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# On the Vanishing of Pontryagin Classes of Para-Sasakian Space Forms

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Abstract. In this note we prove that all the Pontryagin classes of para-Sasakian space forms vanish.

## 1. Introduction and preliminaries

### 1.1. Introduction

The Chern classes of an almost contact manifold $M$ are the Chern classes of its contact distribution (horizontal vector bundle) $H$, and they can be also defined as the Chern classes of the cylinder manifold $M \times \mathbb{R}$ endowed with the canonical almost complex structure induced by the almost contact structure of $M$. For the Sasakian manifolds with constant $\varphi$-sectinal curvature, also called Sasakian space forms, Oproiu proved in [11] that the Chern classes of the horizontal vector bundle are trivial, and consequently, the Pontryagin classes of Sasakian space forms are trivial.

The aim of the present paper is to solve a similar problem in the context of para-contact geometry. More exactly, using the same arguments as in [11], and having in mind some models and constructions often used in para-contact geometry, see for instance [1,3,5,6,9,13,15,16], and also taking into account the study of the Pontryagin classes of a vector bundle endowed with a fiberwise para-complex structure [2], we prove that all the Pontryagin classes of para-Sasakian space forms are trivial.

The structure of the paper is as follows. In the Subsection 1.2, we briefly recall some elementary notions concerning the para-contact and para-Sasakian structures (including the para-Sasakian space forms), which will be used in this paper. In Section 2, we introduce an induced connection on the horizontal bundle of a para-Sasakian manifold, which has the property that is parallel with respect to the natural para-complex structure induced on the horizontal bundle from the almost para-contact structure. Also, in the case of para-Sasakian space forms, we establish a relation for the curvature of this connection, which will be used in the computation of Pontryagin forms. In Section 3, we briefly recall

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the construction of Pontryagin forms of a vector bundle endowed with a fiberwise para-complex structure, see [2], and we apply this construction for the case of horizontal bundle of a para-Sasakian space form endowed with the natural para-complex structure induced from the almost para-contact structure. Next, using the mathematical induction, we prove that all the Pontryagin forms of a para-Sasakian space form are exact, which implies the main result of this note.

## 1.2. Preliminaries

The notion of almost para-contact structure was introduced by Sato. According to his definition [12], an *almost paracontact structure* $(\varphi, \xi, \eta)$ on an odd-dimensional manifold $M$ consists of a $(1,1)$-tensor field $\varphi$, called the *structure endomorphism*, a vector field $\xi$, called the *characteristic vector field* and a 1-form $\eta$, called the *para-contact form*, which satisfy the following conditions:

$$\varphi^2 = \mathrm{Id} - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{1.1}$$

where Id is the identity endomorphism.

Moreover, if $g$ is a pseudo-Riemannian metric on $M$ of signature $(n+1, n)$ such that $g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y)$, for any $X, Y \in \Gamma(TM)$, we shall call $(\varphi, \xi, \eta, g)$ an *almost para-contact metric structure* on $M$. We notice that from the definition it follows

$$\varphi\xi = 0,\ \eta \circ \varphi = 0,\ \eta = i_\xi g,\ g(\xi,\xi) = 1 \text{ and } g(\varphi X, Y) = -g(X, \varphi Y),\ \forall\, X, Y \in \Gamma(TM).$$

For a list of examples of almost para-contact metric structures we refer for instance to [7, 8, 15].

We also notice that $\varphi$ induces on the $2n$-dimensional distribution $H := \ker \eta$ an almost para-complex structure $I = \varphi|_H$, i.e., $I \circ I = \mathrm{Id}|_H$, and the eigensubbundles $H^+$, $H^-$ corresponding to the eigenvalues $1, -1$ of $I$ respectively, have equal dimension $n$. Thus, we have $H = H^+ \oplus H^-$. Moreover, the canonical distribution $H$ is $\varphi$-invariant since $H = \mathrm{Im}\,\varphi$, and taking into account that $\xi$ is orthogonal to $H$, the tangent bundle splits orthogonally:

$$TM = H \oplus \langle \xi \rangle, \tag{1.2}$$

where $\langle \xi \rangle$ is the line distribution spanned by $\xi$. Then

$$v = \eta \otimes \xi, \quad h = \mathrm{Id} - \eta \otimes \xi = \varphi^2 \tag{1.3}$$

are the projections on $V$ and $H$, respectively, and we have

$$h \circ \varphi = \varphi \circ h = \varphi, \quad v \circ \varphi = \varphi \circ v = 0. \tag{1.4}$$

The fundamental 2-form $\Phi(X,Y) = g(X,\varphi Y)$ is non-degenerate on the horizontal bundle $H$ and $\eta \wedge \Phi^n \neq 0$. Moreover, if $g(X,\varphi Y) = d\eta(X,Y)$ (here $d\eta(X,Y) = (1/2)(X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])))$, then $\eta$ is a para-contact form and the almost para-contact metric manifold $(M,\varphi,\xi,\eta,g)$ is a *para-contact metric manifold*. A *para-Sasakian manifold* is a normal para-contact metric manifold, where the normality is referred to the para-complex structure induced on the cylinder manifold $M \times \mathbb{R}$ (for details, see for instance [15]).

Next, we suppose that $M$ is para-Sasakian manifold. Let $g$ be the corresponding pseudo-Riemannian metric and $\nabla$ its Levi-Civita connection. Then, according to [15], we have

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X, \quad (\nabla_X \eta)(Y) = \Phi(X,Y), \tag{1.5}$$

$$\nabla_X \xi = -\varphi X, \quad (\nabla_X \Phi)(Y,Z) = g(X,Y)\eta(Z) - g(X,Z)\eta(Y). \tag{1.6}$$

Also, the para-holomorphic sectional curvature of $M$ is defined as usual in [16]. Moreover, if $M$ has constant para-holomorphic sectional curvature $c \in \mathbb{R}$, then the curvature tensor field $R$ of $\nabla$ is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c-3}{4}\,(g(Y,Z)X - g(X,Z)Y) + \frac{c+1}{4}\,(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\ & + \frac{c+1}{4}\,(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \Phi(Y,Z)\varphi X \\ & - \Phi(X,Z)\varphi Y - 2\Phi(X,Y)\varphi Z) \end{aligned} \tag{1.7}$$

for every $X,Y,Z \in \mathcal{X}(M)$. Such a para-Sasakian manifold (of constant para-holomorphic sectional curvature $c$) is also called a *para-Sasakian space form* and it is denoted by $M(c)$.

## 2. Curvature of an induced connection on the horizontal bundle of a para-Sasakian space form

The Levi-Civita connection $\nabla$ does not preserve the bundles $H$ or $\langle \xi \rangle$, and we denote by $D$ the linear connection induced by $\nabla$ on $H$, that is

$$D_X Z = h\nabla_X Z, \quad \forall\, Z \in \Gamma(H) = \{Z \in \mathcal{X}(M) : hZ = Z\}. \tag{2.1}$$

If we denote by $\varphi|_H$ (or simply by $\varphi$) the restriction of $\varphi$ to $H$, then using (1.5), we have

$$D_X \varphi = 0, \tag{2.2}$$

which says that $D$ is an almost para-complex connection on $(H, I = \varphi|_H)$.

Also, using (2.1), by direct computation, we have

$$R^D(X,Y)Z = h\nabla_X(h\nabla_Y Z) - h\nabla_Y(h\nabla_X Z) - h\nabla_{[X,Y]}Z,$$

and using $h = \mathrm{Id} - \eta \otimes \xi$ it follows

$$R^D(X,Y)Z = hR(X,Y)Z - h\nabla_X(\eta(\nabla_Y Z)\xi) + h\nabla_Y(\eta(\nabla_X Z)\xi).$$

On the other hand,

$$h\nabla_X(\eta(\nabla_Y Z)\xi) = h(\eta(\nabla_Y Z)\nabla_X \xi) = -h(\eta(\nabla_Y Z)\varphi X) = -\eta(\nabla_Y Z)\varphi X.$$

But, taking into account the second relation from (1.5) we have $\eta(\nabla_Y Z) = -\Phi(Y,Z)$ for $Z \in \Gamma(H)$. Hence, we obtain

$$R^D(X,Y)Z = hR(X,Y)Z - \Phi(Y,Z)\varphi X + \Phi(X,Z)\varphi Y, \quad Z \in \Gamma(H). \tag{2.3}$$

Furthermore, using (2.2) we obtain

$$R^D(X,Y)\varphi Z = \varphi R^D(X,Y)Z, \quad Z \in \Gamma(H). \tag{2.4}$$

Let us make now some local considerations. If $(U, x^1, \ldots, x^{2n+1})$ is a local map in $M$, we denote by $\varphi^i_j, \xi^i, \eta_i, h^i_j, g_{ij}, \Phi_{ij}, R^h_{ijk}$ and $R^{D\,h}_{ijk}$ the local components in this map corresponding to $\varphi, \xi, \eta, h, g, \Phi, R$ and $R^D$, respectively. Then, the relations (2.3) and (2.4) read in a local form as

$$R^{Dl}_{ijk} = R^m_{ijk} h^l_m - \Phi_{jk}\varphi^l_i + \Phi_{ik}\varphi^l_j \quad \text{and} \quad R^{Dh}_{ijk}\varphi^l_h = \varphi^h_k R^{Dl}_{ijh}. \tag{2.5}$$

We also consider the square matrix $\Theta = [\Theta^h_k]$ of dimension $2n+1$ with elements given by the following 2-forms:

$$\Theta^l_k = \frac{1}{2}\varphi^l_h R^{D\,h}_{ijk} dx^i \wedge dx^j. \tag{2.6}$$

Using $R^h_{ijk} dx^i \wedge dx^j \wedge dx^k = 0$, which is the local form of the first Bianchi identity verified by $R$, we obtain

$$\Theta^l_k \wedge dx^k = -2\Phi \wedge h^l_i dx^i = -2\Phi \wedge (dx^l - \xi^l \eta). \tag{2.7}$$

Indeed, using the first relation of (2.5), by direct computation we have

$$\begin{aligned}
\Theta^l_k \wedge dx^k &= \frac{1}{2}\left(\varphi^l_h h^h_m R^m_{ijk} - \varphi^l_h \Phi_{jk}\varphi^h_i + \varphi^l_h \Phi_{ik}\varphi^h_j\right) dx^i \wedge dx^j \wedge dx^k \\
&= -\frac{1}{2}\varphi^h_i \varphi^l_h \left(\Phi_{jk} dx^j \wedge dx^k\right) \wedge dx^i - \frac{1}{2}\varphi^h_j \varphi^l_h \left(\Phi_{ik} dx^i \wedge dx^k\right) \wedge dx^j \\
&= -2\varphi^h_i \varphi^l_h \Phi \wedge dx^i = -2\Phi \wedge h^l_i dx^i \\
&= -2\Phi \wedge \left(dx^l - \xi^l \eta\right).
\end{aligned}$$

Now, using the local expression of curvature tensor field of a para-Sasakian space form $M(c)$ given in (1.7), and also using (2.5), (2.6) and $\varphi^2 = h$, $h\varphi = \varphi h = \varphi$, $\varphi\xi = 0$, we obtain

$$\Theta^l_k = -\frac{c+1}{2}\Phi h^l_k + \left[\frac{c-3}{4}\left(g_{jk}\varphi^l_i + \Phi_{jk} h^l_i\right) - \frac{c+1}{4}\eta_j \eta_k \varphi^l_i\right] dx^i \wedge dx^j. \tag{2.8}$$

## 3. Pontryagin classes of para-Sasakian space forms

For the general theory of characteristic classes of vector bundles we refer for instance to [4,14]. In [2] are introduced and studied the $F$-characteristic classes of a vector bundle $\pi\colon E \to M$ endowed with a $F$-structure, that is $F \in \Gamma(M; L(E,E))$, where $L(E,E)$ denotes the vector bundle of the linear maps on each fiber of $E$. In particular, if $E$ is of even rank endowed with a para-complex structure operator $I$, and if $D$ is a para-complex connection on $E$ (that is $DI = 0$), then the Pontryagin ring of $E$ is generated by the de Rham cohomology classes of $M$ represented by the closed $4p$-forms

$$\operatorname{Tr}(IR)^{2p}, \quad p = 0, 1, 2, 3, \dots .$$

Also, in the case when $E$ is of even rank and it is endowed with an almost complex structure operator $J$ and with an almost complex connection $D$ on $E$ (that is $DJ = 0$), the Chern classes of $E$ are studied separately in [10].

In this section, following a similar argument from the study of Chern classes of Sasakian space forms [11], we apply the results from [2] in the study of the Pontryagin classes of para-Sasakian space forms. In our case, the para-complex vector bundle is $H$, the para-complex structure operator is $I = \varphi|_H$ and the curvature operator is $R^D(X,Y)h$. Hence, the Pontryagin ring of $(H, I = \varphi|H)$ is represented in the Rham cohomology of $M$ by the $4p$-forms

$$\operatorname{Tr}(\varphi R^D(\cdot,\cdot)h)^{2p}, \quad p = 0, 1, 2, 3, \dots .$$

Moreover, since $TM = H \oplus \langle\xi\rangle$ and $\langle\xi\rangle$ is trivial the Pontryagin classes of $TM$ and $H$ coincides.

**Lemma 3.1.** *The matrix $\Theta$ satisfies the following relations:*

$$\operatorname{Tr}\Theta = -[n(c+1) + c - 3]\Phi \tag{3.1}$$

*and*

$$\operatorname{Tr}\Theta^2 = -\frac{c+5}{2}\Phi \wedge \operatorname{Tr}\Theta - 2n(c+1)\Phi^2 = \frac{1}{2}[n(c+1)^2 + (c-3)(c+5)]\Phi^2. \tag{3.2}$$

*Proof.* From $h^k_k = 1 - \eta_k\xi^k$ and $\sum_{k=1}^{2n+1}\eta_k\xi^k = 1$ it follows that $\operatorname{Tr} h = \sum_{k=1}^{2n+1}(1 - \eta_k\xi^k) = 2n$. Now, from $\eta \circ \varphi = 0$ we have $\eta_j\eta_k\varphi^k_i dx^i \wedge dx^j = 0$. On the other hand, we have

$$g_{jk}\varphi^k_i dx^i \wedge dx^j = \Phi_{ji} dx^i \wedge dx^j = -2\Phi, \quad \text{and} \quad h^k_i \Phi_{jk} dx^i \wedge dx^j = -2\Phi.$$

Hence, the proof of (3.1) follows by taking the trace in (2.8).

In the sequel we will prove the relation (3.2). Firstly, taking into account (2.4), $\varphi h = h\varphi = \varphi$, $\varphi^2 = h$ and $\Theta = \varphi R^D(\cdot,\cdot)h$, it follows

$$\Theta^l_k \varphi^k_m = \varphi^l_k \Theta^k_m \quad \text{and} \quad \Theta^l_k h^k_m = h^l_k \Theta^k_m = \Theta^l_m. \tag{3.3}$$

Now, using the expression of $\Theta$ from (2.8), we have

$$\Theta^l_h\wedge\Theta^h_k=-\frac{c+1}{2}\Phi\wedge\Theta^l_k+\left[\frac{c-3}{4}\left(g_{jk}\Theta^l_h\varphi^h_i+\Phi_{jk}\Theta^l_hh^h_i\right)-\frac{c+1}{4}\eta_j\eta_k\Theta^l_h\varphi^h_i\right]\wedge dx^i\wedge dx^j.$$

Moreover, using (2.7) and the first relation from (3.3), we obtain

$$\begin{aligned}\Theta^l_h\wedge\Theta^h_k&=-\frac{c+1}{2}\Phi\wedge\Theta^l_k+\frac{c-3}{4}g_{jk}\varphi^l_h\Theta^h_i\wedge dx^i\wedge dx^j+\frac{c-3}{4}\Phi_{jk}\Theta^l_i\wedge dx^i\wedge dx^j\\&\quad-\frac{c+1}{4}\eta_j\eta_k\varphi^l_h\Theta^h_i\wedge dx^i\wedge dx^j\\&=-\frac{c+1}{2}\Phi\wedge\Theta^l_k+\frac{c-3}{4}g_{jk}\varphi^l_h\left(-2\Phi\wedge h^h_idx^i\right)\wedge dx^j\\&\quad+\frac{c-3}{4}\Phi_{jk}\left(-2\Phi\wedge h^l_idx^i\right)\wedge dx^j-\frac{c+1}{4}\eta_j\eta_k\varphi^l_h\left(-2\Phi\wedge h^h_idx^i\right)\wedge dx^j\\&=-\frac{c+1}{2}\Phi\wedge\Theta^l_k-2\Phi\wedge\left[\frac{c-3}{4}\left(g_{jk}\varphi^l_i+\Phi_{jk}h^l_i\right)-\frac{c+1}{4}\eta_j\eta_k\varphi^l_i\right]dx^i\wedge dx^j.\end{aligned}$$

Now, using again (2.8), we have

$$\Theta^l_h\wedge\Theta^h_k=-\frac{c+5}{2}\Phi\wedge\Theta^l_k-(c+1)\Phi^2h^l_k.$$

Finally, taking the trace in the above relation, we obtain (3.2). □

**Theorem 3.2.** *The Pontryagin classes of a para-Sasakian space form $M(c)$ are trivial.*

*Proof.* Following a similar argument used in the case of Sasakian space forms [11], we can prove that all Pontryagin forms of $H$ are powers of $\Phi$. More exactly, using the induction after $k\geq 2$, we obtain

$$\Theta^k=\Phi^{k-1}\wedge(a_k\Theta+b_k\Phi h),\quad\forall\, k\geq 2,$$

where

$$a_k=\sum_{p=0}^{\left[\frac{k-1}{2}\right]}\binom{k-p-1}{p}a^{k-2p-1}b^p,\quad b_k=ba_{k-1},\quad a_2=a=-\frac{c+5}{2},\quad b_2=b=-(c+1),$$

and $\binom{n}{m}$ denotes the binomial coefficients.

Then, the Pontryagin forms of $H$ are

$$\operatorname{Tr}\Theta^k=\Phi^k\wedge(a_k\operatorname{Tr}\Theta+2nb_k\Phi)=[2nb_k-a_k(n(c+1)+c-3)]\Phi^k.$$

Now, taking into account that $\Phi=d\eta$, we have $\Phi^k=d\left(\eta\wedge(d\eta)^{k-1}\right)$, that is, all the Pontryagin forms of $H$ are exact. Thus, the proof is finished. □

*Remark* 3.3. Using a similar argument to [11] (for the case of Sasakian space forms), the above result can be also obtained if we calculate the Pontryagin classes of $M\times\mathbb{R}$ endowed with the natural para-complex structure induced from the normal almost para-contact structure on $M$.

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