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Quasi-periodic Waves and Solitary Waves to a Generalized KdV-Caudrey-Dodd-Gibbon Equation from Fluid Dynamics

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Abstract. In this paper, a generalized KdV-Caudrey-Dodd-Gibbon (KdV-CDG) equation is investigated, which describes certain situations in the fluid mechanics, ocean dynamics and plasma physics. By using Bell polynomials, a lucid and systematic approach is proposed to systematically study its Hirota's bilinear form and N-soliton solution, respectively. Furthermore, based on the Riemann theta function, the one-quasi- and two-quasi-periodic wave solutions are also constructed. Finally, an asymptotic relation of the quasi-periodic wave solutions are strictly analyzed to reveal the relations between quasi-periodic wave solutions and soliton solutions.

1. Introduction

It is well known that nonlinear equations are more and more widely investigated to describe a lot of significant phenomena and dynamic processes in many fields, such as physics, chemistry, biology and mechanics, etc. Consequently, it is meaningful for us to investigate the exact solutions of the nonlinear equations. A good number of effective methods have been come up to obtain the wave solutions. They contains inverse scattering transform [1], Lie group [4], the Darboux transformation [21], Hirota direct method [10, 11], algebro-geometrical approach [3] and Painlevé analysis [17,38], etc. Among these methods, Hirota bilinear method is one of vital convenient approaches used for constructing soliton solutions of the nonlinear equations. Interestingly, the method combining with Riemann theta functions has also been developed to get exact quasi-periodic wave solutions.

In 1980s, Nakamura propose a straight and effective approach to construct quasi-periodic solutions for nonlinear equations in his essay [24]. Combining with Riemann theta functions, one can obtain the quasi-periodic wave solutions of the given nonlinear equations. Recently, Fan and Hon [6,7] extend this method to investigate the discrete Toda

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lattice, $(2+1)$-dimensional Bogoyavlenskii's breaking soliton equation and the asymmetrical Nizhnik-Novikov-Veselov equation. Ma [18, 20] construct one- and two- quasi-periodic wave solutions to a class of $(2+1)$-dimensional Hirota bilinear equations. Chow [5] present the exact quasi-periodic solutions to some evolution equations. Chen, et al. [22, 36] investigate the bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws of some KdV-type equations. In [28–32], Tian, et al. present the methods to construct the quasi-periodic wave solutons of some nonlinear differential equations, discrete soliton equations and supersymmetric equations.

Recently, more and more mathematicians and physicists pay attention to the generalized nonlinear equations because the generalized nonlinear equations could describe more realistic physical phenomena than their constant-coefficient counterparts in various fields [13–15, 25, 33, 34]. In this paper, we consider the following generalized KdV-Caudrey-Dodd-Gibbon (KdV-CDG) equation

$$u_t + (h_1 u_{xx} + h_2 u^2)_x + (h_3 u^3 + h_4 u u_{xx} + h_5 u_{xxxx})_x = 0, \tag{1.1}$$

where u is a function of the variable x, t, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, h_i $(i = 1, 2, 3, 4, 5)$ are arbitrary constants. Obviously, (1.1) can be reduced to the KdV equation, Caudrey-Dodd-Gibbon equation and the Sawada-Kotera equation etc. Some important examples are given as follows.

- The Korteweg-de Vries (KdV) equation [19, 35, 39]

  $$u_t + u_{xxx} + 6uu_x = 0 \tag{1.2}$$

  has been found to model many physical, mechanical and engineering phenomena, such as ion-acoustic waves, geophysical fluid dynamics, lattice dynamics, the jams in congested traffic, etc.

- The Caudrey-Dodd-Gibbon (CDG) equation [26, 37]

  $$u_t + u_{xxxxx} + \alpha u_x u_{xx} + \alpha u u_{xxx} + \frac{\alpha^2}{5} u^2 u_x = 0 \tag{1.3}$$

  is completely integrable and admits multiple-soliton solutions. Meanwhile, (1.3) has the Painlevé property.

- The Sawada-Kotera (SK) equation [16, 27]

  $$u_t + u_{xxxxx} + 15u_x u_{xx} + 15uu_{xxx} + 45u^2 u_x = 0 \tag{1.4}$$

  can be used to describe the evolution of steep waves of short wave-length.

To the best of our knowledge, it has not been studied the quasi-periodic wave solutions for the generalized KdV-CDG equation (1.1). The main purpose of this paper is to apply the Bell polynomial approach to construct its bilinear form, N-soliton solution and quasi-periodic wave solutions. Furthermore, we present an asymptotic relation to reveal the relations between quasi-periodic wave solutions and soliton solutions. In the procedure of applying the Bell polynomials, some relevant constraint condition on (1.1) are naturally found, i.e.,

$$15h_3h_5 = h_4^2, \quad h_1h_4 = 5h_2h_5. \tag{1.5}$$

In the present work, the results of the generalized KdV-CDG equation are investigated under the condition (1.5).

The rest of this paper is organized as follows. In Section 2, the Hirota bilinear form of the generalized KdV-CDG equation is obtained by using the Bell polynomials. In addition, some special cases are also investigated. In Section 3, the N-soliton solutions of generalized KdV-CDG equation are constructed with a detailed proof. After that one can obtain the one-soliton and two-soliton solutions of the equation. Through use of the Riemman theta functions, we obtain the quasi-periodic wave solutions in Section 4. In Section 5, we present a relation between quasi-periodic wave solutions and soliton solutions by a limiting procedure. Finally, some important characters of binary Bell polynomials are briefly introduced.

2. The bilinear form

Based on the Bell polynomials [2, 9, 12], we obtain the bilinear form of the generalized KdV-CDG equation.

By introducing the following transformation

$$u = m(t)q_{2x}, \tag{2.1}$$

where $m = m(t)$ is a free function to be determined with Bell's polynomials. Substituting the transformation (2.1) into (1.1), one obtains

$$\begin{aligned} &m(t)q_{2x,t} + m_t(t)q_{2x} + (h_1m(t)q_{4x} + h_2m(t)^2q_{2x}^2)_x \\ &\quad + (h_3m(t)^3q_{2x}^3 + h_4m(t)^2q_{2x}q_{4x} + h_5m(t)q_{6x})_x = 0. \end{aligned} \tag{2.2}$$

If $m(t) \neq 0$, (2.2) is equivalent to

$$\begin{aligned} &q_{2x,t} + \partial_t(\ln m(t))q_{2x} + (h_1q_{4x} + h_2m(t)q_{2x}^2)_x \\ &\quad + (h_3m(t)^2q_{2x}^3 + h_4m(t)q_{2x}q_{4x} + h_5q_{6x})_x = 0. \end{aligned} \tag{2.3}$$

Integrating (2.3) with respect to x, (2.3) becomes

$$q_{x,t} + \partial_t(\ln m(t))q_x + h_1 q_{4x} + h_2 m(t) q_{2x}^2 + h_3 m(t)^2 q_{2x}^3 + h_4 m(t) q_{2x} q_{4x} + h_5 q_{6x} = \delta,$$

where $\delta = \delta(t)$ is an integration constant. Let

$$h_2 = 3h_1 m^{-1}(t), \quad h_3 = 15h_5(m^{-1}(t))^2 \quad \text{and} \quad h_4 = 15h_5 m^{-1}(t), \tag{2.4}$$

we get

$$q_{x,t} + h_1(q_{4x} + 3q_{2x}^2) + h_5(15q_{2x}^3 + 15q_{2x}q_{4x} + q_{6x}) = \delta.$$

Integrating with the P-polynomials, it shows that

$$E(q) = P_{xt}(q) + h_1 P_{4x}(q) + h_5 P_{6x}(q) = \delta. \tag{2.5}$$

Especially, taking $\delta = 0$, (2.5) can be simplified as follows

$$E(q) \equiv P_{xt}(q) + h_1 P_{4x}(q) + h_5 P_{6x}(q) = 0. \tag{2.6}$$

At last, through use of the property (A.2), we have

$$q = 2\ln f \quad \Longleftrightarrow \quad u = m(t) q_{2x},$$

where $m(t)$ is determined by (2.4) and $f = f(x,t)$. From (2.4), we define $m(t)$ as below

$$m(t) = \begin{cases} 3h_1 h_2^{-1} & \text{when } h_1 h_2 \neq 0, \\ 15h_5 h_4^{-1} & \text{when } h_4 h_5 \neq 0, \\ \sqrt{15h_5 h_3^{-1}} & \text{when } h_3 h_5 > 0. \end{cases}$$

When $h_1 h_2 \neq 0$, $h_4 h_5 \neq 0$, $h_3 h_5 > 0$, these three $m(t)$ are equivalent with each other. In the following derivation process, we take $m(t) = 3h_1 h_2^{-1}$ as an example, i.e.,

$$q = 2\ln f \quad \Longleftrightarrow \quad u = m(t) q_{2x} = 6h_1 h_2^{-1} (\ln f)_{xx}.$$

Using the standard identities of the Hirota D-operator

$$D_x^n D_t^k f(x,t) \cdot g(x,t) = (\partial_x - \partial_{x'})^n (\partial_t - \partial_{t'})^k f(x,t) \cdot g(x',t')\big|_{x=x',t=t'},$$

one can obtain the bilinear representation of the KdV-CDG equation

$$\left(D_x D_t + h_1 D_x^4 + h_5 D_x^6\right) f \cdot f = 0, \tag{2.7}$$

where $h_1 h_2 \neq 0$, and h_1, h_2, h_5 are arbitrary parameters. Formula (2.7) is a new bilinear form. Choosing different coefficients h_i ($i = 1, \ldots, 5$), $m(t)$ may be different. But using the similar way, the bilinear representation can be certainly obtained.

Case 1. Let $h_1 = 1$, $h_2 = 3$, $h_5 = 0$, (1.1) becomes KdV equation (1.2), meanwhile, the Hirota bilinear form (2.7) can be written as

$$\left(D_xD_t + D_x^4\right) f \cdot f = 0, \tag{2.8}$$

which is the same as (2.8) in [30].

Case 2. Let $h_1 = 0$, $h_2 = 0$, $h_3 = \alpha^2/15$, $h_4 = \alpha$, $h_5 = 1$, (1.1) becomes Caudrey-Dodd-Gibbon equation (1.3). Especially, when $\alpha = 30$, by making the transformation $m(t) = 15h_5h_4^{-1} = \sqrt{15h_5h_3^{-1}} = 1/2$ in the above proof procedure, the Hirota bilinear of (1.3) is given by

$$\left(D_xD_t + D_x^6\right) f \cdot f = 0,$$

which is also obtained in [26].

Case 3. Let $h_1 = 0$, $h_2 = 0$, $h_3 = 15$, $h_5 = 1$, $h_4 = 15$, (1.1) becomes Sawada-Kotera equation (1.4). Making the transformation $m(t) = 15h_5h_4^{-1} = \sqrt{15h_5h_3^{-1}} = 1$, and the bilinear form for (1.4) is given by

$$\left(D_xD_t + D_x^6\right) f \cdot f = 0,$$

which is equivalent to (2) in [16].

When $h_1h_2 \neq 0$, the Hirota bilinear form (2.7) admits a more general conclusion and applications than the bilinear form which has definite coefficients, e.g. (2.8). It would make sense to us. When $h_1h_2 = 0$, other cases can be discussed by the similar process. Representatively, we discuss the bilinear form (2.7) in the rest of this paper.

3. The solitary solutions

In this section, we construct the N-soliton solutions of the generalized KdV-CDG equation by using the Hirota bilinear form (2.7) with a detailed proof.

Theorem 3.1. *The N-soliton solution of the generalized KdV-CDG equation is obtained as follows*

$$\begin{aligned} &u = 6h_1h_2^{-1}(\ln f)_{xx}, \\ &f = \sum_{\rho=0,1} \exp\left(\sum_{j=1}^{N} \rho_j\phi_j + \sum_{1\le j<i\le N}^{N} \rho_i\rho_j A_{ij}\right), \end{aligned} \tag{3.1}$$

in which $\phi_i = k_ix + \omega_it + \sigma_i$, $e^{A_{ij}} = \frac{-(\omega_i-\omega_j)(k_i-k_j)-h_1(k_i-k_j)^4-h_5(k_i-k_j)^6}{(\omega_i+\omega_j)(k_i+k_j)+h_1(k_i+k_j)^4+h_5(k_i+k_j)^6}$ with $\omega_i = -h_1k_i^3 - h_5k_i^5$, $(1 \le j < i \le N)$. k_i is a free parameter characterizing the j-th soliton. $\sum_{1\le j<i\le N}^{N}$ is the summation over all possible pairs selected from N elements under the condition $(1 \le j < i \le N)$, and $\sum_{\rho=0,1}$ shows the summation over all possible combinations of $\rho_i, \rho_j = 0, 1$ $(i, j = 1, 2, \ldots, N)$.

Proof. Considering the bilinear form (2.7), (3.1) can be written as

$$\begin{aligned}\sum_{\rho=0,1}\sum_{\rho'=0,1}\mathscr{D}&\left(-\sum_{j=1}^{N}(\rho_j-\rho'_j)(h_1k_j^3+h_5k_j^5),\sum_{j=1}^{N}(\rho_j-\rho'_j)k_j\right)\\ &\times\exp\left(\sum_{j=1}^{N}(\rho_j+\rho'_j)\phi_j+\sum_{1\le j<i\le N}^{N}(\rho_i\rho_j+\rho'_i\rho'_j)A_{ij}\right)=0,\end{aligned}\tag{3.2}$$

and then we let the coefficient of the factor

$$\exp\left(\sum_{j=1}^{N}(\rho_j+\rho'_j)\phi_j\right)=\mathscr{G}(\rho,\rho')=\exp\left(\sum_{j=1}^{m}\phi_j+2\sum_{j=m+1}^{n}\phi_j\right).$$

Moreover, for the left-hand side of (3.2), we introduce the following function

$$\begin{aligned}\mathscr{M}=&\sum_{\rho=0,1}\sum_{\rho'=0,1}\mathscr{G}(\rho,\rho')\mathscr{D}\left(-\sum_{j=1}^{N}(\rho_j-\rho'_j)(h_1k_j^3+h_5k_j^5),\sum_{j=1}^{N}(\rho_j-\rho'_j)k_j\right)\\ &\times\exp\left(\sum_{1\le j<i\le N}^{N}(\rho_i\rho_j+\rho'_i\rho'_j)A_{ij}\right)\\ =&\ 0,\end{aligned}\tag{3.3}$$

where the coefficient $\mathscr{G}(\rho,\rho')$ on behalf of the summations about ρ and ρ' under the conditions provided by the following formulas

$$\rho_j=\begin{cases}1-\rho'_j, & \text{if } 1\le j\le m,\\ \rho'_j=1, & \text{if } m+1\le j\le n,\\ \rho'_j=0, & \text{if } n+1\le j\le N.\end{cases}$$

Now, introducing a new variable

$$\varpi_j=\rho_j-\rho'_j,\tag{3.4}$$

we have

$$\begin{aligned}\exp\left(\sum_{1\le j<i\le N}^{N}(\rho_i\rho_j+\rho'_i\rho'_j)A_{ij}\right)=&\sum_{1\le j<i\le N}^{m}\frac{1}{2}(1+\varpi_i\varpi_j)A_{ij}+\sum_{i=1}^{m}\sum_{j=m+1}^{n}A_{ij}\\ &+\sum_{1\le j<i\le N}^{n}\sum_{j=m+1}^{n}A_{ij}.\end{aligned}\tag{3.5}$$

By considering $\varpi_i,\varpi_j=\pm1$ and the following relations

$$\begin{aligned}&\mathscr{D}(-h_1k_j^3-h_5k_j^5,k_j)=\mathscr{D}(h_1k_j^3+h_5k_j^5,-k_j),\\ &\exp(A_{ij})=-\frac{\mathscr{D}(h_1(k_i^3-k_j^3)+h_5(k_i^5-k_j^5),k_j-k_i)}{\mathscr{D}(-h_1(k_i^3+k_j^3)-h_5(k_i^5+k_j^5),k_i+k_j)},\end{aligned}\tag{3.6}$$

we have

$$\sum_{1\le j<i\le N}^{m} \frac{1}{2}(1+\varpi_i\varpi_j)A_{ij} = -\frac{\mathscr{D}(h_1(k_i^3-k_j^3)+h_5(k_i^5-k_j^5), k_j-k_i)}{\mathscr{D}(-h_1(k_i^3+k_j^3)-h_5(k_i^5+k_j^5), k_i+k_j)} \times \varpi_i\varpi_j. \tag{3.7}$$

Combining equations (3.3), (3.4)–(3.7), we obtain

$$\begin{aligned}\mathscr{M} &= \mathscr{A}\sum_{\varpi=\pm1}\mathscr{D}\left(-\sum_{j=1}^{N}\varpi_j(h_1k_j^3+h_5k_j^5), \sum_{j=1}^{N}\varpi_j k_j\right)\\ &\quad\times\prod_{j<i}^{N}\mathscr{D}\left(h_1(k_i^3-k_j^3)+h_5(k_i^5-k_j^5), k_j-k_i\right)\varpi_i\varpi_j\\ &= 0,\end{aligned} \tag{3.8}$$

where $\mathscr{A}=\mathscr{A}(\exp(A_{ij}))$. There's no relation between the $\mathscr{A}$ and the summation indices ϖ_i $(i=1,2,\ldots,N)$. By considering the bilinear equation (2.7), (3.8) can be rewritten as follows

(3.9)

$$\begin{aligned}\widehat{\mathscr{M}_N} &\equiv \mathscr{A}\sum_{\varpi=\pm1}\left\{-\sum_{i,j=1}^{N}\varpi_i\varpi_j(h_1k_i^3+h_5k_i^5)k_j + h_1\left[\sum_{j=1}^{N}\varpi_j k_j\right]^4 + h_5\left[\sum_{i=1}^{N}\varpi_j k_j\right]^6\right\}\\ &\quad\times\prod_{j<i}^{N}\mathscr{D}(h_1(k_i^3-k_j^3)+h_5(k_i^5-k_j^5), k_j-k_i)\varpi_i\varpi_j\\ &= 0.\end{aligned}$$

This implies that $\widehat{\mathscr{M}_N}(-k_1,-k_2,\ldots,-k_N)=\widehat{\mathscr{M}_N}(k_1,k_2,\ldots,k_N)$ from the above equation (3.9). Suppose $\mathscr{A}\equiv 1$, $k_1=\pm k_2$, we have

$$\begin{aligned}&\widehat{\mathscr{M}_N}(k_1,k_2,\ldots,k_N)\\ &\equiv (132h_1k_1^4+68h_5k_1^6)\prod_{j=3}^{N}(k_1-k_j)^2\\ &\quad\times\left\{(2h_1(k_1-k_j)^2+3h_1k_1k_j+2h_5(k_1-k_j)^4+5h_5(k_1k_j^3+k_1^3k_j-k_1^2k_j^2)\right\}\\ &\quad\times\widehat{\mathscr{M}_{N-2}}(k_3,k_4,\ldots,k_N).\end{aligned} \tag{3.10}$$

Assume that the identity hold for $N-2$. Through using the relevance (3.10), it shows that $\widehat{\mathscr{M}_{N-2}}(k_1,k_2,\ldots,k_N)$ is the factor by the following polynomial

$$\begin{aligned}&\widehat{\mathscr{M}_N}(k_1,k_2,\ldots,k_N)\\ &\equiv \prod_{i=1}^{N}(33h_1k_i^4+17h_5k_i^6)\prod_{i<j}^{N}(k_i-k_j)^2\\ &\quad\times\left\{(2h_1(k_i-k_j)^2+3h_1k_ik_j+2h_5(k_i-k_j)^4+5h_5(k_ik_j^3+k_i^3k_j-k_i^2k_j^2)\right\}\\ &\quad\times\widehat{\mathscr{M}_N}(k_1,k_2,\ldots,k_N).\end{aligned} \tag{3.11}$$

From (3.9) and (3.11), we know that $\widehat{\mathscr{M}_N}(k_1, k_2, \ldots, k_N) = 0$ must be zero when $\mathscr{A} \equiv 1$, $n \geq 2$. From above, we have demonstrated the expression (3.1) is the N-soliton solution for the generalized KdV-CDG equation(1.1). □

Through using Theorem 3.1, the following corollary can be easily obtained.

Corollary 3.2. *When $N = 1$, the one-soliton solution of the generalized KdV-CDG equation* (1.1) *is given by*

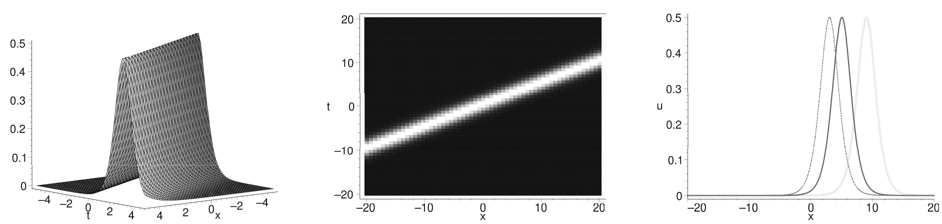
$$u = 6h_1 h_2^{-1} \partial_x^2 \ln(1 + e^{\phi}) = \frac{3}{2} h_1 h_2^{-1} k^2 \sec^2 \frac{kx + \omega t + \sigma}{2}, \tag{3.12}$$

with $\phi = kx - (h_1 k^3 + h_5 k^5)t + \sigma$. When $N = 2$, the two-soliton solution is given by

$$u_1 = 6h_1 h_2^{-1} \partial_x^2 \ln(1 + e^{\phi_1} + e^{\phi_2} + e^{\phi_1 + \phi_2 + A_{12}}), \tag{3.13}$$

with $\phi_i = k_i x + (-h_1 k_i^3 - h_5 k_i^5)t + \sigma_i$, $i = 1, 2$ $e^{A_{12}} = \frac{-(\omega_1 - \omega_2)(k_1 - k_2) - h_1(k_1 - k_2)^4 - h_5(k_1 - k_2)^6}{(\omega_1 + \omega_2)(k_1 + k_2) + h_1(k_1 + k_2)^4 + h_5(k_1 + k_2)^6}$ and $\omega_i = -h_1 k_i^3 - h_5 k_i^5$, $(i = 1, 2)$.

The graphics of the one-soliton and two-soliton wave solutions (3.12) and (3.13) are plotted as Figures 3.1–3.4 by selecting the suitable parameters, respectively.

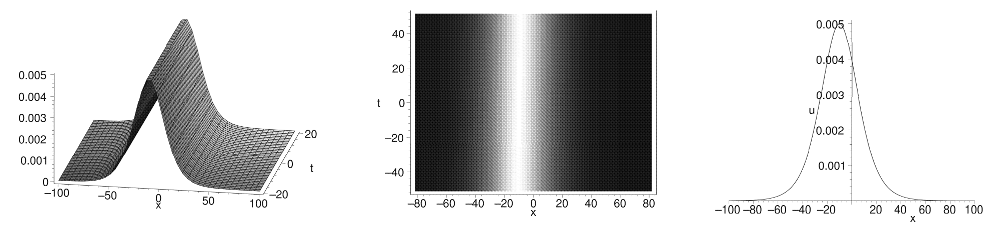


(a) The perspective view of the wave.

(b) The overhead view of the wave.

(c) The wave propagation pattern of the wave along the x taxis.

Figure 3.1: (Color online) Spatial structures of the one-soliton solution (3.12) with the parameters $h_1 = 1$, $h_2 = 3$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $k = 1$ and $\sigma = 1$.

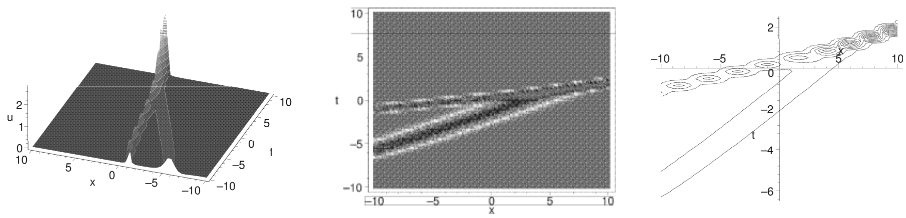


(a) The perspective view of the wave.

(b) The overhead view of the wave.

(c) The wave propagation pattern of the wave along the x taxis.

Figure 3.2: (Color online) Spatial structures of the one-soliton solution (3.12) with the parameters $h_1 = 2$, $h_2 = 6$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $k = 0.1$ and $\sigma = 1$.

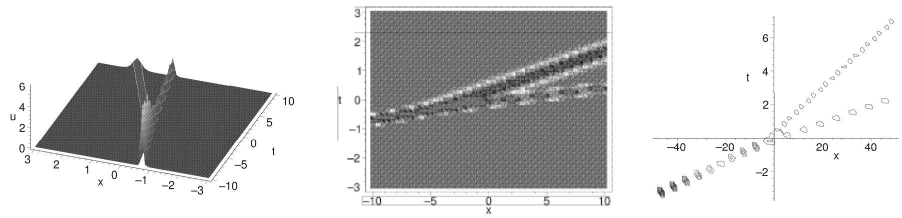


(a) The perspective view of the wave.

(b) The overhead view of the wave.

(c) The corresponding contour plot.

Figure 3.3: (Color online) Spatial structures of the two-soliton solution (3.13) with the parameters $h_1 = 1$, $h_2 = 3$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $k_1 = 2$, $k_2 = -1.5$ and $\sigma_1 = 1$, $\sigma_2 = 0$.



(a) The perspective view of the wave.

(b) The overhead view of the wave.

(c) The corresponding contour plot.

Figure 3.4: (Color online) Spatial structures of the two-soliton solution (3.13) with the parameters $h_1 = 1$, $h_2 = 3$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $k_1 = 2$, $k_2 = -1.5$ and $\sigma_1 = 1$, $\sigma_2 = 1$.

4. Quasi-periodic wave solutions

Firstly, we introduce the multidimensional Riemann theta function [8, 23, 28] of genus N

$$\vartheta(\xi) = \vartheta(\xi, \tau) = \sum_{n \in \mathbb{Z}^N} e^{\pi i \langle n\tau, n\rangle + 2\pi i \langle \xi, n\rangle}, \tag{4.1}$$

in which the integer-valued vector $n = (n_1, \dots, n_N)^T \in \mathbb{Z}^N$ and complex phase variable $\xi = (\xi_1, \dots, \xi_N)^T \in \mathbb{C}^N$. The inner product is defined as

$$\langle f, g\rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N. \tag{4.2}$$

Particularly, when $N = 1$, the Riemann theta function (4.1) becomes

$$\vartheta(\xi, \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2 \tau + 2\pi i n \xi}, \tag{4.3}$$

with the phase variable $\xi = \alpha x + \beta t + \varepsilon$ and $\mathrm{Im}(\tau) > 0$. When $N = 2$, the Riemann theta function (4.1) becomes

$$\vartheta(\xi, \tau) = \vartheta(\xi_1, \xi_2, \tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i \langle \tau n, n\rangle + 2\pi i \langle \xi, n\rangle}, \tag{4.4}$$

with the phase variable $\xi_i = \alpha_i x + \beta_i t + \varepsilon_i$, $i = 1, 2$, $n = (n_1, n_2)^T \in \mathbb{Z}^2$, $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, and $-i\tau$ is a positive definite and real-valued symmetric 2×2 matrix, the τ is given by

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix},$$

which is a symmetric complex matrix and has a positive definite imaginary part. In this paper, we make the matrix τ be pure imaginary matrix to promise the theta function (4.4) real valued.

Furthermore, to construct the quasi-periodic wave solutions of the generalized KdV-CDG equation (1.1), we consider a more generalized form of the bilinear equation (2.7). Suppose (1.1) satisfies the nonzero limiting condition $u \to u_0$ as $|\xi| \to 0$. The solution to (1.1) can be written as

$$u = u_0 + 6h_1 h_2^{-1} \partial_x^2 \ln \vartheta(\xi), \tag{4.5}$$

where u_0 is a constant solution to (1.1). The phase variable ξ is of the form $\xi = (\xi_1, \dots, \xi_N)^T$, $\xi_i = \alpha_i x + \beta_i t + \varepsilon_i$, $i = 1, 2, \dots, N$. Combining (1.1) with (4.5), and integrating with respect to x, we obtain the generalized form as follows

$$\begin{aligned} &\mathscr{M}(D_x, D_t)\vartheta(\xi) \cdot \vartheta(\xi) \\ &= \big(D_x D_t + h_1 D_x^4 + u_0 h_1 D_x^4 + h_5 D_x^6 + u_0 h_5 D_x^6 + c\big)\, \vartheta(\xi) \cdot \vartheta(\xi) \\ &= 0, \end{aligned} \tag{4.6}$$

where $c = c(t)$ is an integration constant.

4.1. One-quasi-periodic wave solutions

According to the bilinear representation (4.6), we construct the one-quasi-periodic wave solutions through use of the Riemann theta function in this subsection.

Considering Theorem 1 in [28], we know that α, β, ε satisfy the following system

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \mathscr{W}(4n\pi i\alpha, 4n\pi i\beta) e^{2n^2\pi i\tau} &= 0, \\ \sum_{n=-\infty}^{+\infty} \mathscr{W}(2\pi i(2n-1)\alpha, 2\pi i(2n-1)\beta) e^{(2n^2-2n+1)\pi i\tau} &= 0. \end{aligned} \tag{4.7}$$

Combining (4.6) with (4.7), we get

$$
\begin{aligned}
\widetilde{\mathscr{W}}(0) = & \sum_{n=-\infty}^{\infty} \big(-16\pi^2 n^2 \alpha\beta + 256 h_1 \pi^4 n^4 \alpha^4 + 256 h_1 u_0 \pi^4 n^4 \alpha^4 \\
& - 4096 h_5 \pi^6 n^6 \alpha^6 - 4096 h_5 u_0 \pi^6 n^6 \alpha^6 + c\big)\, e^{2\pi i n^2 \tau} \\
= & \, 0, \\
\widetilde{\mathscr{W}}(1) = & \sum_{n=-\infty}^{\infty} \big(-4\pi^2 (2n-1)^2 \alpha\beta + 16 h_1 \pi^4 (2n-1)^4 \alpha^4 + 16 h_1 u_0 \pi^4 (2n-1)^4 \alpha^4 \\
& - 64 h_5 \pi^6 (2n-1)^6 \alpha^6 - 64 h_5 u_0 \pi^6 (2n-1)^6 \alpha^6 + c\big)\, e^{\pi i (2n^2-2n+1)\tau} \\
= & \, 0.
\end{aligned} \tag{4.8}
$$

If we set $\mathscr{R} = e^{\pi i \tau}$, (4.8) can be rewritten as

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \beta \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{4.9}
$$

in which

$$
\begin{aligned}
a_{11} &= -\sum_{n=-\infty}^{\infty} 16\pi^2 n^2 \alpha \mathscr{R}^{2n^2}, \\
a_{12} &= \sum_{n=-\infty}^{\infty} \mathscr{R}^{2n^2}, \\
a_{21} &= -\sum_{n=-\infty}^{\infty} 4\pi^2 (2n-1)^2 \alpha \mathscr{R}^{2n^2-2n+1}, \\
a_{22} &= \sum_{n=-\infty}^{\infty} \mathscr{R}^{2n^2-2n+1}, \\
b_1 &= -\sum_{n=-\infty}^{\infty} \big(-256 h_1 \pi^4 n^4 \alpha^4 - 256 h_1 u_0 \pi^4 n^4 \alpha^4 + 4096 h_5 \pi^6 n^6 \alpha^6 \\
&\qquad + 4096 h_5 u_0 \pi^6 n^6 \alpha^6\big)\, \mathscr{R}^{2n^2}, \\
b_2 &= \sum_{n=-\infty}^{\infty} \big(-16 h_1 \pi^4 (2n-1)^4 \alpha^4 - 16 h_1 u_0 \pi^4 (2n-1)^4 \alpha^4 + 64 h_5 \pi^6 (2n-1)^6 \alpha^6 \\
&\qquad + 64 h_5 u_0 \pi^6 (2n-1)^6 \alpha^6\big)\, \mathscr{R}^{2n^2-2n+1},
\end{aligned} \tag{4.10}
$$

Solving the system (4.9), we obtain

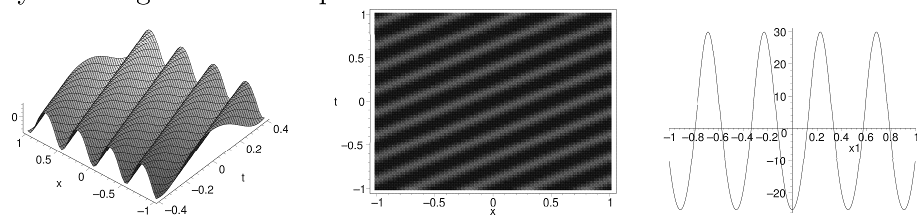
$$
\beta = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}, \quad c = \frac{b_1 a_{21} - b_2 a_{11}}{a_{12} a_{21} - a_{11} a_{22}}, \tag{4.11}
$$

where $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Namely, we obtain the one-quasi-periodic wave solution of the generalized KdV-CDG equation (1.1)

$$
u = u_0 + 6 h_1 h_2^{-1} \partial_x^2 \ln \vartheta(\xi), \tag{4.12}
$$

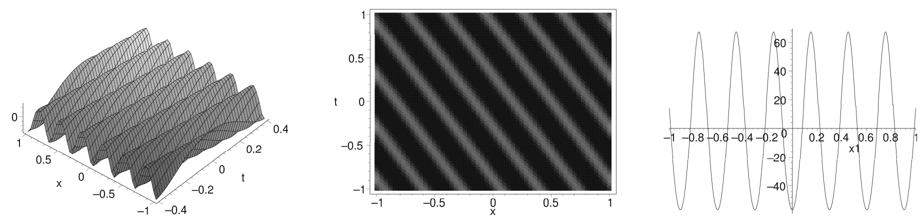
where the vector $(\beta, c)^T$ and the theta function $\vartheta(\xi)$ can be determined by (4.11) and (4.3) respectively. The one-quasi-periodic wave solution is completely depended on arbitrary parameters α, τ and ε.

The graphics of the one-quasi-periodic wave solution (4.12) are plotted as Figures 4.1 and 4.2 by selecting the suitable parameters.



(a) The perspective view of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the x taxis.

Figure 4.1: (Color online) Spatial structures of the one-quasi-periodic wave solution (4.12) with the parameters $h_1 = 1$, $h_2 = 3$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $\tau = i$, $\alpha = 2$, $u_0 = 0$ and $\varepsilon = 0$.



(a) The perspective view of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the x taxis.

Figure 4.2: (Color online) Spatial structures of the one-quasi-periodic wave solution (4.12) with the parameters $h_1 = 1$, $h_2 = 3$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $\tau = i$, $\alpha = 3$, $u_0 = 0$ and $\varepsilon = 0$.

4.2. Two-quasi-periodic wave solutions

In this subsection, we search for the two-quasi-periodic wave solutions for (1.1) by a similar way. By considering Theorem 2 in [28], α_i, β_i, ε_i should satisfy the following system

$$
\begin{aligned}
\widetilde{\mathscr{W}}(0,0) &= \sum_{n\in\mathbb{Z}^2} \mathscr{W}(2\pi i \langle 2n-\theta_1, \alpha\rangle, 2\pi i \langle 2n-\theta_1, \beta\rangle) e^{\pi i[\langle\tau(n-\theta_1), n-\theta_1\rangle + \langle\tau n, n\rangle]} = 0,\\
\widetilde{\mathscr{W}}(1,0) &= \sum_{n\in\mathbb{Z}^2} \mathscr{W}(2\pi i \langle 2n-\theta_2, \alpha\rangle, 2\pi i \langle 2n-\theta_2, \beta\rangle) e^{\pi i[\langle\tau(n-\theta_2), n-\theta_2\rangle + \langle\tau n, n\rangle]} = 0,\\
\widetilde{\mathscr{W}}(0,1) &= \sum_{n\in\mathbb{Z}^2} \mathscr{W}(2\pi i \langle 2n-\theta_3, \alpha\rangle, 2\pi i \langle 2n-\theta_3, \beta\rangle) e^{\pi i[\langle\tau(n-\theta_3), n-\theta_3\rangle + \langle\tau n, n\rangle]} = 0,\\
\widetilde{\mathscr{W}}(1,1) &= \sum_{n\in\mathbb{Z}^2} \mathscr{W}(2\pi i \langle 2n-\theta_4, \alpha\rangle, 2\pi i \langle 2n-\theta_4, \beta\rangle) e^{\pi i[\langle\tau(n-\theta_4), n-\theta_4\rangle + \langle\tau n, n\rangle]} = 0,
\end{aligned}
\tag{4.13}
$$

where $\theta_i = (\theta_i^1, \theta_i^2)^T$, $i = 1, 2, 3, 4$, $\theta_1 = (0, 0)^T$, $\theta_2 = (1, 0)^T$, $\theta_3 = (0, 1)^T$ and $\theta_4 = (1, 1)^T$.

Combining (4.13) with (4.6), (4.13) can be rewritten as

$$\sum_{n\in\mathbb{Z}^2}\Big[-4\pi^2\langle 2n-\theta_1,\alpha\rangle\langle 2n-\theta_1,\beta\rangle+16h_1\pi^4\langle 2n-\theta_1,\alpha\rangle^4+16h_1u_0\pi^4\langle 2n-\theta_1,\alpha\rangle^4$$
$$-64h_5\pi^6\langle 2n-\theta_1,\alpha\rangle^6-64h_5u_0\pi^6\langle 2n-\theta_1,\alpha\rangle^6+c\Big]e^{\pi i[\langle\tau(n-\theta_1),n-\theta_1\rangle+\langle\tau n,n\rangle]}=0,$$
$$\sum_{n\in\mathbb{Z}^2}\Big[-4\pi^2\langle 2n-\theta_2,\alpha\rangle\langle 2n-\theta_2,\beta\rangle+16h_1\pi^4\langle 2n-\theta_2,\alpha\rangle^4+16h_1u_0\pi^4\langle 2n-\theta_2,\alpha\rangle^4$$
$$-64h_5\pi^6\langle 2n-\theta_2,\alpha\rangle^6-64h_5u_0\pi^6\langle 2n-\theta_2,\alpha\rangle^6+c\Big]e^{\pi i[\langle\tau(n-\theta_2),n-\theta_2\rangle+\langle\tau n,n\rangle]}=0,$$
$$\sum_{n\in\mathbb{Z}^2}\Big[-4\pi^2\langle 2n-\theta_3,\alpha\rangle\langle 2n-\theta_3,\beta\rangle+16h_1\pi^4\langle 2n-\theta_3,\alpha\rangle^4+16h_1u_0\pi^4\langle 2n-\theta_3,\alpha\rangle^4$$
$$-64h_5\pi^6\langle 2n-\theta_3,\alpha\rangle^6-64h_5u_0\pi^6\langle 2n-\theta_3,\alpha\rangle^6+c\Big]e^{\pi i[\langle\tau(n-\theta_3),n-\theta_3\rangle+\langle\tau n,n\rangle]}=0,$$
$$\sum_{n\in\mathbb{Z}^2}\Big[-4\pi^2\langle 2n-\theta_4,\alpha\rangle\langle 2n-\theta_4,\beta\rangle+16h_1\pi^4\langle 2n-\theta_4,\alpha\rangle^4+16h_1u_0\pi^4\langle 2n-\theta_4,\alpha\rangle^4$$
$$-64h_5\pi^6\langle 2n-\theta_4,\alpha\rangle^6-64h_5u_0\pi^6\langle 2n-\theta_4,\alpha\rangle^6+c\Big]e^{\pi i[\langle\tau(n-\theta_4),n-\theta_4\rangle+\langle\tau n,n\rangle]}=0.$$

Similarly, by considering the following system

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14}\\ h_{21} & h_{22} & h_{23} & h_{24}\\ h_{31} & h_{32} & h_{33} & h_{34}\\ h_{41} & h_{42} & h_{43} & h_{44}\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ u_0\\ c\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\\ b_3\\ b_4\end{pmatrix}, \tag{4.14}$$

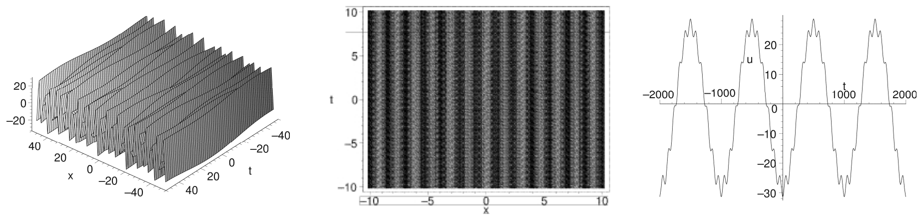
we obtain

$$\begin{aligned}
h_{i1}&=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,\alpha\rangle(2n_1-\theta_i^1)\Im_i(n),\\
h_{i2}&=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,\alpha\rangle(2n_2-\theta_i^2)\Im_i(n),\\
h_{i3}&=\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Big(16h_1\pi^4\langle 2n-\theta_i,\alpha\rangle^4-64h_5\pi^6\langle 2n-\theta_i,\alpha\rangle^6\Big)\Im_i(n),\\
h_{i4}&=\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Im_i(n),\\
b_i&=16\pi^4\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Big(-h_1\langle 2n-\theta_i,\alpha\rangle^4+4h_5\pi^2\langle 2n-\theta_i,\alpha\rangle^6\Big)\Im_i(n)\\
\Im_i(n)&=\mathscr{R}_1^{n_1^2+(n_1-\theta_i^1)^2}\mathscr{R}_2^{n_2^2+(n_2-\theta_i^2)^2}\mathscr{R}_3^{n_1n_2+(n_1-\theta_i^1)(n_2-\theta_i^2)},\\
\mathscr{R}_1&=e^{\pi i\tau_{11}},\quad \mathscr{R}_2=e^{\pi i\tau_{22}},\quad \mathscr{R}_3=e^{2\pi i\tau_{12}},\quad i=1,2,3,4,
\end{aligned} \tag{4.15}$$

where $\theta_i = (\theta_i^1, \theta_i^2)^T$, $i = 1, 2, 3, 4$, $\theta_1 = (0, 0)^T$, $\theta_2 = (1, 0)^T$, $\theta_3 = (0, 1)^T$, $\theta_4 = (1, 1)^T$, and α_i, τ_{ij}, ε_i $(i, j = 1, 2)$ are free parameters. Solving system (4.14), based on the formulas (4.15), we can obtain the vector $(\beta_1, \beta_2, u_0, c)^T$. Furthermore the theta function $\vartheta(\xi)$ can be identified by virtue of the vector $(\beta_1, \beta_2, u_0, c)^T$. Then, we get the two-quasi-periodic wave solution of the generalized KdV-CDG equation (1.1) as follows

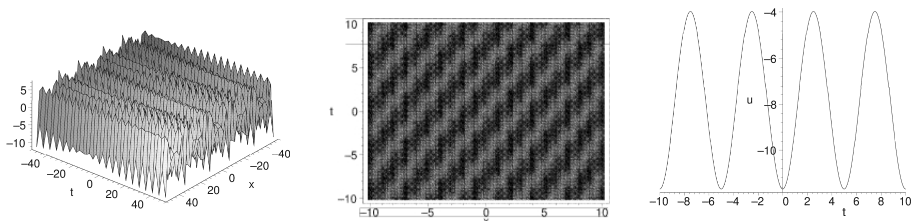
$$u = u_0 + 6h_1 h_2^{-1} \partial_x^2 \ln \vartheta(\xi_1, \xi_2, \tau). \tag{4.16}$$

The graphics of the two-quasi-periodic wave solution (4.16) are plotted as Figures 4.3 and 4.4 by selecting the suitable parameters, respectively.



(a) The perspective view of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the t taxis.

Figure 4.3: (Color online) Spatial structures of the two-quasi-periodic wave solution (4.16) with the parameters $h_1 = 2$, $h_2 = 6$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $\alpha_1 = -2$, $\alpha_2 = 1$, $u_0 = 0$, $\tau_{11} = i$, $\tau_{12} = 0.5i$, $\tau_{22} = 2i$ and $\varepsilon_1 = 1$, $\varepsilon_2 = 0$.



(a) The perspective view of the wave. (b) The overhead view of the wave. (c) The wave propagation pattern of the wave along the t taxis.

Figure 4.4: (Color online) Spatial structures of the two-quasi-periodic wave solution (4.16) with the parameters $h_1 = 2$, $h_2 = 6$, $h_3 = 15$, $h_4 = 15$, $h_5 = 1$, $\alpha_1 = -1$, $\alpha_2 = 2$, $u_0 = 0$, $\tau_{11} = i$, $\tau_{12} = 0.5i$, $\tau_{22} = 2i$ and $\varepsilon_1 = 1$, $\varepsilon_2 = 1$.

5. Limiting behavior of quasi-periodic wave solutions

In this section, we research the asymptotic behaviour of the quasi-periodic solutions. In what follows, we obtain that the one-quasi- and two-quasi-periodic wave solutions (4.12), (4.16) that expressed by $\vartheta(\xi, \tau)$, $\vartheta(\xi_1, \xi_2, \tau)$ can be degenerated into the one- and two-soliton solutions (3.12), (3.13) as the amplitude $\mathscr{R} \to 0$, respectively.

Theorem 5.1. *If the vector* $(\beta, c)^T$ *is a solution of the system* (4.9)*, and for the one-quasi-periodic wave solution* (4.12)*, we take*

$$u_0 = 0, \quad \alpha = \frac{k}{2\pi i}, \quad \varepsilon = \frac{\sigma - \pi i \tau}{2\pi i}, \tag{5.1}$$

where k*,* σ *are free constants and lie on* (3.13)*. Then we have the limiting properties as the following*

$$c \to 0, \quad \xi \to \frac{\phi - \pi i \tau}{2\pi i}, \quad \vartheta(\xi, \tau) \to 1 + e^{\phi}, \quad \text{when } \mathscr{R} \to 0,$$

which means that the quasi-one-periodic wave solution (4.12) *can be degenerated into the one-soliton solution* (3.12) *by a small amplitude limit* $\mathscr{R} \to 0$.

Proof. Expanding the matrix elements a_{ij} $(i = 1, 2)$ and b_i $(i = 1, 2)$ in terms of $\mathscr{R}$ based on (4.10) we obtain

$$\begin{aligned}
a_{11} &= -32\pi^2\alpha(\mathscr{R}^2 + 4\mathscr{R}^8 + \cdots + n^2\mathscr{R}^{2n^2} + \cdots),\\
a_{12} &= 1 + 2(\mathscr{R}^2 + \mathscr{R}^8 + \cdots + \mathscr{R}^{2n^2} + \cdots),\\
a_{21} &= -8\pi^2\alpha(\mathscr{R} + 9\mathscr{R}^5 + \cdots + (2n-1)^2\mathscr{R}^{2n^2-2n+1} + \cdots),\\
a_{22} &= 2(\mathscr{R} + \mathscr{R}^5 + \cdots + \mathscr{R}^{2n^2-2n+1} + \cdots),\\
b_1 &= 512\pi^4\alpha^4\big[(-h_1 - h_1u_0 + 16h_5\pi^2\alpha^2 + 16h_5u_0\pi^2\alpha^2)\mathscr{R}^2\\
&\qquad + 16(-h_1 - h_1u_0 + 32h_5\pi^2\alpha^2 + 32h_5u_0\pi^2\alpha^2)\mathscr{R}^8 + \cdots\\
&\qquad + (-h_1n^4 - h_1u_0n^4 + 16h_5n^6\pi^2\alpha^2 + 16h_5u_0n^6\pi^2\alpha^2)\mathscr{R}^{2n^2} + \cdots\big],\\
b_2 &= 32\pi^4\alpha^4\big[(-h_1 - h_1u_0 + 4h_5\pi^2\alpha^2 + 4h_5u_0\pi^2\alpha^2)\mathscr{R}\\
&\qquad + 81(-h_1 - h_1u_0 + 36h_5\pi^2\alpha^2 + 36h_5u_0\pi^2\alpha^2)\mathscr{R}^5 + \cdots\\
&\qquad + \big(-h_1(2n-1)^4 - h_1u_0(2n-1)^4 + 4h_5\pi^2(2n-1)^6\alpha^2\\
&\qquad + 4h_5u_0\pi^2(2n-1)^6\alpha^2\big)\mathscr{R}^{2n^2-2n+1} + \cdots\big].
\end{aligned} \tag{5.2}$$

From (5.2), based on (4.10) and (4.12) in [28], we can get the following formulas

$$\begin{aligned}
&A_0 = \begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0\\ -8\pi^2\alpha & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -32\pi^2\alpha & 2\\ 0 & 0 \end{pmatrix},\\
&A_5 = \begin{pmatrix} 0 & 0\\ -72\pi^2\alpha & 2 \end{pmatrix}, \quad A_3 = A_4 = 0, \ \ldots,\\
&B_0 = 0, \quad B_1 = \big(0, 32\pi^4\alpha^4(-h_1 - h_1u_0 + 4h_5\pi^2\alpha^2 + 4h_5u_0\pi^2\alpha^2)\big)^T,\\
&B_2 = \big(512\pi^4\alpha^4(-h_1 - h_1u_0 + 16h_5\pi^2\alpha^2 + 16h_5u_0\pi^2\alpha^2), 0\big)^T, \quad B_3 = 0,\\
&B_5 = \big(0, 2592\pi^4\alpha^4(-h_1 - h_1u_0 + 36h_5\pi^2\alpha^2 + 36h_5u_0\pi^2\alpha^2)\big)^T, \quad B_4 = 0, \ldots.
\end{aligned} \tag{5.3}$$

According to the Proposition 3 in [28], and combining with system (5.3), we have

$$X_0 = \begin{pmatrix} 4\pi^2\alpha^3(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \\ 0 \end{pmatrix},$$
$$X_2 = \begin{pmatrix} 32\pi^2\alpha^3(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \\ 128\pi^4\alpha^4(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \end{pmatrix},$$
$$X_4 = \begin{pmatrix} (480h_1\pi^2\alpha^3 - 12288h_5\pi^4\alpha^5)(1+u_0) \\ 768\pi^4\alpha^4(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \end{pmatrix}, \quad X_1 = X_3 = 0, \ldots.$$

From (4.11) in [28], the following formulas can be obtained

$$\begin{aligned} \begin{pmatrix} \beta \\ c \end{pmatrix} &= \begin{pmatrix} 4\pi^2\alpha^3(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 32\pi^2\alpha^3(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \\ 128\pi^4\alpha^4(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \end{pmatrix} \mathscr{R}^2 \\ &\quad + \begin{pmatrix} (480h_1\pi^2\alpha^3 - 12288h_5\pi^4\alpha^5)(1+u_0) \\ 768\pi^4\alpha^4(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2) \end{pmatrix} \mathscr{R}^4 + o(\mathscr{R}^4), \end{aligned}$$

namely

$$\begin{aligned} \beta &= 4\pi^2\alpha^3\left(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2\right) \\ &\quad + 32\pi^2\alpha^3\left(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2\right)\mathscr{R}^2 \\ &\quad + \left(480h_1\pi^2\alpha^3 - 12288h_5\pi^4\alpha^5\right)(1+u_0)\mathscr{R}^4 + o(\mathscr{R}^4), \\ c &= 128\pi^4\alpha^4\left(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2\right)\mathscr{R}^2 \\ &\quad + 768\pi^4\alpha^4\left(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2\right)\mathscr{R}^4 + o(\mathscr{R}^4). \end{aligned}$$

By considering the formulas (5.1), we have

$$c \to 0, \quad \beta \to 4\pi^2\alpha^3(h_1 + h_1 u_0 - 4h_5\pi^2\alpha^2 - 4h_5 u_0\pi^2\alpha^2), \quad \text{if } \mathscr{R} \to 0, \tag{5.4}$$

which means

$$2\pi i\beta \to -h_1 k^3 - h_5 k^5. \tag{5.5}$$

Furthermore, the quasi-periodic function $\vartheta(\xi)$ can be rewritten as

$$\vartheta(\xi,\tau) = 1 + \left(e^{2\pi i\xi} + e^{-2\pi i\xi}\right)\mathscr{R} + \left(e^{4\pi i\xi} + e^{-4\pi i\xi}\right)\mathscr{R}^4 + \cdots. \tag{5.6}$$

Considering the transformation (5.1), we get

$$\begin{aligned} \vartheta(\xi,\tau) &= 1+e^{\widetilde{\xi}}+\left(e^{-\widetilde{\xi}}+e^{2\widetilde{\xi}}\right)\mathscr{R}^2+\left(e^{-2\widetilde{\xi}}+e^{3\widetilde{\xi}}\right)\mathscr{R}^6+\cdots\to 1+e^{\widetilde{\xi}},\ \text{when } \mathscr{R}\to 0,\\ \widetilde{\xi} &= 2\pi i\xi+\pi i\tau = kx+2\pi i\beta t+\sigma. \end{aligned} \tag{5.7}$$

From (5.4) to (5.7), we obtain

$$\begin{aligned} &\widetilde{\xi}\to kx+(-h_1k^3-h_5k^5)t+\sigma=\phi, \quad \text{when } \mathscr{R}\to 0,\\ &\xi\to\frac{\phi-\pi i\tau}{2\pi i}, \quad \text{when } \mathscr{R}\to 0. \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8), we further have

$$\vartheta(\xi)\to 1+e^{\phi}, \quad \text{when } \mathscr{R}\to 0.$$

From all the above, this implies that the one-quasi-periodic solution (4.12) tends to the one-soliton solution (3.12) under a small amplitude limit $\mathscr{R}\to 0$. □

Theorem 5.2. *If the vector* $(\beta_1,\beta_2,u_0,c)^T$ *is a solution of the system* (4.14)*, and for the two-quasi-periodic wave solution* (4.16)*, we consider*

$$\alpha_i=\frac{k_i}{2\pi i}, \quad \varepsilon_i=\frac{\sigma_i-\pi i\tau_{ii}}{2\pi i}, \quad \tau_{12}=\frac{A_{12}}{2\pi i}, \quad i=1,2, \tag{5.9}$$

where k_i*,* σ_i*,* A_{12}*,* $i=1,2$ *are depended on* (3.13) *and* k_i*,* σ_i *are free constants. And then we have the limiting properties as follows*

$$\begin{aligned} &u_0\to 0, \quad c\to 0, \quad \xi_i\to\frac{\phi_i-\pi i\tau_{ii}}{2\pi i}, \quad i=1,2,\\ &\vartheta(\xi_1,\xi_2,\tau)\to 1+e^{\phi_1}+e^{\phi_2}+e^{\phi_1+\phi_2+A_{12}}, \quad \text{as } \mathscr{R}_1,\mathscr{R}_2\to 0. \end{aligned}$$

This means that the two-quasi-periodic wave solution (4.16) *tends to the two-soliton solution* (3.13) *under a small amplitude limit, that is,* $(u,\mathscr{R}_1,\mathscr{R}_2)\to(u_1,0,0)$*.*

Proof. First, we write the functions H, b, $(\beta_1,\beta_2,u_0,c)^T$ as the series about $\mathscr{R}$

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14}\\ h_{21} & h_{22} & h_{23} & h_{24}\\ h_{31} & h_{32} & h_{33} & h_{34}\\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} = H_0+H_1\mathscr{R}_1+H_2\mathscr{R}_2+H_3\mathscr{R}_1^2+H_4\mathscr{R}_2^2+H_5\mathscr{R}_1\mathscr{R}_2+\cdots, \tag{5.10}$$

$$(\beta_1,\beta_2,u_0,c)^T=\Lambda_0+\Lambda_1\mathscr{R}_1+\Lambda_2\mathscr{R}_2+\Lambda_3\mathscr{R}_1^2+\Lambda_4\mathscr{R}_2^2+\Lambda_5\mathscr{R}_1\mathscr{R}_2+\cdots, \tag{5.11}$$

$$(b_1,b_2,b_3,b_4)^T=B_1\mathscr{R}_1+B_2\mathscr{R}_2+B_3\mathscr{R}_1^2+B_4\mathscr{R}_2^2+B_5\mathscr{R}_1\mathscr{R}_2+\cdots. \tag{5.12}$$

From (4.15) and (5.10)–(5.12), we can obtain

$$
\begin{aligned}
H = {} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -8\pi^2\alpha_1 & 0 & 32h_1\pi^4\alpha_1^4 - 128h_5\pi^6\alpha_1^6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathscr{R}_1 \\
& + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -8\pi^2\alpha_2 & 32h_1\pi^4\alpha_2^4 - 128h_5\pi^6\alpha_2^6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathscr{R}_2 \\
& + \begin{pmatrix} -32\pi^2\alpha_1 & 0 & 512h_1\pi^4\alpha_1^4 - 8192h_5\pi^6\alpha_1^6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathscr{R}_1^2 \\
& + \begin{pmatrix} 0 & -32\pi^2\alpha_2 & 512h_1\pi^4\alpha_2^4 - 8192h_5\pi^6\alpha_2^6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathscr{R}_2^2 \\
& + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T_1 & T_2 & T_3 & T_4 \end{pmatrix} \mathscr{R}_1\mathscr{R}_2 + o(\mathscr{R}_1^i\mathscr{R}_2^j), \quad i+j \geq 2,
\end{aligned}
$$

$$
\begin{aligned}
b = {} & \begin{pmatrix} 0 \\ \Upsilon_1 \\ 0 \\ 0 \end{pmatrix} \mathscr{R}_1 + \begin{pmatrix} 0 \\ 0 \\ \Upsilon_2 \\ 0 \end{pmatrix} \mathscr{R}_2 + \begin{pmatrix} \Upsilon_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathscr{R}_1^2 \\
& + \begin{pmatrix} \Upsilon_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathscr{R}_2^2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Upsilon_5 \end{pmatrix} \mathscr{R}_1\mathscr{R}_2 + o(\mathscr{R}_1^i\mathscr{R}_2^j), \quad i+j \geq 3,
\end{aligned}
$$

$$\begin{pmatrix}\beta_1\\ \beta_2\\ u_0\\ c\end{pmatrix}=\begin{pmatrix}\beta_1^{(00)}\\ \beta_2^{(00)}\\ u_0^{(00)}\\ c^{(00)}\end{pmatrix}+\begin{pmatrix}\beta_1^{(11)}\\ \beta_2^{(11)}\\ u_0^{(11)}\\ c^{(11)}\end{pmatrix}\mathscr{R}_1+\begin{pmatrix}\beta_1^{(21)}\\ \beta_2^{(21)}\\ u_0^{(21)}\\ c^{(21)}\end{pmatrix}\mathscr{R}_2+\begin{pmatrix}\beta_1^{(12)}\\ \beta_2^{(12)}\\ u_0^{(12)}\\ c^{(12)}\end{pmatrix}\mathscr{R}_1^2$$
$$+\begin{pmatrix}\beta_1^{(22)}\\ \beta_2^{(22)}\\ u_0^{(22)}\\ c^{(22)}\end{pmatrix}\mathscr{R}_2^2+\begin{pmatrix}\beta_1^{(2)}\\ \beta_2^{(2)}\\ u_0^{(2)}\\ c^{(2)}\end{pmatrix}\mathscr{R}_1\mathscr{R}_2+o(\mathscr{R}_1^i\mathscr{R}_2^j),\quad i+j\geq 2,$$

with

$$\begin{aligned}
T_1&=-8\pi^2(\alpha_1-\alpha_2)-8\pi^2(\alpha_1+\alpha_2)\mathscr{R}_3,\\
T_2&=8\pi^2(\alpha_1-\alpha_2)-8\pi^2(\alpha_1+\alpha_2)\mathscr{R}_3,\\
T_3&=32\pi^4(\alpha_1-\alpha_2)^4\left(h_1-4h_5\pi^2(\alpha_1-\alpha_2)^2\right)\\
&\quad+32\pi^4(\alpha_1+\alpha_2)^4\left(h_1-4h_5\pi^2(\alpha_1+\alpha_2)^2\right)\mathscr{R}_3,\\
T_4&=2+2\mathscr{R}_3,\\
\Upsilon_1&=-32h_1\pi^4\alpha_1^4+128h_5\pi^6\alpha_1^6,\\
\Upsilon_2&=-32h_1\pi^4\alpha_2^4+128h_5\pi^6\alpha_2^6\\
\Upsilon_3&=-512h_1\pi^4\alpha_1^4+8192h_5\pi^6\alpha_1^6,\\
\Upsilon_4&=-512h_1\pi^4\alpha_2^4+8192h_5\pi^6\alpha_2^6,\\
\Upsilon_5&=-32\pi^4(\alpha_1-\alpha_2)^4\left(h_1-4h_5\pi^2(\alpha_1-\alpha_2)^2\right)\\
&\quad-32\pi^4(\alpha_1+\alpha_2)^4\left(h_1-4h_5\pi^2(\alpha_1+\alpha_2)^2\right)\mathscr{R}_3.
\end{aligned}$$

Meanwhile, from the system (4.14) and (5.10)–(5.12), we have

$$\begin{aligned}
&H_0\Lambda_0=0, && H_1\Lambda_0+H_0\Lambda_1=B_1,\\
&H_0\Lambda_2+H_2\Lambda_0=B_2, && H_0\Lambda_3+H_1\Lambda_1+H_3\Lambda_0=B_3,\\
&H_0\Lambda_4+H_2\Lambda_2+H_4\Lambda_0=B_4, && H_0\Lambda_5+H_1\Lambda_2+H_2\Lambda_1+H_5\Lambda_0=B_5.
\end{aligned}\tag{5.13}$$

Furthermore, we have the following formulas based on (5.13)

$$\begin{gathered}
c^{(00)}=c^{(11)}=c^{(21)}=c^{(2)}=c^{(3)}=0,\\
-8\pi^2\alpha_1\beta_1^{(00)}+(32h_1\pi^4\alpha_1^4-128h_5\pi^6\alpha_1^6)u_0^{(00)}=\Upsilon_1,\\
-8\pi^2\alpha_2\beta_2^{(00)}+(32h_1\pi^4\alpha_2^4-128h_5\pi^6\alpha_2^6)u_0^{(00)}=\Upsilon_2,\\
c^{(12)}-32\pi^2\alpha_1\beta_1^{(00)}+(512h_1\pi^4\alpha_1^4-8192h_5\pi^6\alpha_1^6)u_0^{(00)}=\Upsilon_3,
\end{gathered}$$

$$
\begin{aligned}
-8\pi^2\alpha_1\beta_1^{(11)} + (32h_1\pi^4\alpha_1^4 - 128h_5\pi^6\alpha_1^6)u_0^{(11)} &= 0,\\
c^{(22)} - 32\pi^2\alpha_2\beta_2^{(00)} + (512h_1\pi^4\alpha_2^4 - 8192h_5\pi^6\alpha_2^6)u_0^{(00)} &= \Upsilon_4,\\
-8\pi^2\alpha_2\beta_2^{(21)} + (32h_1\pi^4\alpha_2^4 - 128h_5\pi^6\alpha_2^6)u_0^{(21)} &= 0,\\
-8\pi^2\alpha_1\beta_1^{(21)} + (32h_1\pi^4\alpha_1^4 - 128h_5\pi^6\alpha_1^6)u_0^{(21)} &= 0,\\
-8\pi^2\alpha_2\beta_2^{(11)} + (32h_1\pi^4\alpha_2^4 - 128h_5\pi^6\alpha_2^6)u_0^{(11)} &= 0,\\
T_1\beta_1^{(00)} + T_2\beta_2^{(00)} + T_3u_0^{(00)} &= \Upsilon_5.
\end{aligned}
$$

Considering $u_0^{(00)} = 0$ yields

$$
\begin{aligned}
u_0 &= o(\mathscr{R}_1, \mathscr{R}_2) \to 0,\\
c &= \left(-384h_1\pi^4\alpha_1^4 + 7680h_5\pi^6\alpha_1^6\right)\mathscr{R}_1^2 + \left(-384h_1\pi^4\alpha_2^4 + 7680h_5\pi^6\alpha_2^6\right)\mathscr{R}_2^2\\
&\quad + o(\mathscr{R}_1\mathscr{R}_2) \to 0,\\
\beta_1 &= 4h_1\pi^2\alpha_1^3 - 16h_5\pi^4\alpha_1^5 + o(\mathscr{R}_1\mathscr{R}_2) \to 4h_1\pi^2\alpha_1^3 - 16h_5\pi^4\alpha_1^5,\\
\beta_2 &= 4h_1\pi^2\alpha_2^3 - 16h_5\pi^4\alpha_2^5 + o(\mathscr{R}_1\mathscr{R}_2) \to 4h_1\pi^2\alpha_2^3 - 16h_5\pi^4\alpha_2^5,
\end{aligned}
\tag{5.14}
$$

when $(\mathscr{R}_1, \mathscr{R}_2) \to (0,0)$. Combining with (5.9) yields

$$
2\pi i\beta_1 \to -h_1k_1^3 - h_5k_1^5, \quad 2\pi i\beta_2 \to -h_1k_2^3 - h_5k_2^5, \quad \text{when } (\mathscr{R}_1, \mathscr{R}_2) \to (0,0). \tag{5.15}
$$

Furthermore, the quasi-periodic wave function $\vartheta(\xi_1, \xi_2, \tau)$ can be rewritten as

$$
\begin{aligned}
\vartheta(\xi_1, \xi_2, \tau) &= 1 + \left(e^{2\pi i\xi_1} + e^{-2\pi i\xi_1}\right)e^{\pi i\tau_{11}} + \left(e^{2\pi i\xi_2} + e^{-2\pi i\xi_2}\right)e^{\pi i\tau_{22}}\\
&\quad + \left(e^{2\pi i(\xi_1+\xi_2)} + e^{-2\pi i(\xi_1+\xi_2)}\right)e^{\pi i(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots .
\end{aligned}
\tag{5.16}
$$

Under transformation (5.9), we have

$$
\begin{aligned}
\vartheta(\xi_1, \xi_2, \tau) &= 1 + e^{\widetilde{\xi}_1} + e^{\widetilde{\xi}_2} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+2\pi i\tau_{12}} + \mathscr{R}_1^2e^{-\widetilde{\xi}_1} + \mathscr{R}_2^2e^{-\widetilde{\xi}_2}\\
&\quad + \mathscr{R}_1^2\mathscr{R}_2^2e^{-\widetilde{\xi}_1-\widetilde{\xi}_2+2\pi i\tau_{12}} + \cdots\\
&\to 1 + e^{\widetilde{\xi}_1} + e^{\widetilde{\xi}_2} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+A_{12}},
\end{aligned}
\tag{5.17}
$$

as $\mathscr{R}_1, \mathscr{R}_2 \to 0$, with $\widetilde{\xi}_i = k_ix + 2\pi i\beta_it + \sigma_i$, $i = 1,2$. From (5.14) to (5.17), we obtain

$$
\widetilde{\xi}_i \to k_ix + \omega_it + \sigma_i = \phi_i, \quad \xi_i \to \frac{\phi_i - \pi i\tau_{ii}}{2\pi i} \quad \text{as } \mathscr{R}_1, \mathscr{R}_2 \to 0. \tag{5.18}
$$

Combining (5.17) and (5.18), we have

$$
\vartheta(\xi_1, \xi_2, \tau) \to 1 + e^{\phi_1} + e^{\phi_2} + e^{\phi_1+\phi_2+A_{12}}, \quad \text{as } \mathscr{R}_1, \mathscr{R}_2 \to 0.
$$

This implies that the two-quasi-periodic wave solution (4.16) degenerated to the two-soliton solution (3.13) under a small amplitude limit, that is, $(u, \mathscr{R}_1, \mathscr{R}_2) \to (u_1, 0, 0)$. □

6. Conclusions and discussions

In this paper, we investigate a generalized KdV-Caudrey-Dodd-Gibbon (KdV-CDG) equation, which can be used to describe certain situations from the fluid mechanics, ocean dynamics and plasma physics. Based on Bell polynomials, we derive the Hirota bilinear form of the generalized KdV-CDG equation, based on which, we present its N-soliton solutions with a detailed proof. Furthermore, the quasi-periodic wave solutions are obtained by using the properties of Riemman theta functions. And then we investigate the limiting behavior of the quasi-periodic solutions. As a result, we obtain the relationship between the soliton solutions and the quasi-periodic solutions, that is the one-quasi- and two-quasi-periodic wave solution that expressed by $\vartheta(\xi,\tau)$ $\vartheta(\xi_1,\xi_2,\tau)$ can be degenerated into the one- and two-soliton solutions as the amplitude $\mathscr{R}\to 0$, respectively. Finally, we hope that the discussed method is much meaningful for us to do further research nonlinear problems in mathematical physics.

Appendix: Multidimensional Bell polynomials

We narrate some necessary notations on multidimensional binary Bell polynomials concisely as follows, for details refer, for example, to Lembert and Gilson's work [2, 9, 12].

Let $f = f(x_1, x_2, \ldots, x_n)$ be a $\mathbb{C}^\infty$ multi-variables function. Considering $n = 1$, Bell polynomials are given as follows

$$Y_{nx}(f) \equiv Y_n(f_1,\ldots,f_n) = \sum \frac{n!}{s_1!\cdots s_n!(1!)^{s_1}\cdots(n!)^{s_n}} f_1^{s_1}\cdots f_n^{s_n}, \quad n = \sum_{k=1}^{n} k s_k,$$

$$Y_x(f) = f_x,\ Y_{2x}(f) = f_{2x} + f_x^2,\ Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3,\ \ldots.$$

To correlate the Bell polynomials with the Hirota D-operator, one can define the multidimensional binary Bell polynomials as follows [9]

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon,\omega) = Y_{n_1,\ldots,n_r}(f)\big|_{f_{l_1x_1,\ldots,l_rx_r}} = \begin{cases} \upsilon_{l_1x_1,\ldots,l_rx_r} & l_1+\cdots+l_r \text{ is odd},\\ \omega_{l_1x_1,\ldots,l_rx_r} & l_1+\cdots+l_r \text{ is even},\end{cases}$$

$$\mathscr{Y}_x(\upsilon,\omega) = \upsilon_x, \quad \mathscr{Y}_{2x}(\upsilon,\omega) = \upsilon_x^2 + \omega_{2x}, \quad \mathscr{Y}_{x,t}(\upsilon,\omega) = \upsilon_x\upsilon_t + \omega_{xt},$$
$$\mathscr{Y}_{3x}(\upsilon,\omega) = \upsilon_{3x} + 3\upsilon_x\omega_{2x} + \upsilon_x^3,\ \ldots.$$

The above formulas inherit the easily recognizable partial structure of the Bell polynomials.

The relationship between the $\mathscr{Y}$-polynomials and the Hirota bilinear equation $D_{x_1}^{n_1}\cdots D_{x_1}^{n_r}F\cdot G$ [10] can be presented by the following formula [9]

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon = \ln F/G, \omega = \ln FG) = (FG)^{-1} D_{x_1}^{n_1}\cdots D_{x_r}^{n_r} F\cdot G, \tag{A.1}$$

with F and G both the functions about x and t. Particularly, when $F = G$, the formula (A.1) becomes

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot F = \mathscr{Y}(0, q = 2\ln F) = \begin{cases} 0 & n_1+\cdots+n_r \text{ is odd,} \\ P_{n_1x_1,\ldots,n_rx_r}(q) & n_1+\cdots+n_r \text{ is even,} \end{cases}$$

where the P-polynomials can be substituted by an equally recognizable even part partitional structure

$$P_{2x}(q) = q_{2x},\ P_{x,t}(q) = q_{xt},\ P_{4x}(q) = q_{4x} + 3q_{2x}^2,\ P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3,\ \ldots.$$

The binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon,\omega)$ can be separated into P-polynomials and Y-polynomials

$$\begin{aligned} &(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G \\ &= \mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon,\omega)\big|_{\upsilon=\ln F/G,\omega=\ln FG} \\ &= \mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon,\upsilon+q)\big|_{\upsilon=\ln F/G,\omega=\ln FG} \\ &= \sum_{n_1+\cdots+n_r=\text{even}}\sum_{l_1=0}^{n_1}\cdots\sum_{l_r=0}^{n_r}\prod_{i=0}^{r}\binom{n_i}{l_i}P_{l_1x_1,\ldots,l_rx_r}(q)Y_{(n_1-l_1)x_1,\ldots,(n_r-l_r)x_r}(\upsilon). \end{aligned}$$

The multidimensional Bell polynomials have the following key property

$$Y_{n_1x_1,\ldots,n_rx_r}(\upsilon)\big|_{\upsilon=\ln\psi} = \psi_{n_1x_1,\ldots,n_rx_r}/\psi, \tag{A.2}$$

which shows that the binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon,\omega)$ can be linearized through use of the Hopf-Cole transformation $\upsilon = \ln\psi$, that is, $\psi = F/G$.

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