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Orthogonality Preservers of JB*-triple-valued Functions

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Abstract. Let X, Y be locally compact Hausdorff spaces, and V, W be JB*-triples such that W is prime. Let $T\colon C_0(X,V) \to C_0(Y,W)$ be an orthogonality preserving linear operator with dense range. We show that T can be written as a weighted composition operator.

1. Introduction

A complex Banach space V is a *JB*-triple* if and only if its open unit ball is a symmetric manifold. This is equivalent to the condition that V admits a continuous triple product

$$\{\cdot,\cdot,\cdot\} : V^3 \to V,$$

which is symmetric and linear in the outer variables, conjugate linear in the middle variable, and satisfies

(i) $\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\}$,

(ii) $\|\exp it(a \square a)\| = 1$ for all $t \in \mathbb{R}$,

(iii) $a \square a$ has a nonnegative spectrum,

(iv) $\|a \square a\| = \|a\|^2$,

for $a,b,c,d,e \in V$. Here, the *box operator* $a \square b\colon V \to V$ is defined by

$$(a \square b)(\cdot) = \{a,b,\cdot\}.$$

See the monographs of Chu [5] and Upmeier [12] for details.

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A C*-algebra $\mathcal{A}$ is a JB*-triple when it is equipped with the triple product

$$\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a) \quad (a,b,c \in \mathcal{A}).$$

Other examples of JB*-triples are all Hilbert spaces, the Banach spaces of bounded linear operators between Hilbert spaces, and some exceptional Jordan algebras.

Let V be a JB*-triple, we call two elements a, b in V *orthogonal* if $a \square b = 0$ (see [8, p. 194]). Note that $a \square b = 0$ if and only if $b \square a = 0$ (see [5, Lemma 1.2.32]). Two elements a, b in a C*-algebra $\mathcal{A}$ are orthogonal exactly when $ab^* = b^*a = 0$ (see [2, p. 221]).

Following Wolff [13] and Wong [14, 15], Burgos, Fernández-Polo, Gacés, Moreno and Peralta [2] studied orthogonality preserving linear operators between C*-algebras, JB*-algebras and JB*-triples. Burgos, Garcés and Peralta [4] studied the automatic continuity of biorthogonality linear preservers between JB*-triples, while Tsai [11] and Leung, Tsai and Wong [10] studied the cases of zero product preservers of CCR C*-algebras and W*-algebras, respectively. Roughly speaking, various zero products and orthogonality preservers arise from Jordan or algebra (*-)homomorphisms followed by multipliers.

In this paper, we will study those linear maps T from $C_0(X,V)$ into $C_0(X,W)$ preserving (JB*-triple) orthogonality, that is,

$$f(x) \square g(x) = 0 \text{ for all } x \in X \quad \Longrightarrow \quad (Tf)(y) \square (Tg)(y) = 0 \text{ for all } y \in Y.$$

Here, X, Y are locally compact spaces and V, W are JB*-triples. We show that T can be written as a weighted composition operator if T has dense range and W is prime in Theorem 2.1. This supplements results in [9, 11]. We also establish the automatic continuity of such bijective preservers in Theorem 2.3 when V, W are prime von Neumann algebras.

2. Results

Recall that a *triple ideal* of V is a subspace J of V such that $\{a,b,c\} \in J$ whenever one of a, b and c belongs to J. A JB*-triple V is said to be *prime* if $J = \{0\}$ or $K = \{0\}$ whenever J, K are norm closed triple ideals of V with $J \cap K = \{0\}$ (see [1]).

Let X be a locally compact Hausdorff space. Denote by $X_\infty = X \cup \{\infty\}$ the one-point compactification of X. In case X is already compact, ∞ is an isolated point in X_∞. For a JB*-triple V, let

$$C_0(X,V) = \{f \in C(X,V) : f(\infty) = 0\}$$

be the Banach space of all continuous vector-valued functions from X into V vanishing at infinity.

Denote by $\operatorname{coz}(f) = \{x \in X : f(x) \neq 0\}$ the *cozero set* of an f in $C_0(X,V)$. We note that a subset U of X is the cozero set of a continuous function in $C_0(X,V)$ if and only if

U is σ-compact and open. For a σ-compact open subset U of X, denote by $C_0(U, V)$ the subspace of all f in $C_0(X, V)$ with $\mathrm{coz}(f) \subset U$.

Theorem 2.1. *Suppose that X, Y are locally compact spaces and V, W are JB*-triples with W being prime. Assume that T is a bounded linear orthogonality preserver from $C_0(X, V)$ into $C_0(Y, W)$ with dense range. Then there exist a continuous map $\sigma\colon Y \to X$ and, for each y in Y, a bounded linear orthogonal preserving map $J_y\colon V \to W$ with dense range such that*

$$Tf(y) = J_y f(\sigma(y)), \quad \forall\, f \in C_0(X, V),\ y \in Y.$$

Proof. For any y in Y, define

$$S_y = \{x \in X_\infty : \text{for each } \sigma\text{-compact open neighbourhood } U \text{ of } x,$$
$$\text{there exists an } f \in C_0(U, V) \text{ such that } Tf(y) \neq 0\}.$$

Claim 1. $S_y \neq \emptyset$.

Suppose on the contrary that for each $x \in X_\infty$, there is a σ-compact open neighbourhood U of x such that $(Tf)(y) = 0$ for all f in $C_0(U, V)$. Since X_∞ is compact, there exist σ-compact open neighbourhoods $U_0, U_1, \ldots, U_n$ of $x_0 = \infty, x_1, \ldots, x_n$, respectively, such that $X_\infty = U_0 \cup U_1 \cup \cdots \cup U_n$. Let $1 = f_0 + f_1 + \cdots + f_n$ be a continuous partition of unity such that $\mathrm{coz}(f_i) \subset U_i$ for each $i = 0, 1, 2, \ldots, n$. Then for each $f \in C_0(X, V)$, one can derive that

$$Tf(y) = T(f_0 f + f_1 f + \cdots + f_n f)(y) = 0,$$

since $\mathrm{coz}(f_i f) \subset U_i$ for $i = 0, 1, 2, \ldots, n$. This conflicts with the assumption that T has dense range.

Claim 2. S_y is a singleton.

Suppose on the contrary that two distinct points $x_1, x_2 \in S_y$. Let Z_1 and Z_2 be disjoint σ-compact open neighbourhoods of x_1 and x_2, respectively. We have

$$(Tf_1) \,\square\, (Tf_2) = 0, \quad \forall\, f_i \in C_0(Z_i, V),\ i = 1, 2. \tag{2.1}$$

Let I_1 be the closed subtriple of the JB*-triple W spanned by the set

$$\left\{T(f')(y) : f' \in C_0(X, V) \text{ with a compact support contained in } Z_1\right\}.$$

Let Z_0 be a nonempty open set in X with compact closure $\overline{Z_0} \subset Z_1$. Let g be in $C_0(X)$ with a compact support contained in Z_1 such that $g = 1$ on Z_0. For each f in $C_0(X, V)$, we have $\mathrm{coz}(f(1-g)) \cap Z_0 = \emptyset$. Therefore, $T(f(1-g))(y) \,\square\, T(f')(y) = 0$, and thus,

$$\begin{aligned} &(Tf)(y) \,\square\, (Tf')(y) = T(fg)(y) \,\square\, (Tf')(y), \\ &(Tf')(y) \,\square\, (Tf)(y) = (Tf')(y) \,\square\, T(fg)(y), \quad \forall\, f' \in C_0(Z_0, V). \end{aligned} \tag{2.2}$$

Let $f' \in C_0(X,V)$ with compact support $\overline{Z_0} \subset Z_1$ and $g \in C_0(X)$ as above. For any h_1, h_2 in $C_0(X,V)$, it follows from (2.2) that

$$\begin{aligned} &\{T(h_1)(y), T(f')(y), T(h_2)(y)\} = T(h_1)(y) \,\square\, T(f')(y)(T(h_2)(y)) \\ &= T(gh_1)(y) \,\square\, T(f')(y)(T(h_2)(y)) = \{T(gh_1)(y), T(f')(y), T(h_2)(y)\} \\ &= \{T(h_2)(y), T(f')(y), T(gh_1)(y)\} = T(h_2)(y) \,\square\, T(f')(y)(T(gh_1)(y)) \\ &= T(gh_2)(y) \,\square\, T(f')(y)(T(gh_1)(y)) = \{T(gh_2)(y), T(f')(y), T(gh_1)(y)\} \in I_1. \end{aligned}$$

Let $g' \in C_0(X)$ with a compact support contained in Z_1 such that $g' = 1$ on the compact support of g. A similar argument gives

$$\begin{aligned} &\{T(f')(y), T(h_1)(y), T(h_2)(y)\} = T(f')(y) \,\square\, T(h_1)(y)(T(h_2)(y)) \\ &= T(f')(y) \,\square\, T(gh_1)(y)(T(h_2)(y)) = \{T(f')(y), T(gh_1)(y), T(h_2)(y)\} \\ &= \{T(h_2)(y), T(gh_1)(y), T(f')(y)\} = T(h_2)(y) \,\square\, T(gh_1)(y)(T(f')(y)) \\ &= T(g'h_2)(y) \,\square\, T(gh_1)(y)(T(f')(y)) = \{T(g'h_2)(y), T(gh_1)(y), T(f')(y)\} \in I_1. \end{aligned}$$

So I_1 is a triple ideal of V.

Similarly, we get another triple ideal of W, which is the closed span of

$$\{T(f')(y) : f' \in C_0(X,V) \text{ with a compact support contained in } Z_2\}.$$

By (2.1), we have $I_1 \,\square\, I_2 = 0$, and hence $I_1 \cap I_2 = \{0\}$. By the primeness assumption on W, we have $I_1 = \{0\}$ or $I_2 = \{0\}$. But both possibilities conflict with the construction of S_y.

Claim 3. If $S_y = \{x\}$, then we have that

$$f(x) = 0 \quad \Longrightarrow \quad Tf(y) = 0.$$

By Urysohn's Lemma and the boundedness of T, we can assume that f vanishes in a neighborhood of x. Thus, $x \notin \overline{\operatorname{coz}(f)}$, which is compact in X_∞. For any x' in $\overline{\operatorname{coz}(f)}$, and thus $x' \notin S_y$, there is a σ-compact open neighborhood U' of x' such that $(Tk)(y) = 0$ for each $k \in C_0(U',V)$. By a compactness argument similar to the one proving Claim 1, one can derive that $Tf(y) = 0$.

It follows from Claim 3 and the assumption on the denseness of the range of T that $S_y \neq \{\infty\}$ for all y in Y. Denote by $\sigma(y) = x$ if $S_y = \{x\}$ (as in [7], x is called the *support point* of $\delta_y \circ T$). Consequently, there is a linear map $J_y \colon V \to W$ such that

$$Tf(y) = J_y f(\sigma(y)), \quad \forall\, y \in Y,\ f \in C_0(X,V).$$

For any $y \in Y$ and any a, b in V with $a \,\square\, b = 0$. Choose a function $\widetilde{g} \in C_0(X)$ such that $\widetilde{g}(\sigma(y)) = 1$. Then

$$T(\widetilde{g}a)(y) = J_y(a)\widetilde{g}(\sigma(y)) = J_y(a)$$

and

$$T(\widetilde{g}b)(y) = J_y(b)\widetilde{g}(\sigma(y)) = J_y(b).$$

Since $(\widetilde{g}a)(x) \,\square\, (\widetilde{g}b)(x) = 0$ for all x in X, one can derive that

$$T(\widetilde{g}a)(y) \,\square\, T(\widetilde{g}b)(y) = 0, \quad \forall\, y \in Y.$$

Therefore, $J_y(a) \,\square\, J_y(b) = 0$, and hence J_y is orthogonality preserving.

The boundedness and dense range properties of each J_y follow easily from those properties of T. To complete the proof, we show that $\sigma\colon Y \to X$ is continuous. Suppose on the contrary that $\{y_\lambda\}$ is a net converging to y_0 in Y such that $\sigma(y_\lambda) \to x_0 \neq \sigma(y_0)$ in X_∞. Then there exist disjoint neighborhoods U_1 and U_2 of x_0 and $\sigma(y_0)$ in X_∞, respectively, and an index λ_0 such that $\sigma(y_\lambda) \in U_1$ for all $\lambda \geq \lambda_0$. Let $f \in C_0(X,V)$ such that $\operatorname{coz}(f) \subset U_2$ and $(Tf)(y_0) \neq 0$. Because $\sigma(y_\lambda) \notin \overline{\operatorname{coz}(f)}$ for all $\lambda \geq \lambda_0$, it follows from Claim 3 that $(Tf)(y_\lambda) = 0$ for all $\lambda \geq \lambda_0$. This contradicts to the continuity of Tf at y_0. □

Without the primeness assumption, we cannot guarantee the conclusion of Theorem 2.1.

Example 2.2. Let $X = \{-1, 1\}$ and $Y = \{0\}$ in discrete topology. Let $V = \mathbb{C}$ and $W = \mathbb{C} \oplus_\infty \mathbb{C}$ be the one and two dimensional W*-algebras, respectively. Let $T\colon C(X,V) \to C(Y,W)$ be defined by

$$Tf(0) = f(-1) \oplus f(1), \quad \forall\, f \in C(X,V).$$

Then both T and T^{-1} are isometric orthogonality preserving linear maps. However, $S_0 = \{-1, 1\}$, in the notations of the proof of Theorem 2.1, and T is not a weighted composition operator. Note that W is not prime while V is.

Utilizing Theorem 2.1, when V, W are prime C*-algebras, we get the following supplement to those results in [9, 11] dealing with various zero product preserving linear maps. We note that for a C*-algebra V, by [5, Remark 3.1.17], the primeness of V as a C*-algebra coincides with the primeness of V as a JB*-triple.

Theorem 2.3. *Suppose that X, Y are locally compact spaces and V, W are prime C*-algebras. Let $T\colon C_0(X,V) \to C_0(X,W)$ be a bijective linear map preserving orthogonality in two ways, i.e.,*

$$\begin{aligned} & f(x)^* g(x) = f(x) g(x)^* = 0 \quad \text{on } X \\ \iff \quad & Tf(y)^* Tg(y) = Tf(y) Tg(y)^* = 0 \quad \text{on } Y. \end{aligned}$$

Assume any one of the following conditions holds.

(1) *V is a properly infinite unital C*-algebra.*

(2) *V, W are von Neumann algebras.*

Then T is automatically continuous.

Proof. Recall that in a C*-algebra, two elements a, b are orthogonal exactly when $a \,\square\, b = 0$. As in proving Theorem 2.1 (up to Claim 2 without assuming the boundedness of T), one can find a map $\sigma\colon Y \to X$ such that $Tf(y) = 0$ whenever f vanishes in a neighborhood of $\sigma(y)$ for any y in Y. Following the proof of [6, Theorem 2.3], which is also valid when X, Y are locally compact, one can derive that $f(\sigma(y)) = 0$ implies $Tf(y) = 0$, when both T and T^{-1} preserve orthogonality. It is then routine to see that σ is a homeomorphism. Furthermore, there exists a bijective linear map $J_y\colon V \to W$ preserving orthogonality in both directions for each y in Y such that

$$Tf(y) = J_y f(\sigma(y)), \quad \forall\, y \in Y,\ f \in C_0(X,V).$$

It follows from [3, Corollary 16 and Theorem 19] that J_y is continuous (and assumes a canonical form) under the assumptions in either case.

Let $\{f_n\}$ be a sequence of functions converging to f_0 in $C_0(X,V)$. Suppose that Tf_n converges to g_0 in $C_0(Y,W)$. Consequently, $Tf_n(y)$ converges to $g_0(y)$ in W for each y in Y. That is, $J_y f_n(\sigma(y)) \to g_0(y)$ for any y in Y. Since $f_n(\sigma(y)) \to f_0(\sigma(y))$ and J_y is continuous, $J_y f_n(\sigma(y)) \to J_y f_0(\sigma(y)) = Tf_0(y)$. This ensures that $Tf_0 = g_0$. Therefore, T has a closed graph, and hence T is continuous. □

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