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ANALYTIC SPACES DEFINED BY SYMMETRIC NORMING FUNCTIONS

Mark C. Ho and Mu Ming Wong

Abstract. Let c_0 be the space of sequences converging to 0. A symmetric norming function (or briefly, s.n. function) is a function Φ from c_0 into nonnegative numbers with the properties of that in a norm, a normalizing criteria: $\Phi(1,0,0,\cdots)=1$, and the symmetric condition: $\Phi(x_1,x_2,\cdots)=\Phi(x_1^*, x_2^*,\cdots)$, where $x_1^*, x_2^*,\cdots$ is the nonincreasing rearrangement of $|x_1|,|x_2|,\cdots$. In this paper, we will define spaces of analytic functions based on s.n. functions, which are generalization of the space B_1^+ in [2].

1. INTRODUCTION

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on $\mathcal{H}$. Let $\mathfrak{S}$ be a proper (two-sided) ideal of $\mathcal{B}$. It is well-know that $\mathcal{F}\subseteq\mathfrak{S}\subseteq\mathfrak{S}_\infty$, where $\mathcal{F}=\mathcal{F}(\mathcal{H})$ is the ideal of finite rank operators and $\mathfrak{S}_\infty$ is the ideal of compact operators. A norm $\|\cdot\|_s$ defined on $\mathfrak{S}$ is called a *symmetric norm* if it is a usual norm with the additional properties:
(1) $\|ATB\|_s\le\|A\|\|T\|_s\|B\|$ for $A,B\in\mathcal{B}$ and $T\in\mathfrak{S}$;
(2) for any rank one operator T, $\|T\|_s=\|T\|=s_1(T)$.
We shall say that $\mathfrak{S}$ is a *symmetrically-normed ideal* (or briefly, s.n. ideal) if $\mathfrak{S}$ is complete with respect to the norm $\|\cdot\|_s$.

On the other hand, let c_0 be the space of real sequences which converge to 0, and $\hat{c}=\{(x_1,x_2,\cdots)\in c_0 : x_n=0$ for all but finitely many $n\}$. A function $\Phi:\hat{c}\to\mathbb{R}$ is called a *symmetric norming function* (or briefly, s.n. function) if

(a) $\Phi(x)>0$ for $x\in\hat{c}$, $x\neq 0$;

(b) $\Phi(\alpha x)=|\alpha|\Phi(x)$ for any $\alpha\in\mathbb{R}$, $x\in\hat{c}$;

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(c) $\Phi(x+y) \leq \Phi(x) + \Phi(y)$, $x, y \in \hat{c}$;

(d) $\Phi(1, 0, 0, \cdots) = 1$;

(e) $\Phi(x_1, \cdots, x_n, 0, 0, \cdots) = \Phi(|x_{\sigma(1)}|, \cdots, |x_{\sigma(n)}|, 0, 0, \cdots)$, for any n, and any permutation σ of $1, 2, \cdots, n$.

Now consider $x = (x_1, x_2, \cdots) \in c_0$. Write $x^{(n)} = (x_1, \cdots, x_n, 0, 0, \cdots)$ and define

$$c_\Phi = \left\{ x \in c_0 : \sup_n \Phi(x^{(n)}) < \infty \right\}.$$

It is well-known that $\Phi(x^{(n)})$ is nondecreasing (Lemma 3.2, Chap. III, [4]), and therefore we may define $\Phi(x) = \lim_{n\to\infty} \Phi(x^{(n)})$, $x \in c_\Phi$. Now let $T \in \mathcal{B}$. The *singular values* of T is the nonincreasing sequence of nonnegative numbers $\{s_n(T)\}$ $(n = 1, 2, \cdots)$

$$s_n(T) = \inf\{\|T - S\| : \text{rank}(S) < n\}, \quad n = 1, 2, \cdots .$$

Note that $s_n(T) \searrow 0$ if and only if $T \in \mathfrak{S}_\infty$. There are other alternatives to describe the $s_n(T)$'s. For instance, one can show that the $s_n(T)$'s are in fact the eigenvalues of $|T| = (T^*T)^{1/2}$. We say that T is in the *symmetrically-normed ideal* $\mathfrak{S}_\Phi$ *generated by* Φ if $(s_0(T), s_1(T), \cdots) \in c_\Phi$, with norm

$$\|T\|_\Phi = \Phi(s_0(T), s_1(T), \cdots)$$

since Φ induces a symmetric norm on $\mathfrak{S}_\Phi$ (Theorem 4.1, Chap. III, [4]), and we use $\mathfrak{S}_\Phi^{(0)}$ to denote the closure of $\mathcal{F}$ in $\mathfrak{S}_\Phi$, which is itself a s.n. ideal.

The concept of s.n. functions is introduced for the purpose of classifying operator ideals on Hilbert spaces with norms invariant with respect to unitary dilation (See [4]). For example, the usual Schatten p-classes $\mathfrak{S}_p$, $1 \leq p \leq \infty$ are defined by the s.n. functions $\Phi_p(x_1, x_2, \cdots) = (\sum |x_n|^p)^{1/p}$ and $\mathfrak{S}_p^{(0)} = \mathfrak{S}_p$, and the so-called *binormalizing* s.n. ideals $\mathfrak{S}_\Pi$ are defined by the functions of form

$$\Phi_\Pi(x) = \sup_n \frac{\sum_1^n x_k^*}{\sum_1^n \pi_k},$$

where $x_1^*, x_2^*, \cdots$ is the nonincreasing rearrangement of $|x_1|, |x_2|, \cdots$, and $\Pi = \{\pi_n\}$ is a nonincreasing nonnegative sequence with $\pi_1 = 1$ and $\sum \pi_n = +\infty$.

The problems of characterizing special classes of operators in various s.n. ideals, especially in the Schatten classes, have always attracted the attentions from the functional analysts. In our case, the class of interest here is the *Hankel operators* on $H^2 = H^2(\mathbb{D})$, the Hardy space on the unit disc $\mathbb{D}$. Let $f \in H^2$, then the operator h_f on H^2 is called a Hankel operator if the matrix of h_f with respect to the standard basis $\{1, z, z^2, \cdots\}$ in H^2 is given by

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & \cdots & \cdots \\ a_2 & a_3 & \cdots & \cdots & \cdots \\ a_3 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $f(z) = \sum_0^\infty a_n z^n$ is the Taylor series of f. For the detail of Hankel operators, we refer the readers to [7]. In [6], Peller proved a remarkable result stating that $h_f \in \mathfrak{S}_p$, $1 \le p < \infty$ if and only if $f \in B_p$, where B_p is the analytic *Besov* space defined by

$$\left\{ f \text{ analytic on } \mathbb{D} : \int_{\mathbb{D}} |f''(z)|^p (1 - |z|^2)^{2p-2} dxdy < \infty \right\}.$$

Later, in [2] and [3], the spaces B_1^+ and B_L are defined based on the s.n. function Φ_Π to describe f for which the corresponding Hankel operator h_f belonging to $\mathfrak{S}_\Pi$, when Π satisfies a so-called *regular* conditions. Our goal for this paper is to generalize these analytic spaces for all s.n. functions.

2. Spaces of Analytic Functions Related to Symmetric Norming Functions

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $z, \omega \in \mathbb{D}$. Let $\rho(z, \omega)$ be the hyperbolic distance between z and ω, i.e.,

$$\rho(z, \omega) = \frac{1}{2} \log \frac{1 + \left| \dfrac{z - \omega}{1 - \overline{z}\omega} \right|}{1 - \left| \dfrac{z - \omega}{1 - \overline{z}\omega} \right|}.$$

Also, for $r > 0$, we denote the hyperbolic ball with center at z and radius r by $D(z, r)$, i.e., $D(z, r) = \{\omega : \rho(z, \omega) < r\}$. On the other hand, let $K(z, \omega) = (1 - z\overline{\omega})^{-2}$ be the Bergmann kernel on $\mathbb{D}$ and $dv(z) = \frac{1}{\pi} dxdy$ the normalized Lebesgue area measure on $\mathbb{D}$. The followings are some useful facts in analytic function theory on $\mathbb{D}$ concerning the hyperbolic metric that will be considered in our later discussion:

Given $r, s > 0$, there is a $C > 0$ depending only on r and s so that

F1. $C^{-1}(1 - |a|^2)^2 \le |D(z, r)| \le C(1 - |a|^2)^2$, where $|D(z, r)|$ is the area of $D(z, r)$, for all $z \in D(a, r)$ and $a \in \mathbb{D}$.

F2. $C^{-1}|D(z,r)| \le |D(\omega,s)| \le C|D(z,r)|$ if $\rho(\omega,z) < r$.

F3. $C^{-1}K(a,\omega) \le K(z,\omega) \le CK(a,\omega)$ for all $\omega \in \mathbb{D}$ if $z \in D(a,r)$.

F4. (Subnormality)

$$h(a) \le \frac{C}{|D(a,r)|}\int_{D(a,r)} h(z)dv(z)$$

for any nonnegative subharmonic function h on $\mathbb{D}$.

F5. There is a sequence $\{\omega_n\}$ in $\mathbb{D}$ and measurable sets $D_n \subseteq \mathbb{D}$ so that (1) $|\omega_n| \to 1$ and $\bigcup_{n=1}^{\infty} D_n = \mathbb{D}$ (2) $D(\omega_n, \frac{r}{4}) \subseteq D_n \subseteq D(\omega_n, r)$ for $n \ge 1$ (3) $D_n \cap D_m = \emptyset$ if $n \neq m$ and (4) there is a $N \in \mathbb{N}$ depending only on r such that any z in $\mathbb{D}$ belongs to at most N of the sets $\{D(\omega_n, 2r)\}$.

The reader can find details about these properties in, for instance, [8].

Now let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$ and Φ be a s.n. function, as mentioned earlier. Pick $r > 0$, and choose a cover $\mathcal{D} = \{D_n\}$ of $\mathbb{D}$ with respect to some sequence $\{\omega_n\}$ in $\mathbb{D}$, as in **F5**. Fixing a nonnegative integer k, and let us consider the space of analytic functions on $\mathbb{D}$ as follows:

$$B_{\Phi,\mathcal{D}} = \{f \in \mathcal{H}(\mathbb{D}) : (\lambda_1(f), \lambda_2(f), \cdots) \in c_\Phi\},$$

where $\lambda_n(f) = \sup\{|f''(z)|(1-|z|^2)^2 : z \in D_n\}$, equipped with norm

$$\|f\|_{\Phi,\mathcal{D}} = |f(0)| + |f'(0)| + \Phi(\lambda_1(f), \lambda_2(f), \cdots).$$

It is clear from the definition that $B_{\Phi,\mathcal{D}} \subseteq \mathcal{B}_0$, where $\mathcal{B}_0$ is the little Bloch space. It is also clear that if $f_k \to f$ uniformly on compact subsets of $\mathbb{D}$, then $\lambda_n(f_k) \to \lambda_n(f)$ for each n.

Theorem 2.1. *$B_{\Phi,\mathcal{D}}$ with the norm above is a Banach space.*

Proof. Let $\{f_i\}$ be a Cauchy sequence in $B_{\Phi,\mathcal{D}}$ and $K \subset \mathbb{D}$ be compact. Choose n_0 large enough such that $\bigcup_{z\in K} D(z,r) \subseteq \bigcup_{n=1}^{n_0} D_n$. Hence, by the subnormality of $|f''|$ and **F1-F5**, for any $z \in K$, we have

$$\begin{aligned}
|f_i''(z) - f_j''(z)| &\le \frac{C}{|D(z,r)|}\int_{D(z,r)} |f_i''(\omega) - f_j''(\omega)|dv(\omega) \\
&\le C_K \sum_{n=1}^{n_0} \int_{D_n} |f_i''(\omega) - f_j''(\omega)|(1-|\omega|^2)^2 d\lambda(\omega) \\
&\le C_K C_1 n_0 \max_{1\le n\le n_0} \lambda_n(f_i - f_j)
\end{aligned}$$

for some $C, C_1 > 0$ depending only on r, with $(\inf_{z\in K}|D(z,r)|)C_K \geq C$. Here $d\lambda(z) = K(z,z)dv(z)$. It follows that for all i, j, there exists $\tilde{C}_K > 0$ such that

$$|f_i''(z) - f_j''(z)| \leq \tilde{C}_K \|f_i - f_j\|_{\Phi,\mathcal{D}}$$

by properties of s.n. functions. This shows that $\{f_i\}$ is uniformly Cauchy on compact subsets of $\mathbb{D}$. Therefore f_i converges uniformly on compacts to an analytic function on $\mathbb{D}$. The fact that $f \in B_{\Phi,\mathcal{D}}$ is clear. Indeed, for any fixed n, we have

$$\Phi((\lambda_1(f), \lambda_2(f), \cdots)^{(n)}) \leq \\ \Phi((\lambda_1(f - f_i), \lambda_2(f - f_i), \cdots)^{(n)}) + \Phi((\lambda_1(f_i), \lambda_2(f_i), \cdots)^{(n)})$$

for any i. Since f_i converges uniformly on compact subsets of $\mathbb{D}$, we have, by letting $i \to \infty$, that for all n,

$$\Phi((\lambda_1(f), \lambda_2(f), \cdots)^{(n)}) \leq M,$$

where $M \geq \sup_i \|f_i\|_{\Phi,\mathcal{D}}$. This means that $(\lambda_1(f), \lambda_2(f), \cdots) \in c_\Phi$. ■

As much as the fact that $B_{\Phi,\mathcal{D}}$ being a Banach space is now established, it may appear, at least in the surface, that the space $B_{\Phi,\mathcal{D}}$ depends on both the s.n. function Φ and $\mathcal{D}$. We will show, in the next result, that the "dependence" of $B_{\Phi,\mathcal{D}}$ on $\mathcal{D}$ can be removed:

Proposition 2.2. *$B_{\Phi,\mathcal{D}}$ does not depend on r, or the cover $\mathcal{D}$.*

Proof. In order to prove this, we need a couple of lemmas, which also turn out to be very useful throughout the remaining of this article:

Lemma 2.3. *Let $p > 0$, $r, \tilde{r} > 0$ and $\mathcal{D} = \{D_n\}$, $\tilde{\mathcal{D}} = \{\tilde{D}_n\}$ be the corresponding decompositions of $\mathbb{D}$ (see* **F5***), with the interpolating sequences $\{\omega_n\}$, $\{\tilde{\omega}_n\}$, respectively. Let $h \geq 0$ be subharmonic on $\mathbb{D}$. Then*

$$\sum_{k=1}^{n} \left(\sup_{z\in D_k} h(z)(1 - |z|^2)^{2p} \right)^* \approx \sum_{k=1}^{n} \left(\int_{\tilde{D}_k} h(z)(1 - |z|^2)^{2p} d\lambda(z) \right)^*,$$

where $a_n \approx b_n$ means that there exists a $C > 0$ such that $C^{-1}a_n \leq b_n \leq Cb_n$ for all n (Recalling that $x_1^*, x_2^*, \cdots$ is the nonincreasing rearrangement of $|x_1|, |x_2|, \cdots$).

Proof of Lemma 2.3. Since the hyperbolic metric ρ is invariant under Mӧbius transformation, there exists $M \in \mathbb{N}$, depending only on $r, \tilde{r}$, such that there are at most M points in $D(z, 2\epsilon)$ that are at least δ apart, in the hyperbolic metric ρ, for any $z \in \mathbb{D}$, where $\epsilon \geq \max(r, \tilde{r})$ and $0 < \delta \leq \min(\frac{r}{2}, \frac{\tilde{r}}{2})$.

For each n, let $l(n)$ be the cardinality of the set $\mathcal{A}_n = \{\tilde{D}_k : \tilde{D}_k \cap D(\omega_n, 2r) \neq \emptyset\}$. Then $l(n) \leq M$, and we may write $\mathcal{A}_n = \{\tilde{D}_{n,i} : 1 \leq i \leq M\}$, where $\tilde{D}_{n,i} = \emptyset$ if $i > l(n)$. Since h is continuous, there exists $z_n \in \overline{D_n}$ (the closure of D_n) for each n so that

$$\sup_{z \in D_n} h(z)(1-|z|^2)^{2p} = h(z_n)(1-|z_n|^2)^{2p}.$$

This means, by the subnormality of h and **F1, F2**, that there is a $C > 0$, depending only on r and $\tilde{r}$, so that

$$\sup_{z \in D_n} h(z)(1-|z|^2)^{2p} \leq C \int_{D(z_n, \frac{r}{4})} h(z)(1-|z|^2)^{2p} d\lambda(z),$$

which implies that

$$\begin{aligned}\sup_{z \in D_n} h(z)(1-|z|^2)^{2p} &\leq C \sum_{i=1}^{M} \int_{\tilde{D}_{n,i}} h(z)(1-|z|^2)^{2p} d\lambda(z) \\ &\leq CM \max_{1 \leq i \leq M} \left\{ \int_{\tilde{D}_{n,i}} h(z)(1-|z|^2)^{2p} d\lambda(z) \right\}.\end{aligned}$$

Now for each fixed i, $\tilde{D}_{n,i}$ can appear at most M times in $\tilde{\mathcal{D}}$ as n runs through $\mathbb{N}$. Thus

$$\sum_{k=1}^{n} \left(\sup_{z \in D_k} h(z)(1-|z|^2)^{2p} \right)^* \leq CM^2 \sum_{k=1}^{n} \left(\int_{\tilde{D}_k} h(z)(1-|z|^2)^{2p} d\lambda(z) \right)^*$$

for each n. The other half of the inequalities, i.e.,

$$\sum_{k=1}^{n} \left(\int_{\tilde{D}_k} h(z)(1-|z|^2)^{2p} d\lambda(z) \right)^* \leq C_1 M^2 \sum_{k=1}^{n} \left(\sup_{z \in D_k} h(z)(1-|z|^2)^{2p} \right)^*$$

for each n, and for some $C_1 > 0$, can be derived similarly. ■

The second lemma, due to K. Fan (See [1] or Lemma 3.1, Chap. III, [4]), is stated as follow:

Lemma 2.4. *Suppose that* $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots) \in \hat{c}$. *If* $x_1 \geq x_2 \geq \cdots \geq 0$, $y_1 \geq y_2 \geq \cdots \geq 0$ *and*

$$\sum_{k=1}^{n} x_k \leq \sum_{k=1}^{n} y_k, \;\; n = 1, 2, \cdots,$$

then for any s.n. function Φ *we have* $\Phi(x) \leq \Phi(y)$.

Proof of Proposition 2.2. By applying Lemma 2.3 to the subharmonic function $|f''|$, we have

$$\sum_{k=1}^{n} \lambda_k(f)^* \le C \sum_{k=1}^{n} \tilde{\lambda}_k(f)^*, \; n = 1, 2, \cdots,$$

where $\tilde{\lambda}_n(f) = \sup\{|f''(z)|(1-|z|^2)^2 : z \in \tilde{D}_n\}$. Therefore, as a consequence of Lemma 2.4

$$\Phi(\lambda_1(f), \lambda_2(f), \cdots) \le C\Phi(\tilde{\lambda}_1(f), \tilde{\lambda}_2(f), \cdots)$$

for some $C > 0$. This means that $\|f\|_{B_{\Phi,\mathcal{D}}} \le C_1 \|f\|_{B_{\Phi,\tilde{\mathcal{D}}}}$ for some $C_1 > 0$. On the other hand, it is also clear that we can show, by similar argument, that $\|f\|_{B_{\Phi,\tilde{\mathcal{D}}}} \le C_2 \|f\|_{B_{\Phi,\mathcal{D}}}$ for some $C_2 > 0$. This proves that $B_{\Phi,\mathcal{D}}$ is isomorphic to $B_{\Phi,\tilde{\mathcal{D}}}$. ∎

From now on we shall replace $B_{\Phi,\mathcal{D}}$ by B_Φ, in view of Proposition 2.2.

Example Let $1 \le p \le \infty$ and pick a decomposition $\mathcal{D}$ of $\mathbb{D}$ described in **F5**. Recall the s.n. function Φ_p defined in the previous section. Then the space B_{Φ_p} is precisely the analytic Besov space B_p. Indeed, for $1 \le p < \infty$, since

$$\int_{D_n} |f''(z)|^p (1-|z|^2)^{2p} d\lambda(z) \le C\lambda_n(f)^p$$

for some $C > 0$ depending only on r, we have $B_p \subseteq B_{\Phi_p}$. On the other hand, since $|f''|^p$ is subharmonic for $p > 0$, we have

$$\sum_{k=1}^{n} \lambda_k(f)^{*p} \le C \sum_{k=1}^{n} \left(\int_{D_k} |f''(z)|^p (1-|z|^2)^{2p} d\lambda(z) \right)^* \text{ for all } n,$$

by Lemma 2.3. This shows that $B_{\Phi_p} \subseteq B_p$. When $p = \infty$, we have $B_{\Phi_\infty} = \mathcal{B}_0$ since Φ_∞ is given by $\Phi_\infty(x) = \|x\|_\infty$, $x \in c_0$.

As a consequence of the above examples, we have

Corollary 2.5. *Let Φ be a s.n. function on $\hat{c}$. Then $B_1 \subseteq B_\Phi \subseteq \mathcal{B}_0$.*

Proof. Since by the properties of s.n. function, we have $\Phi_\infty(x) \le \Phi(x) \le \Phi_1(x)$ for any $x \in \hat{c}$. ∎

In the theory of s.n. ideals, it is known that two s.n. ideals are the same if and only if their norms are equivalent. An analog of this fact can also be derived for these spaces of analytic functions associated with s.n. functions:

Corollary 2.6. *Let Φ_1 and Φ_2 be s.n. functions on $\hat{c}$. Suppose that $B_{\Phi_1} = B_{\Phi_2}$ elementwise. Then B_{Φ_1} is isomorphic to B_{Φ_2}.*

Proof. Fix a decomposition $\mathcal{D}$ of $\mathbb{D}$ as described in **F5**, for all the spaces involved. Let B be the set of elements of B_{Φ_1} or, what is the same, B_{Φ_2}. Equip B with the norm

$$\|f\| = \max\{\|f\|_{\Phi_1}, \|f\|_{\Phi_2}\}.$$

It is clear that $\|\cdot\|$ is the same with the norm generated by the s.n. function

$$\Phi(x) = \max\{\Phi_1(x), \Phi_2(x)\},\ x \in \hat{c},$$

i.e., B is in fact the space B_Φ, which is, by Theorem 2.1, complete. On the other hand, we see that, by the open mapping theorem, the identity map is an isomorphism from $(B, \|\cdot\|)$ onto $(B_{\Phi_1}, \|\cdot\|_{\Phi_1})$ and in the same time an isomorphism from $(B, \|\cdot\|)$ onto $(B_{\Phi_2}, \|\cdot\|_{\Phi_2})$. This completes the proof. ∎

Another well-known fact about the s.n. ideals is that $\mathfrak{S}_\Phi$ is separable if $\Phi(x_{n+1}, x_{n+2}, \cdots) \to 0$ (or, equivalently, $\Phi(x^*_{n+1}, x^*_{n+2}, \cdots) \to 0$) as $n \to \infty$ for every $(x_1, x_2, \cdots) \in c_\Phi$, and that $\mathcal{F}$ is dense in $\mathfrak{S}_\Phi$. Here we would like to prove a similar result for B_Φ:

Proposition 2.7. *Suppose that Φ is a s.n. function such that for every $(x_1, x_2, \cdots) \in c_\Phi$ we have $\Phi(x_{n+1}, x_{n+2}, \cdots) \to 0$ as $n \to \infty$. Then B_Φ is separable. Moreover, the disk algebra $A(\mathbb{D})$ is dense in B_Φ.*

Proof. In view of Proposition 2.2, let us first introduce a special decomposition of $\mathbb{D}$ satisfying the conditions in **F5**: Let $I = [a, b) \subseteq [0, 1)$ and $|I| = b - a$. Associate I a subset $W(I)$ (a "*window*") in $\mathbb{D}$ defined by

$$W(I) := \left\{ z : \frac{1}{2}|I| < 1 - |z| \le |I| \text{ and } \arg(z) \in 2\pi I \right\}.$$

A $\omega \in W(I)$ is called the *center* of $W(I)$ if $1 - |\omega| = \frac{3}{4}|I|$ and $\arg(\omega) = (a + b)\pi$ (or, the geometric center of $W(I)$). Now let us consider an interval $I \subseteq [0, 1)$ with dyadic endpoints, then the associated $W(I)$ has the form

$$W_{n,k} = \left\{ z : \frac{1}{2^{n+1}} \le 1 - |z| < \frac{1}{2^n} \text{ and } \frac{k-1}{2^{n-1}}\pi \le \arg(z) < \frac{k}{2^{n-1}}\pi \right\}$$

where $1 \le k \le 2^n$ and $n \ge 0$. It is known that the collection $\mathfrak{D} = \{W_{n,k}\}$ forms a decomposition of the unit disc $\mathbb{D}$ with their centers (The so-called dyadic decomposition of $\mathbb{D}$) satisfying the conditions described in **F5** with suitable $r > 0$

(see, for example, [5]). We shall rewrite $\mathfrak{D} = \{D_n : n = 1, 2, \cdots\}$, with the corresponding centers $\{\omega_n : n = 1, 2, \cdots\}$.

Now suppose that $f \in B_\Phi$ and $r_i \nearrow 1$. Set $f_i(z) = f(r_i z)$. We will show that $f_i \to f$ in B_Φ. Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ so that

$$\Phi(\lambda_n(f), \lambda_{n+1}(f), \cdots) < \varepsilon$$

if $n \geq n_0$. Pick $\frac{1}{2} < R < 1$ such that $\{z \in \mathbb{C} : R \leq |z| < 1\} \subseteq \bigcup_{n \geq n_0} D_n$. Furthermore, choose $n_1 > n_0$ so that if $n \geq n_1$, then $\rho(\zeta, z) > 2r$ whenever $\zeta \in D_n$ and $|z| \leq \frac{R+1}{2}$. Since $f_i'' \to f''$ uniformly on compact subsets of $\mathbb{D}$, we may choose large i such that $1 - r_i < \varepsilon$,

$$\Phi((\lambda_1(f_i - f), \lambda_2(f_i - f), \cdots)^{(n_1)}) < \varepsilon$$

and $r_i(R+1) > 2R$. Now choose $z_n \in \overline{D_n}$ such that $\lambda_n(f_i) = |f_i''(z_n)|(1-|z_n|^2)^2$, one has

$$\begin{aligned} \lambda_n(f_i) &\leq C \int_{D(z_n, \frac{r}{4})} |f_i''(z)| dv(z) \\ &\leq C \sum_{D_k \in \mathcal{A}_n} \int_{D_k} |f_i''(z)| dv(z) \end{aligned}$$

for some $C > 0$ depending only on r, where again, the cardinality of the set $\mathcal{A}_n = \{D_k : D_k \cap D(\omega_n, r) \neq \emptyset\}$ does not exceed M for some $M \in \mathbb{N}$. Also note that by the choice of $\mathcal{A}_n$, $D_k \subseteq \{z \in \mathbb{C} : \frac{R+1}{2} \leq |z| < 1\}$ if $D_k \in \mathcal{A}_n$ and $n \geq n_1$, which means that there exists $n_0 \leq n_2 \leq n_1$ so that $D_k \in \{D_m : m \geq n_2\} \subseteq \{z : \frac{R+1}{2} \leq |z| < 1\}$ if $D_k \in \mathcal{A}_n$ and $n \geq n_1$. It follows then from the definition of f_i that

$$\begin{aligned} \lambda_n(f_i) &\leq C \sum_{D_k \in \mathcal{A}_n} r_i^{-2} \int_{r_i D_k} |f''(z)| dv(z) \\ &\leq 2C \sum_{D_k \in \mathcal{A}_n} \int_{r_i D_k} |f''(z)| dv(z) \\ &\leq 2CM \max_{D_k \in \mathcal{A}_n} \int_{r_i D_k} |f''(z)| dv(z). \end{aligned}$$

However, since the $\{r_i D_k\}$ are pairwise disjoint and the set $\mathcal{B}_k = \{D_n : D_n \cap r_i D_k \neq \emptyset\}$ has at most two elements, we see that

$$\sum_{k=n_2}^{n_2+j} \left(\int_{r_i D_k} |f''(z)| dv(z) \right)^* \leq 2 \sum_{n=n_0}^{n_0+j} \left(\int_{D_n} |f''(z)| dv(z) \right)^*$$

for $j = 1, 2, \cdots$, by the choice of n_2 and r_i. Now each D_k appears in at most N different $\mathcal{A}_n$'s (see **F5**). Therefore

$$\sum_{n=n_1}^{n_1+j} \lambda_n(f_i)^* \leq 4CMN \sum_{n=n_0}^{n_0+j} \left(\int_{D_n} |f''(z)| dv(z) \right)^* \leq 4C_1 MN \sum_{n=n_0}^{n_0+j} \lambda_n(f)^*$$

for $j = 1, 2, \cdots$, and for some $C_1 > 0$ depending on r. Hence, by Lemma 2.4, we have

$$\Phi(\lambda_{n_1}(f_i)^*, \lambda_{n_1+1}(f_i)^*, \cdots) \leq 4C_1 MN \Phi(\lambda_{n_0}(f)^*, \lambda_{n_0+1}(f)^*, \cdots) < 4C_1 MN \varepsilon.$$

This implies that

$$\Phi(\lambda_1(f_i - f), \lambda_2(f_i - f), \cdots) \leq 2(1 + 2C_1 MN)\varepsilon,$$

or, $\|f_i - f\|_\Phi \leq (|f'(0)| + 2(1 + 2C_1 MN))\varepsilon$. This completes the proof. ∎

Conjecture. B_Φ is separable if and only if $\Phi(x_{n+1}, x_{n+2}, \cdots) \to 0$.

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Mark C. Ho
Department of Applied Mathematics,
National Sun Yat-Sen University,
Kaohsiung 804, Taiwan
E-mail: hom@math.nsysu.edu.tw

Mu-Ming Wong
Department of Information Technology,
Meiho Institute of Technology,
23, Ping-Guang Road,
Nei-Pu Village, Ping-Tong 912,
Taiwan
E-mail: x2119@mail.meiho.edu.tw