1}(U_{B_n})$ $(= \sup \varphi^{-1}(U_{B_n}))$. Clearly, $\varphi^{-1}(U_{\cup B_n})$ is an upper bounded of $\{\varphi^{-1}(U_{B_n})\}_n$. Suppose that U_A is anothre upper bound of $\{\varphi^{-1}(U_{B_n})\}_n$. Since Φ is surjective, there is a B in $\mathcal{B}$ such that $\Phi(B) = A$. By assumption, $\varphi^{-1}(U_{B_n}) \subset U_A = \varphi^{-1}(U_B)$, then $U_{B_n} \subset U_B$ for all $n \in \mathbb{N}$. Since $U_{\cup B_n} = \sup U_{B_n}$, we have $\varphi^{-1}(U_{\cup B_n}) \subset \varphi^{-1}(U_B) = U_A$. Therefore, $\varphi^{-1}(U_{\cup B_n}) = \sup \varphi^{-1}(U_{B_n})$. This establishes the claim.

Finally, observe that, for all B in $\mathcal{B}$,

$$\begin{aligned}\widehat{T\chi_B} &= \widehat{T}\widehat{\chi_B} = \widehat{h} \cdot (\widehat{\chi_B} \circ \varphi) = \widehat{h} \cdot (\chi_{U_B} \circ \varphi) = \widehat{h} \cdot \chi_{\varphi^{-1}(U_B)} \\ &= \widehat{h} \cdot \chi_{U_{\Phi(B)}} = \widehat{h} \cdot \widehat{\chi_{\Phi(B)}} = \widehat{h \cdot \chi_{\Phi(B)}}.\end{aligned}$$

Hence, $T\chi_B = h \cdot \chi_{\Phi(B)}$ for all B in $\mathcal{B}$.

As an immediate consequence of Theorems 2 and 4, we have the following

Theorem 5. *Every* σ*-finite measure space solves the Lamperti problem.*

3. The Banach-stone Problem

In this section, we devote to the Banach-Stone problem. First, let us to see a special case.

Proposition 6. *If* $1 \leq p,\ q \leq \infty$*, and* $T : \ell^p \to \ell^q$ *is a bounded disjointness preserving operator, then there exist a map* $\varphi : \mathbb{N} \to \mathbb{N}$ *and a function* h *in* ℓ^∞ *such that,*

$$Tx = h \cdot (x \circ \varphi) \text{ for all } x \in \ell^p.$$

Proof. For each m in $\mathbb{N}$, define $e_m : \mathbb{N} \to \mathbb{N}$ by $e_m(n) = 1$ if $m = n$, and $e_m(n) = 0$ otherwise. For each n in $\mathbb{N}$, define δ_n to be the linear functional on ℓ^q sending e_n to 1, and e_m to 0 if $m \neq n$. For each n in $\mathbb{N}$, define $\Phi(n) = \{m : Te_m(n) \neq 0\}$. Since T is disjoint preserving, $\Phi(n)$ contains at most one element.

Let $\mathbf{N}_0 = \{n : \Phi(n) = \emptyset\}$. It is easy to see that $\{n : \delta_n \circ T = 0\} \subset \mathbf{N}_0$. On the other hand, let $n \in \mathbf{N}_0$. Since the linear span of $\{em : m \in \mathbb{N}\}$ is weak$^a st$-dense in ℓ^p, $\delta_n \circ T = 0$. Hence, $\mathbf{N}_0 = \{n : \delta_n \circ T = 0\}$.

Let n_0 be a fixed natural number. Define $\varphi : \mathbb{N} \to \mathbb{N}$ by $\varphi(n) = m$ where m is the unique element in $\Phi(n)$ if $n \in \mathbb{N} \setminus \mathbf{N}_0$, and $\varphi(n) = n_0$ if $n \in \mathbf{N}_0$.

If $n \in \mathbb{N} \setminus \mathbf{N}_0$, then there is a scalar $\alpha_n \neq 0$ such that $\delta_n \circ T = \alpha_n \cdot \delta_{\varphi(n)}$. To see this, if $n \in \mathbb{N}\setminus\mathbf{N}_0$, then it is easy to see that ker ($\delta_n \circ T$) is a nontrivial subspace of ℓ^p. Since $\delta_n \circ T$ and $\delta_{\varphi(n)}$ are linear functionals defined on ℓ^p, it suffices to show that ker ($\delta_n \circ T$) $=$ ker $\delta_{\varphi(n)}$. Let $x \in$ ker $\delta_{\varphi(n)}$, that is, $x(\varphi(n)) = 0$. Since $Te_{\varphi(n)}(n) \neq 0$ and T is disjointness-preserving, we have $Tx(n) = 0$, that is, $x \in \ker(\delta_n \circ T)$. That is, $\emptyset \neq$ ker $\delta_{\varphi(n)} \subset$ ker ($\delta_n \circ T$). Notice that ker $\delta_{\varphi(n)}$ and ker ($\delta_n \circ T$) have same codimension. Therefore, ker ($\delta_n \circ T$) $=$ ker $\delta_{\varphi(n)}$. This establishes the claim.

Now, let $h : \mathbb{N} \to \mathbb{K}$ be defined by $h(n) = \alpha_n$ if $n \in \mathbb{N} \setminus \mathbf{N}_0$, and $h(n) = 0$ otherwise. Then $\delta_n \circ T = h(n) \cdot \delta_{\varphi(n)}$ and thus $(Tx)(n) = h(n) \cdot x(\varphi(n))$ for all x in ℓ^p and all natural numbers n. Also, $|h(n)| = |h(n) \cdot e_{\varphi(n)}(\varphi(n))| = |Te_{\varphi(n)}(n)| \leq \|Te_{\varphi(n)}\|_q \leq \|T\|$ for all natural numbers n. Thus $\|h\|_\infty = \|T\|$ and this completes the proof.

Now, we consider the Banach-Stone problem in the general case. Observe that the gap between Lamperti map and BS map is the extent to which whethera regular set homomorphism can be induced by a point map. To be more precise, we introduce the following notion.

For a measure space $(X, \mathcal{B}, \mu)$ a member B of $\mathcal{B}$ is an *atom* of μ if $\mu(C) = \mu(B)$ or $\mu(C) = 0$ for all C in $\mathcal{B}$ with $C \subset B$. We call $(X, \mathcal{B}, \mu)$ an *atom-free* measure space if μ posseses no atom. We call a measure subspace X' of X a *maximal atom free* subspace if X' is atom-free and $X \setminus X'$ is a disjoint union of atoms of μ.

We say that an atom-free measure space $(X, \mathcal{B}, \mu)$ has *Sikorski's property* if, for an arbitrary measure space $(Y, \mathcal{A}, \nu)$, every regular set homomorphism $\Phi : \mathcal{B} \to \mathcal{A}$ with $\Phi(X) = Y$ is induced by a measurable point map φ, that is, $\Phi(B) = \varphi^{-1}(B)$ for any B in $\mathcal{B}$.

Remark. It is known (cf., [13, p. 397]) that $([0,1], \mathcal{B}_{[0,1]}, \mu)$ has Sikorski's property, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra and μ is any σ-finite regular measure on $\mathcal{B}_{[0,1]}$.

Theorem 7. *Every σ-finite measure space $(X, \mathcal{B}, \mu)$ which has Sikorski's* *property solves the Banach-Stone problem.*

Proof. Let $\{a_i\}_{i\in I}$ be an arbitrary maximal set of atoms of μ. Since μ is σ-finite, I is at most countable. Let $X_1 = \bigcup_{i\in I} a_i$, then X_1 works exactly as a subset of $\mathbb{N}$. (For the Banach-Stone problem on $\mathbb{N}$, see Proposition 6.) Thus, without loss of generality, we may assume μ is atom-free. For an arbitrary measure space $(Y, \mathcal{A}, \nu)$, let $T : L^p(X) \to L^q(Y)$ be a (surjective when $p = q = \infty$) bounded disjointness preserving operator. It demands to show that T is a BS map. By Theorem 5, there exist a regular set homomorphism $\Phi : \mathcal{B} \to \mathcal{A}$ and a $h \in L^q(Y)$ such that $T\chi_B = h \cdot \chi_{\Phi(B)}$ for all B in $\mathcal{B}$ with $\mu(B) < \infty$.

There exists a measurable mapping $\varphi : Y \to X$ such that $\Phi(B) = \varphi^{-1}(B)$ for all B in $\mathcal{B}$ with $\mu(B) < \infty$. Consequently, $T\chi_B = h \cdot (\chi_B \circ \varphi)$.

In case of $p = q = \infty$, it is known that $\Phi(X) = Y$ by Theorem 4. By the Sikorski's property, the claim is done.

For the case, $1 \le p, q < \infty$ and $p, q \ne 2$, it is not necessarily true that $\Phi(X) = Y$. Let $Y_0 = \Phi(X)$ be a measurable set. Let $\mathcal{A}_0 = \{Y_0 \cap A : A \in \mathcal{A}\}$ and $\nu_0 = \nu|_{\mathcal{A}_0}$. Define $\Phi : \mathcal{B} \to \mathcal{A}_0$ by $\Phi_0(B) = Y_0 \cap \Phi(B)$ for all B in $\mathcal{B}$. It is easy to see that Φ_0 is a regualr set homomorphism satisfying $\Phi_0(X) = Y_0$. By assumption, there exists a measurable mapping $\varphi_0 : Y_0 \to X$ such that $\Phi(B) = \varphi_0^{-1}(B)$ for all B in $\mathcal{B}$. It follows that, for all B in $\mathcal{B}$ with $\mu(B) < \infty$, $(T\chi_B)|_{Y_0} = (h\cdot\chi_{\Phi(B)})|_{Y_0} = h|_{Y_0} \cdot \chi_{\Phi_0(B)} = h_0 \cdot \chi_{\varphi^{-1}(B)} = h_0 \cdot (\chi_B \circ \varphi_0)$ where $h_0 = h|_{Y_0} : Y_0 \to \mathbb{K}$. Since the support of $(T\chi_X)$ is contained in $\Phi(X) = Y_0$, we can redefine $h : Y \to \mathbb{K}$ by $h(y) = h_0(y)$ on Y_0 and $h(y) = 0$ otherwise. We can also extend φ_0 to $\varphi : Y \to X$ by $\varphi(y) = \varphi_0(y)$ on Y_0 and $\varphi(y) = x_0$ otherwise for some fixed x_0 in X. Then both h and φ are measurable and $T\chi_B = h \cdot (\chi_B \circ \varphi)$. This establishes the claim.

Now, if s is any simple function which vanishes outside a set of finite measure, we have $Ts = h \cdot (s \circ \varphi)$ by the linearity of T. Let f be in $L^p(X)$. By passing to a sequence of simple functions which approximate f, we have $Tf = h \cdot (f \circ \varphi)$. Hence, T is a BS map.

In the following, we shall give a σ-finite measure space which has Sikorski's property.

Definition 3.1. A totally ordered space $(X, \le)$ is said to be *Dedekind complete* if every bounded below nonempty subset has an infimum in X. A σ-algebra $\mathcal{B}$ is called *the order σ-algebra* of X if it is generated by all order intervals $(a, b) = \{x \in X : a < x < b\}$. A *totally ordered measure space* is a totally ordered space with the order σ-algebra. A measure space $(X, \mathcal{B}, \mu)$ is said to be *μ-separable* if $(X, \mathcal{B}, \mu)$ is totally ordered and contains a countable subset D of X such that $(a, b) \cap D \ne \emptyset$ for all a, b in X with $\mu((a, b)) \ne 0$. In this case, D is called an *order-μ-dense* subset of X.

Proposition 8. *Let $(X, \le)$ be totally ordered and Dedekind complete. If*

$(X, \mathcal{B}, \mu)$ is a σ-finite μ-separable measure space, then $(X, \mathcal{B}, \mu)$ has Sikorski's property.

Proof. Let $\{a_i\}_{i \in I}$ be an arbitrary maximal set of atoms of μ. Since μ is σ-finite, I is at most countable. Let $X' = X \setminus \bigcup_{i \in I} a_i$. X' is a maximal atom-free measure subspace of X. It suffices to show that X' has Sikorski's property. Without loss of generality, we may assume X is atom-free.

Given a measure space $(Y, \mathcal{A}, \nu)$ and a regular set homomorphism $\Phi : \mathcal{B} \to \mathcal{A}$ with $\Phi(X) = Y$. Let D be a countable order-μ-dense subset of X. Since X is an atom-free, $\bigcap_{\alpha \in D}(-\infty, \alpha)$ is a null set and $\bigcup_{\alpha \in D}(-\infty, \alpha) = X$. For each α in D, define $B_\alpha = \Phi(-\infty, \alpha)$. Then $B_\alpha \subset B_\beta$, $\alpha \leq \beta$, and $\bigcap_{\alpha \in D} B_\alpha = \emptyset$, $\bigcup_{\alpha \in D} B_\alpha = Y$. Let $\varphi : Y \to X$ be defined by $\varphi(y) = \inf\{\alpha \in D : y \in B_\alpha\}$ for all y in Y. Since X is (Dedekind) complete, φ is well-defined.

For all x in X (we may assume $(-\infty, x) \neq \emptyset$), it is easy to see that $\{y \in Y : \varphi(y) < x\} = \bigcup_{\gamma < x,\ \gamma \in D} B_\gamma$. Since D is order-μ-dense in X, the set $(-\infty, x) \setminus \bigcup_{\gamma < x,\ \gamma \in D}(-\infty, \gamma)$ is at most a null set in X.

By the facts $\bigcup_{\gamma < x,\ \gamma \in D}(-\infty, \gamma) \subset (-\infty, x)$ and Φ is regular set homomorphism, we have that $\bigcup_{\gamma < x,\ \gamma \in D} B_\gamma = \Phi(\bigcup_{\gamma < x,\ \gamma \in D}(-\infty, \gamma)) = \Phi(-\infty, x)$. Then $\Phi(-\infty, x) = \{y \in Y : \varphi(y) < x\} = \varphi^{-1}(-\infty, x)$. Therefore, the family $\mathcal{B}' = \{B \subset X : \Phi(B) = \varphi^{-1}(B)\}$ contains all order intervals in X. However $\mathcal{B}'$ is a σ-algebra. It follows that $\mathcal{B} \subseteq \mathcal{B}'$, i.e., $\Phi(B) = \varphi^{-1}(B)$ for all B in $\mathcal{B}$. And then φ is measurable. This complete the proof.

By Theorem 7 and Proposition 8, we get a generalization of Banach's result.

Theorem 9. *Every σ-finite μ-separable measure space $(X, \mathcal{B}, \mu)$, that $(X, \leq)$ is totally ordered and Dedeking complete, solves the Banach-Stone problem.*

To end this paper, we give some examples.

Example 10. Let $\mathbb{R}^n$ be equipped with usual norm $\|\cdot\|$ and μ which is Lebesgue measure restricted to the Borel σ-algebra. Define "$<$" such that $\mathbf{a} = (a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n) = \mathbf{b}$ if and only if $\|\mathbf{a}\| < \|\mathbf{b}\|$ or, otherwise, there exists an i such that $a_j = b_j$ for all $j < i$ and $a_i < b_i$. Then $\mathbb{R}^n$ becomes to be totally ordered and Dedekind complete. Moreover, the σ-algebra $\mathcal{B}$ generated by all the order intervals induced by $<$ is exactly the usual Borel σ-algebra for $\mathbb{R}^n$. Let D be the set $\{(d_1, d_2, \ldots, d_n) | d_i \in \mathbb{Q} \text{ for all } i\}$. Then D is countable and order-μ-dense in X. Thus X is μ-separable. Hence $(X, \mathcal{B}, \mu)$solves the Banach-Stone problem.

Example 11. For the Hilbert cube (that is, $\{x \in l^2 : |x_n| \leq \frac{1}{n}\}$ in norm topology) with usual norm $\|\cdot\|$, define "$<$" s.t. $\mathbf{a} = (a_1, a_2, \ldots) < (b_1, b_2, \ldots) = \mathbf{b}$ if and only if $\|\mathbf{a}\| < \|\mathbf{b}\|$ or, otherwise, there exists an i such that $a_j = b_j$ for all

$j < i$ and $a_i < b_i$. Then the Hilbert cube becomes a Dedekind complete totally ordered space and the σ-algebra $\mathcal{B}$ generated by all the order intervals induced by $<$ is exactly the usual Borel σ-algebra for Hilbert cube. Let μ be any σ-finite measure such that $\mu(S_r) = 0$ when S_r is a maximal atom-free part of the intersection of $\{x \in l^2 : \|x\|_2 = r\}$ and the Hilber cube. Let D be the set $\{(d_1, d_2, \ldots)|d_i \in \mathbb{Q}$ for all $i\}$ together with all atoms. Then D is countable and order-μ-dense in X. Thus X is μ-separable. Hence solves the Banach-Stone problem.

Let $(X, \mathcal{B}, \mu)$ be a measure space. In case $p = 2$, even though $(X, \mathcal{B}, \mu)$ has Sikorski's property, not every (surjective) linear isometry $T : L^2(X) \to L^2(X)$ is a BS map. It may also happen that T is not disjointness preserving and not even a Lamperti map.

Example 12. Consider $X = [-\pi, \pi]$. Let $e_1(x) = \frac{1}{\sqrt{2\pi}}$ and $e_{2n}(x) = \frac{\cos nx}{\sqrt{\pi}}$, $e_{2n+1}(x) = \frac{\sin nx}{\sqrt{\pi}}$ for $n = 1, 2, \ldots$. Let $\{p_1(x), p_2(x), p_3(x), \ldots\}$ be the collection of Legendre polynomials (they can be easily computed by the Gram-Schmidt precess that, for example, $p_1(x) = \frac{1}{\sqrt{2\pi}}$, $p_2(x) = \sqrt{\frac{3}{2\pi^3}}x$ and $p_3(x) = \frac{1}{\ell}(x^2 - \frac{2\pi^3}{3})$, where $\ell = \sqrt{\frac{2\pi^5}{5} - \frac{8\pi^6}{9} + \frac{4\pi^7}{9}}$). Then the two families of functions $\{e_1(x), e_2(x), e_3(x), \ldots\}$ and $\{p_1(x), p_2(x), p_3(x), \ldots\}$ are both orthonormal bases of $L^2[-\pi, \pi]$. Let $T : L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ be the surjective linear isometry such that $Tp_n = e_n$ for all $n = 1, 2, \ldots$.

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