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# Characterizations of Tori in 3-spheres

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Abstract. Using the $II$-metric and the $II$-Gauss map on a surface derived from the non-degenerate second fundamental form of a surface in the sphere, we establish some characterizations of compact surfaces including the spheres and the tori in the 3-dimensional unit sphere.

## 1. Introduction

It is interesting to look at differential geometry of surfaces of the unit 3-sphere $\mathbb{S}^3(1)$ in the 4-dimensional Euclidean space $\mathbb{E}^4$. According to [5, p. 138], there are no complete surfaces immersed in $\mathbb{S}^3(1)$ with constant extrinsic Gaussian curvature $K_{\text{ext}} < -1$ and $-1 < K_{\text{ext}} < 0$. In fact, $K_{\text{ext}}$ is derived from the determinant of the second fundamental form of a surface of $\mathbb{S}^3(1)$. There are infinitely many complete flat surfaces in $\mathbb{S}^3(1)$, for example, tori $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, the product of plane circles, are good examples of those, where $a^2 + b^2 = 1$. Among them, the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is a unique torus immersed in $\mathbb{S}^3(1)$, which is minimal and allows closed geodesics of the surface mapped onto closed curves of finite type in $\mathbb{S}^3(1)$. The Clifford torus was studied in [3] in terms of the notion of finite-type immersion. By definition, a finite type immersion $x\colon M \to \mathbb{E}^m$ of a submanifold $M$ in a Euclidean space $\mathbb{E}^m$ can be represented as a decomposition of the eigenvectors of the Laplace operator $\Delta$ of $M$ in the following

$$x = x_0 + x_1 + \cdots + x_k,$$

where $x_0$ is a constant vector and $x_1, \ldots, x_k$ are non-constant vectors satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, 2, \ldots, k$. In particular, if all of $\lambda_1, \ldots, \lambda_k$ are different, it is called $k$-type or the submanifold $M$ is said to be of $k$-type (cf. [1, 2]).

On the other hand, it is also interesting to look at the Gauss map which satisfies some differential equations and how it is related to characterize nice surfaces. For example, how can we say about the surfaces of $\mathbb{S}^3(1)$ if the Gauss map is of finite type or harmonic?

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Interestingly, there do not exist surfaces of $\mathbb{S}^3(1)$ with harmonic Gauss map, which will be shown in Section 3. If a surface has the non-degenerate second fundamental form, we can define a non-degenerate metric, possibly a time-like metric induced from the shape operator and the induced Riemannian metric. We also define a formal Gauss map which is called the $II$-Gauss map and the Laplace operator associated with the metric, which is called the $II$-Laplace operator.

In the present paper, we consider some obstruction theorems for surfaces in $\mathbb{S}^3(1)$ regarding the usual Gauss map related to the usual Laplacian and the isometric immersion with respect to the $II$-Laplacian, and we find the necessary and sufficient condition for a compact surface $M$ in $\mathbb{S}^3(1)$ with non-degenerate second fundamental form to be a sphere or a torus in $\mathbb{S}^3(1)$.

## 2. Preliminaries

Let $\mathbb{E}^4$ be the 4-dimensional Euclidean space with the canonical metric tensor $\langle \cdot, \cdot \rangle$ and $\mathbb{S}^3(1)$ the unit hypersphere centered at the origin in $\mathbb{E}^4$.

Let $M$ be a surface in $\mathbb{S}^3(1)$. We denote the Levi-Civita connection by $\widetilde{\nabla}$ of $\mathbb{S}^3(1)$ and the induced connection $\nabla$ of $M$ and $D$ for the normal connection of $M$. We use the same notation $\langle \cdot, \cdot \rangle$ as the canonical metric tensors of $\mathbb{E}^4$, $\mathbb{S}^3(1)$ and $M$. The shape operator (or the Weingarten map) $S\colon TM \to TM$ of $M$ is defined by $S(X) = -\widetilde{\nabla}_X N$ for a tangent vector field $X$ of $M$, where $TM$ is the tangent bundle of $M$ and $N$ the unit normal frame associated with the orientation of $M$ in $\mathbb{S}^3(1)$. Let $H$ and $K_{\text{ext}}$ be the mean curvature and the extrinsic Gaussian curvature of $M$ in $\mathbb{S}^3(1)$ defined by $H = \frac{1}{2}\operatorname{tr} S$ and $K_{\text{ext}} = \det S$ of $M$, respectively. $M$ is said to be flat if its Gaussian curvature $K = 1 + K_{\text{ext}}$ vanishes identically and $M$ is minimal in $\mathbb{S}^3(1)$ if $H \equiv 0$. The Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is minimal in $\mathbb{S}^3(1)$ and flat in $\mathbb{E}^4$, which is of 1-type in $\mathbb{E}^4$ (see [1, 2]).

The Gauss and Codazzi equations of $M$ in $\mathbb{S}^3(1)$ are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle SX, Y \rangle N, \tag{2.1}$$

$$(\nabla_X S)Y = (\nabla_Y S)X \tag{2.2}$$

for the vector fields $X$, $Y$ and $Z$ tangent to $M$.

Suppose that $p$ is not a flat point in $M$. Then, we can choose a coordinate patch $x(s,t)$ on a neighborhood $\mathcal{U}$ around $p$ such that $x_s$, $x_t$ are in the principal directions. Then, we have

$$S = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

with respect to the coordinate frame $\{x_s, x_t\}$. Thus, $H = (\kappa_1 + \kappa_2)/2$ and $K = 1 + \kappa_1\kappa_2$.

We put $\langle x_s, x_s\rangle = E$ and $\langle x_t, x_t\rangle = G$. Define a symmetric tensor $h$ by

$$h(X,Y) = \langle SX, Y\rangle \,. \tag{2.3}$$

Suppose that $h$ is non-degenerate. Then, $h$ is regarded as a non-degenerate metric on $M$, which is called the $II$-metric with representation given by

$$h = \begin{pmatrix} \kappa_1 E & 0 \\ 0 & \kappa_2 G \end{pmatrix}. \tag{2.4}$$

According to [5, p. 138], if $M$ is a compact surface with constant Gaussian curvature $K \geq 1$, $M$ is totally umbilical in $\mathbb{S}^3(1)$. Thus, we mainly focus on the case that $\kappa_1\kappa_2 \neq 0$ on $M$ in (2.4).

## 3. Obstruction theorems of surfaces in $\mathbb{S}^3(1)$

Let $M$ be a surface of $\mathbb{S}^3(1)$. Choose the isothermal coordinate system $(s,t)$ of a point in $M$ such that its metric is represented by $d\widetilde{s}^2 = \lambda(ds^2 + dt^2)$, where $\lambda > 0$. Then, we can have the Gauss map $G$ of $M$ in $\mathbb{E}^4$ defined by

$$G = \frac{1}{\lambda}(x_s \wedge x_t), \tag{3.1}$$

where $x\colon M \to \mathbb{S}^3(1) \subset \mathbb{E}^4$ is the isometric immersion of $M$ into $\mathbb{S}^3(1)$, $x_s = \partial x/\partial s$ and $x_t = \partial x/\partial t$. In this case, the Laplace operator $\Delta$ is given by

$$\Delta = -\frac{1}{\lambda}\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right). \tag{3.2}$$

It is straightforward to compute

$$x_{ss} = \frac{\lambda_s}{2\lambda}x_s - \frac{\lambda_t}{2\lambda}x_t + a\lambda N - \lambda x, \tag{3.3}$$

$$x_{st} = \frac{\lambda_t}{2\lambda}x_s + \frac{\lambda_s}{2\lambda}x_t + b\lambda N, \tag{3.4}$$

$$x_{tt} = -\frac{\lambda_s}{2\lambda}x_s + \frac{\lambda_t}{2\lambda}x_t + c\lambda N - \lambda x, \tag{3.5}$$

where $N$ is the unit normal vector field of $M$ in $\mathbb{S}^3(1)$ and $S = \left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)$ is the shape operator of $M$ with respect to $N$.

Then, using (3.1)–(3.5), we have

$$\begin{aligned} \Delta G = -\frac{1}{\lambda}\Big\{ &-(a^2 + 2b^2 + c^2 + 2)x_s \wedge x_t \\ &+ \left(a_s + \frac{\lambda_s}{2\lambda}a + \frac{\lambda_t}{\lambda}b + b_t - \frac{\lambda_s}{2\lambda}c\right) N \wedge x_t \\ &+ \left(\frac{\lambda_t}{2\lambda}a - b_s - \frac{\lambda_s}{\lambda}b - c_t - \frac{\lambda_t}{2\lambda}c\right) N \wedge x_s \Big\}, \end{aligned} \tag{3.6}$$

from which, it is impossible for the Gauss map $G$ to be harmonic, i.e., $\Delta G = 0$. Thus, we have

**Theorem 3.1.** *There do not exist surfaces of* $\mathbb{S}^3(1)$ *with harmonic Gauss map.*

*Remark* 3.2. Equation (3.6) and Theorem 3.1 can be obtained from [4, Lemma 3.2].

*Remark* 3.3. The torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ satisfies $\Delta G = \mu G$ for some non-zero real number $\mu$.

Now, let us consider a surface $M$ of $\mathbb{S}^3(1)$ with non-degenerate second fundamental form, i.e., $\kappa_1\kappa_2 \neq 0$. Let $x = x(s,t)$ be a surface patch of $M$ such that $x_s$ and $x_t$ are in the principal directions. Then, we have a non-degenerate metric $h$ on $M$ defined by (2.4). It is easy to derive

$$\nabla_{x_s} x_s = \frac{E_s}{2E} x_s - \frac{E_t}{2G} x_t, \tag{3.7}$$

$$\nabla_{x_s} x_t = \frac{E_t}{2E} x_s + \frac{G_s}{2G} x_t, \tag{3.8}$$

$$\nabla_{x_t} x_t = -\frac{G_s}{2E} x_s + \frac{G_t}{2G} x_t. \tag{3.9}$$

Without loss of generality, we may regard as $\kappa_1 > 0$. We put

$$h_{11} = \kappa_1 E = a^2, \quad h_{12} = h_{21} = 0, \quad h_{22} = \kappa_2 G = \varepsilon b^2 \tag{3.10}$$

for some positive functions $a$ and $b$, where $\varepsilon = \pm 1$ depending upon the signature of $h_{22}$. Then, we have the equations of Gauss

$$x_{ss} = \tilde{\nabla}_{x_s} x_{x_s} = \frac{E_s}{2E} x_s - \frac{E_t}{2G} x_t + a^2 N - Ex, \tag{3.11}$$

$$x_{st} = \tilde{\nabla}_{x_s} x_t = \frac{E_t}{2E} x_s + \frac{G_s}{2G} x_t, \tag{3.12}$$

$$x_{tt} = \tilde{\nabla}_{x_t} x_t = -\frac{G_s}{2E} x_s + \frac{G_t}{2G} x_t + \varepsilon b^2 N - Gx. \tag{3.13}$$

We then define the $II$-Laplace operator $\Delta^{II}$ with respect to the metric $h$ by

$$\begin{aligned} \Delta^{II} &= -\frac{1}{\sqrt{|\det h|}} \left\{ \frac{\partial}{\partial s} \left( \sqrt{|\det h|} \frac{1}{a^2} \frac{\partial}{\partial s} \right) + \frac{\partial}{\partial t} \left( \varepsilon \sqrt{|\det h|} \frac{1}{b^2} \frac{\partial}{\partial t} \right) \right\} \\ &= -\frac{1}{ab} \left\{ \frac{\partial}{\partial s} \left( \frac{b}{a} \frac{\partial}{\partial s} \right) + \varepsilon \frac{\partial}{\partial t} \left( \frac{a}{b} \frac{\partial}{\partial t} \right) \right\}. \end{aligned} \tag{3.14}$$

If we put $f = b/a$, then (3.14) can be written as

$$\Delta^{II} = -\frac{1}{ab} \left\{ f_s \frac{\partial}{\partial s} + f \frac{\partial^2}{\partial s^2} + \varepsilon \left( \frac{1}{f} \right)_t \frac{\partial}{\partial t} + \varepsilon \left( \frac{1}{f} \right) \frac{\partial^2}{\partial t^2} \right\}. \tag{3.15}$$

Using (3.11), (3.12) and (3.13), we have

**Lemma 3.4.** *Let $M$ be a surface of $\mathbb{S}^3(1)$ with non-degenerate second fundamental form. Then, we have*

$$\begin{aligned}\Delta^{II}x = -\frac{1}{ab}\Bigg\{&\left(f_s + f\frac{E_s}{2E} - \frac{\varepsilon}{f}\frac{G_s}{2E}\right)x_s + \left(-\frac{fE_t}{2G} + \left(\frac{\varepsilon}{f}\right)_t + \frac{\varepsilon}{f}\frac{G_t}{2G}\right)x_t \\ &+ 2abN - \left(fE + \frac{\varepsilon G}{f}\right)x\Bigg\}.\end{aligned} \tag{3.16}$$

We then have immediately from Lemma 3.4

**Proposition 3.5.** *There do not exist surfaces of $\mathbb{S}^3(1)$ with non-degenerate second fundamental form satisfying $\Delta^{II}x = \lambda_1 x$ for some function $\lambda_1$.*

## 4. Some characterization of Clifford torus

First of all, we prove

**Lemma 4.1.** *Let $M$ be a surface in $\mathbb{S}^3(1)$ with non-degenerate second fundamental form. If $\Delta^{II}x$ is parallel to the unit normal vector field or $\Delta^{II}N$ is parallel to $x$, the induced $II$-metric $h$ is time-like.*

*Proof.* Without loss of generality, we may assume $\kappa_1 > 0$ as in the previous section.

Suppose that $\Delta^{II}x$, the $II$-Laplacian of the immersion $x$, is parallel to the unit normal vector field $N$, that is,

$$\Delta^{II}x = \lambda N \tag{4.1}$$

for some function $\lambda$. By straightforward computation, we have

$$\begin{aligned}\Delta^{II}x = -\frac{1}{ab}\Bigg\{&\left(f_s + f\frac{E_s}{2E} - \varepsilon\frac{1}{f}\frac{G_s}{2E}\right)x_s + \left(-f\frac{E_t}{2G} + \left(\frac{\varepsilon}{f}\right)_t + \frac{\varepsilon}{f}\frac{G_t}{2G}\right)x_t \\ &+ 2abN - \left(fE + \frac{\varepsilon G}{f}\right)x\Bigg\},\end{aligned} \tag{4.2}$$

where $a$ and $b$ are some positive functions given in the previous section by $a^2 = \kappa_1 E$, $\varepsilon b^2 = \kappa_2 G$ and $f = b/a$.

Since $\Delta^{II}x$ is parallel to the unit normal vector field $N$, we have from (4.2)

$$f_s + f\frac{E_s}{2E} - \frac{\varepsilon}{f}\frac{G_s}{2E} = 0, \tag{4.3}$$

$$-f\frac{E_t}{2G} + \left(\frac{\varepsilon}{f}\right)_t + \frac{\varepsilon}{f}\frac{G_t}{2G} = 0, \tag{4.4}$$

$$fE + \varepsilon\frac{G}{f} = 0. \tag{4.5}$$

Since $f > 0$, $E > 0$ and $G > 0$, (4.5) implies $\varepsilon = -1$.

Suppose that $\Delta^{II} N$, the $II$-Laplacian of the unit normal vector field $N$ of $M$ in $\mathbb{S}^3(1)$, is parallel to the position vector $x$ which is identified with the immersion $x$, that is,

$$\Delta^{II} N = \mu x \tag{4.6}$$

for some function $\mu$. Using (3.7)–(3.9) and (3.15), we get

$$\begin{aligned} \Delta^{II} N = -\frac{1}{ab}\bigg\{ & \left(-(f\kappa_1)_s + f\kappa_1 \frac{E_s}{2E} - \varepsilon \frac{1}{f}\frac{G_s}{2E}\kappa_2\right) x_s \\ & + \left( f\kappa_1 \frac{E_t}{2G} - \left(\frac{\varepsilon\kappa_2}{f}\right)_t - \frac{\varepsilon\kappa_2}{f}\frac{G_t}{2G}\right) x_t \\ & + 2ab(\kappa_1+\kappa_2)N - \left(f\kappa_1 E + \frac{\varepsilon}{f}\kappa_2 G\right) x \bigg\}. \end{aligned} \tag{4.7}$$

Since $\Delta^{II} N \wedge x = 0$, (4.7) implies $\kappa_1 + \kappa_2 = 0$ and $\varepsilon = -1$. $\square$

We now prove

**Theorem 4.2.** *Let $M$ be a compact surface in $\mathbb{S}^3(1)$ with non-degenerate second fundamental form. Then, the following are equivalent:*

(1) *$\Delta^{II} x$ is parallel to the unit normal vector field $N$.*

(2) *$\Delta^{II} N$ is parallel to the position vector field $x$.*

(3) *$M$ is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

(4) *$M$ is flat and of 1-type.*

*Proof.* Due to [3], (3) and (4) are equivalent. Now, we show that (1), (2) and (3) are equivalent.

First, we show (1) and (3) are equivalent. Suppose that $\Delta^{II} x$ is parallel to $N$. According to Lemma 4.1, the induced $II$-metric $h$ is time-like, that is, $\varepsilon = -1$ and thus (4.5) yields

$$G = f^2 E. \tag{4.8}$$

Differentiating (4.8) with respect to $s$ and putting it into (4.3), we get

$$f_s + f\frac{E_s}{2E} = 0,$$

which implies

$$f = C_1(t)/\sqrt{E} \tag{4.9}$$

for some positive function $C_1(t)$ of $t$. Together with (4.8) and (4.9), we see that

$$G = C_1^2(t), \tag{4.10}$$

that is, $G$ is a function of $t$. If we solve (4.5) and make use of (4.9), we have

$$f = C_2(s)\sqrt{EG}$$

for some positive function $C_2(s)$ of $s$. Together this with (4.8), we see that

$$E = 1/C_2(s) \tag{4.11}$$

is a function of $s$. Then, (3.12) implies

$$x_{st} = 0.$$

So, the tangent vector field $x_s$ depends only on $s$. By (3.11) and (4.11), $\kappa_1 N - x = \big(x_{ss} - \frac{E_s}{2E}x_s\big)/E$ depends only on the variable $s$. Thus, we have

$$0 = (\kappa_1 N - x)_t = (\kappa_1)_t N - (1 + \kappa_1\kappa_2)x_t = (\kappa_1)_t N - K x_t.$$

Also, the tangent vector field $x_t$ depends only on the variable $t$. Similarly as before, (3.13) and (4.10) yield

$$0 = (\kappa_2 N - x)_s = (\kappa_2)_s N - (1 + \kappa_1\kappa_2)x_s = (\kappa_2)_s N - K x_s.$$

Therefore, we have that $(\kappa_1)_t = (\kappa_2)_s = 0$ and $K = 0$. Differentiating $K = 0$ with respect to $s$ and $t$, and using the fact that the second fundamental form of $M$ in $\mathbb{S}^3(1)$ is non-degenerate, we see that the principal curvatures $\kappa_1$ and $\kappa_2$ are constant. Thus, $M$ is isoparametric. Since $M$ is compact, $M$ is a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$.

Then, the immersion $x$ of $M$ into $\mathbb{E}^4$ can be written as

$$x(s,t) = (r_1 \cos s, r_1 \sin s, r_2 \cos t, r_2 \sin t). \tag{4.12}$$

In a natural manner, we may choose a unit normal vector field $N$ in $\mathbb{S}^3(1)$ by

$$N = (-r_2 \cos s, -r_2 \sin s, r_1 \cos t, r_1 \sin t), \tag{4.13}$$

from which, the shape operator $S$ and $II$-metric $h$ of $M$ in $\mathbb{S}^3(1)$ are respectively given by

$$S = \begin{pmatrix} r_2/r_1 & 0 \\ 0 & -r_1/r_2 \end{pmatrix} \tag{4.14}$$

and

$$h = \begin{pmatrix} r_1 r_2 & 0 \\ 0 & -r_1 r_2 \end{pmatrix}. \tag{4.15}$$

Thus, we have

$$\Delta^{II} x = \left( \frac{1}{r_2} \cos s, \frac{1}{r_2} \sin s, -\frac{1}{r_1} \cos t, -\frac{1}{r_1} \sin t \right).$$

Since $\Delta^{II} x$ and $N$ are parallel, it is easy to see that $r_1 = r_2$. Hence, $M$ is represented as the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ which is minimal in $\mathbb{S}^3(1)$.

It is straightforward that $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ satisfies

$$\Delta^{II} x = -2N.$$

Now, suppose that

$$\Delta^{II} N = \mu x \tag{4.16}$$

for some function $\mu$. By Lemma 4.1, we have $\varepsilon = -1$ and $\kappa_1 + \kappa_2 = 0$. From (4.7) together with (4.16), we get

$$-(f\kappa_1)_s + f\kappa_1 \frac{E_s}{2E} - \frac{1}{f} \frac{G_s}{2E} \kappa_1 = 0, \tag{4.17}$$

$$f\kappa_1 \frac{E_t}{2G} - \left( \frac{\kappa_1}{f} \right)_t - \frac{\kappa_1}{f} \frac{G_t}{2G} = 0 \tag{4.18}$$

because of $\kappa_1 + \kappa_2 = 0$. The solution of (4.17) is given by

$$f\kappa_1 = \widetilde{C}_0(t) \sqrt{\frac{E}{G}} \tag{4.19}$$

for some positive function $\widetilde{C}_0(t)$ of $t$, or, equivalently

$$G^2 \kappa_1^2 = C_0(t) E^2, \tag{4.20}$$

where $C_0(t) = \widetilde{C}_0(t)^2$. From (4.18), we have $\kappa_1/f = \widetilde{C}_1(s)\sqrt{E/G}$ for some function $\widetilde{C}_1(s)$ of $s$. If we use $f = b/a$, we have $\kappa_1^2 = C_1(s)$, where $C_1(s) = \widetilde{C}_1(s)^2$. Therefore, $\kappa_1$ is a positive function of $s$ only. Then, (4.18) reduces to

$$f \frac{E_t}{2G} - \left( \frac{1}{f} \right)_t - \frac{1}{f} \frac{G_t}{2G} = 0,$$

or,

$$\frac{f_t}{f} - \frac{G_t}{2G} + \frac{E_t}{2E} = 0, \tag{4.21}$$

which yields

$$f^2 = \frac{b^2}{a^2} = C_3(t)\frac{E}{G} \tag{4.22}$$

for some positive function $C_3(t)$ of $t$. Together with (4.20) and (4.21), we see that $\kappa_1$ is a constant. Therefore, $M$ is an isoparametric minimal surface in $\mathbb{S}^3(1)$ which is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.

Conversely, one can easily show that the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ satisfies $\Delta^{II} N = -2x$. □

## 5. Spheres and tori in $\mathbb{S}^3(1)$

In this section we consider a compact surface $M$ with non-degenerate second fundamental form in $\mathbb{S}^3(1)$. Then, we have a non-degenerate metric $h$ induced from the Riemannian metric and the shape operator $S$ by (2.3).

Firstly, we prove

**Theorem 5.1.** *Let $M$ be a compact surface of $\mathbb{S}^3(1)$ with non-degenerate second fundamental form. Then, $M$ is a sphere $\mathbb{S}^2(r)$ with $0 < r < 1$ if and only if $M$ satisfies $\Delta^{II} x = \lambda_2 \boldsymbol{H}$ for some non-zero constant $\lambda_2$, where $\boldsymbol{H}$ is the mean curvature vector field in $\mathbb{E}^4$ defined by $\boldsymbol{H} = HN - x$.*

*Proof.* Suppose that a compact surface $M$ of $\mathbb{S}^3(1)$ is a sphere $\mathbb{S}^2(r)$ with $0 < r < 1$. Without loss of generality, we may choose the unit normal vector field $N$ so that the principal curvature $\kappa = 1/r$ is positive. Then, we have $a^2 = \kappa E$, $b^2 = \kappa G$ and $f = b/a = \sqrt{G/E}$. Putting these into equation (3.16) in Lemma 3.4, we see that $\Delta^{II} x = -2N + (2/\kappa)x = \lambda_2(\kappa N - x) = \lambda_2 \boldsymbol{H}$ with $\lambda_2 = -2/\kappa$.

Suppose that $M$ satisfies $\Delta^{II} x = \lambda_2 \boldsymbol{H}$ for some non-zero constant $\lambda_2$. It follows from (3.16) that the mean curvature $H = -2/\lambda_2$ is constant and $K_{\text{ext}} = H^2 \neq 0$. Therefore, the Gauss curvature $K$ of $M$ satisfies $K = 1 + K_{\text{ext}} > 1$ and thus $M$ is totally umbilic. Since the second fundamental form of $M$ is non-degenerate, the principal curvature is a non-zero constant. Thus, $M$ is a sphere $\mathbb{S}^2(r)$ with $0 < r < 1$. □

We now define the $II$-Gauss map $\widetilde{G}$ as follows

$$\widetilde{G} = \frac{x_s \wedge x_t}{\sqrt{|\det h|}} = \frac{x_s \wedge x_t}{ab}, \tag{5.1}$$

where $\kappa_1 E = a^2$, $\kappa_2 G = \varepsilon b^2$, $\kappa_1 > 0$ and $\kappa_2$ are the principal curvatures, $E = \langle x_s, x_s \rangle$,

$G=\langle x_t,x_t\rangle$, $a>0$, $b>0$ and $\varepsilon=\pm1$. Then, (3.14) implies

$$
\begin{aligned}
\Delta^{II}\widetilde{G} = \Bigg[ & f\left\{g_s+g\left(\frac{E_s}{2E}+\frac{G_s}{2G}\right)-\frac{a^3}{bE}-\frac{E}{ab}\right\}+f_sg+\left(\frac{\varepsilon}{f}\right)_t q \\
& +\frac{\varepsilon}{f}\left\{q_t-\frac{G}{ab}+q\left(\frac{E_t}{2E}+\frac{G_t}{2G}\right)-\frac{b^3}{aG}\right\}\Bigg] x_s\wedge x_t \\
& +\left\{\frac{f_s}{f}+f\left(a^2g+\left(\frac{1}{f}\right)_s\right)\right\}N\wedge x_t \\
& +\left\{-f_s\frac{E}{ab}-f\left(gE+\left(\frac{E}{ab}\right)_s+\frac{E}{ab}\frac{G_s}{2G}\right)+\frac{\varepsilon}{f}\frac{G}{ab}\frac{G_s}{2G}\right\}x\wedge x_t \\
& +\left\{\frac{E_t}{2E}-f\left(\frac{1}{f}\right)_t-\frac{1}{f}\left(qb^2+f_t+f\frac{E_t}{2E}\right)\right\}N\wedge x_s \\
& +\left\{-f\frac{E}{ab}\frac{E_t}{2E}+\left(\frac{\varepsilon}{f}\right)_t\frac{G}{ab}+\frac{\varepsilon}{f}\left(qG+\left(\frac{G}{ab}\right)_t+\frac{G}{ab}\frac{E_t}{2E}\right)\right\}x\wedge x_s,
\end{aligned}
\tag{5.2}
$$

where $g=(1/ab)_s+(1/2ab)(E_s/E+G_s/G)$ and $q=(1/ab)_t+(1/2ab)(E_t/E+G_t/G)$.

We now prove

**Lemma 5.2.** *Let $M$ be a compact surface in $\mathbb{S}^3(1)$ with non-degenerate second fundamental form. If $M$ admits $II$-harmonic $II$-Gauss map, then $M$ is flat.*

*Proof.* Suppose that the $II$-Gauss map $\widetilde{G}$ is $II$-harmonic, that is, $\Delta^{II}\widetilde{G}=0$. Since the vectors $x_s\wedge x_t$, $N\wedge x_t$, $x\wedge x_t$, $N\wedge x_s$ and $x\wedge x_s$ are linearly independent, we have from (5.2) that

$$
\begin{aligned}
& f\left\{g_s+g\left(\frac{E_s}{2E}+\frac{G_s}{2G}\right)-\frac{a^3}{bE}-\frac{E}{ab}\right\}+f_sg \\
& \quad +\left(\frac{\varepsilon}{f}\right)_t q+\frac{\varepsilon}{f}\left\{q_t-\frac{G}{ab}+q\left(\frac{E_t}{2E}+\frac{G_t}{2G}\right)-\frac{b^3}{aG}\right\}=0,
\end{aligned}
\tag{5.3}
$$

$$\frac{f_s}{f}+f\left(a^2g+\left(\frac{1}{f}\right)_s\right)=0, \tag{5.4}$$

$$-f_s\frac{E}{ab}-f\left(gE+\left(\frac{E}{ab}\right)_s+\frac{E}{ab}\frac{G_s}{2G}\right)+\frac{\varepsilon}{f}\frac{G}{ab}\frac{G_s}{2G}=0, \tag{5.5}$$

$$\frac{E_t}{2E}-f\left(\frac{1}{f}\right)_t-\frac{1}{f}\left(qb^2+f_t+f\frac{E_t}{2E}\right)=0, \tag{5.6}$$

$$-f\frac{E}{ab}\frac{E_t}{2E}+\left(\frac{\varepsilon}{f}\right)_t\frac{G}{ab}+\frac{\varepsilon}{f}\left(qG+\left(\frac{G}{ab}\right)_t+\frac{G}{ab}\frac{E_t}{2E}\right)=0. \tag{5.7}$$

It follows from (5.4) that $a^2g=0$. Since $a$ is a positive function, we get

$$g=0. \tag{5.8}$$

Equation (5.6) yields $b^2q=0$ and thus

$$q=0. \tag{5.9}$$

Therefore, $\sqrt{EG}/ab$ is constant and the Gauss curvature $K$ of $M$ in $\mathbb{E}^4$ is a constant given by $K = 1 + \kappa_1\kappa_2 = 1 + \varepsilon\frac{a^2b^2}{EG}$.

On the other hand, equation (5.3) with $g = q = 0$ implies

$$-\frac{a^2}{E} - \frac{E}{a^2} - \varepsilon\left(\frac{b^2}{G} + \frac{G}{b^2}\right) = 0, \tag{5.10}$$

from which, we get $\varepsilon = -1$.

Equation (5.10) with $\varepsilon = -1$ gives

$$(\kappa_1 + \kappa_2)(1 + \kappa_1\kappa_2) = K(\kappa_1 + \kappa_2) = 0. \tag{5.11}$$

Suppose that $K \neq 0$. Then, (5.11) implies that $M$ is minimal and $x\colon M \to \mathbb{E}^4$ is of 1-type immersion. Since the Gauss curvature $K$ is constant, the principal curvatures $\kappa_1$ and $\kappa_2$ are constant. Thus, $M$ is isoparametric in $\mathbb{S}^3(1)$ and $M$ is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$, a contradiction. Therefore, $K = 0$ and $M$ is flat. □

Together with Lemma 5.2, a compact and finite-type surface $M$ with non-degenerate second fundamental form and $II$-harmonic $II$-Gauss map is a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ (see [3]).

Conversely, we can easily verify that the $II$-Gauss map of a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ is $II$-harmonic.

Hence, we have

**Theorem 5.3.** *Let $M$ be a compact surface with non-degenerate second fundamental form in $\mathbb{S}^3(1)$. The following are equivalent:*

(1) *The $II$-Gauss map $\widetilde{G}$ is $II$-harmonic.*

(2) *$M$ is a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$.*

(3) *$M$ is flat and of finite-type.*

*Remark* 5.4. In Theorem 5.3, in case of 1-type, $M$ is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$. If $M$ is of 2-type, $M$ is $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ ($r_1 \neq 1/\sqrt{2}$).

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