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On the r-th Root Partition Function

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Abstract. The well known partition function $p(n)$ has a long research history, where $p(n)$ denotes the number of solutions of the equation $n = a_1 + \cdots + a_k$ with integers $1 \le a_1 \le \cdots \le a_k$. In this paper, we investigate a new partition function. For any real number $r > 1$, let $p_r(n)$ be the number of solutions of the equation $n = \lfloor \sqrt[r]{a_1} \rfloor + \cdots + \lfloor \sqrt[r]{a_k} \rfloor$ with integers $1 \le a_1 \le \cdots \le a_k$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. In this paper, it is proved that $\exp(c_1 n^{r/(r+1)}) \le p_r(n) \le \exp(c_2 n^{r/(r+1)})$ for two positive constants c_1 and c_2 (depending only r).

1. Introduction

Let $f(n)$ be a real valued arithmetic function and $q_f(n)$ be the number of solutions to the equation

$$n = \lfloor f(a_1) \rfloor + \lfloor f(a_2) \rfloor + \cdots + \lfloor f(a_k) \rfloor \tag{1.1}$$

with integers $1 \le a_1 \le \cdots \le a_k$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. We call (1.1) a *f-partition* of n and $q_f(n)$ the *f-partition function*. For $f(n) = \sqrt[r]{n}$ (where $\sqrt[r]{n}$ stands for $n^{1/r}$), let $p_r(n) = q_f(n)$, where r is a positive real number. That is, $p_r(n)$ is the number of solutions to the equation

$$n = \lfloor \sqrt[r]{a_1} \rfloor + \cdots + \lfloor \sqrt[r]{a_k} \rfloor \tag{1.2}$$

with integers $1 \le a_1 \le \cdots \le a_k$. We call (1.2) an *$r$-th root partition* of n and $p_r(n)$ the *r-th root partition function*. It is known that, when $r = 2$, there exist two explicit positive constants c_1' and c_2' such that

$$\exp(c_1' n^{2/3}) \le p_2(n) \le \exp(c_2' n^{2/3})$$

for all integers $n \ge 1$ (see [1] and [2]).

In this paper, the following results are proved.

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Theorem 1.1. *Let $f(n)$ be a real valued arithmetic function and*

$$w(z) = 1 + \sum_{n=1}^{\infty} q_f(n) z^n, \quad 0 < z < 1. \tag{1.3}$$

Suppose that the series in (1.3) *is convergent. Then*

$$w(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-\Delta_f(n)},$$

where $\Delta_f(n) = \#\{m : \lfloor f(m) \rfloor = n\}$.

Theorem 1.2. *For any real number $r > 1$, there exist two explicit positive constants c_1 and c_2 (depending only r) such that*

$$\exp(c_1 n^{r/(r+1)}) \le p_r(n) \le \exp(c_2 n^{r/(r+1)})$$

for all integers $n \ge 1$.

Theorem 1.2 is mentioned in [2]. We believe that the f-partition will bring extensive study as $p(n)$.

Throughout this paper, the numbers n, k, m, etc. are positive integers.

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

A partition $\boldsymbol{b}$ of n is

$$n = b_1 + b_2 + \cdots + b_k \tag{2.1}$$

with integers $1 \le b_1 \le \cdots \le b_k$. Now we consider the f-partitions of n corresponding to (2.1)

$$n = \lfloor f(a_1) \rfloor + \lfloor f(a_2) \rfloor + \cdots + \lfloor f(a_k) \rfloor$$

with integers $1 \le a_1 \le \cdots \le a_k$ such that $b_1 = \lfloor f(a_{i_1}) \rfloor, \dots, b_k = \lfloor f(a_{i_k}) \rfloor$, where $i_1, i_2, \dots, i_k$ is a permutation of $1, 2, \dots, k$.

For a partition $\boldsymbol{b}$ of n, it is possible that there is more than one vector $(a_1, a_2, \dots, a_k)$ corresponding to $\boldsymbol{b}$. Fix a partition $\boldsymbol{b}$ of n. If m occurs h_m times in the partition $\boldsymbol{b}$ of n, then there exist some integers $j_1 < j_2 < \cdots < j_{h_m}$ such that

$$\lfloor f(a_{j_1}) \rfloor = \lfloor f(a_{j_2}) \rfloor = \cdots = \lfloor f(a_{j_{h_m}}) \rfloor = m.$$

Then $a_{j_1}, a_{j_2}, \dots, a_{j_{h_m}} \in \{t : \lfloor f(t) \rfloor = m\}$ are integers which are subjected to $1 \le a_{j_1} \le a_{j_2} \le \cdots \le a_{j_{h_m}}$. So the number of vectors $(a_{j_1}, a_{j_2}, \dots, a_{j_{h_m}})$ corresponding to $\boldsymbol{b}$ is equal to the number of nonnegative integral solutions to the equation

$$x_1 + x_2 + \cdots + x_{\Delta_f(m)} = h_m. \tag{2.2}$$

Let $R(\Delta_f(m), h_m)$ denote the number of nonnegative integral solutions to (2.2). If $h_m \neq 0$ and $\Delta_f(m) = 0$, let $R(\Delta_f(m), h_m) = 0$. If $h_m = \Delta_f(m) = 0$, let $R(\Delta_f(m), h_m) = 1$. Hence

$$q_f(n) = \sum_{\boldsymbol{b} \in P(n)} \prod_{m=1}^{n} R(\Delta_f(m), h_m),$$

where $P(n)$ is the set of all partitions of n. It is clear that $h_1 + 2h_2 + \cdots + nh_n = n$. Therefore,

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} q_f(n) z^n &= 1 + \sum_{n=1}^{\infty} \left(\sum_{\boldsymbol{b} \in P(n)} \prod_{m=1}^{n} R(\Delta_f(m), h_m) \right) z^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\sum_{\boldsymbol{b} \in P(n)} \prod_{m=1}^{n} R(\Delta_f(m), h_m) \right) z^{h_1 + 2h_2 + \cdots + nh_n} \\
&= 1 + \sum_{n=1}^{\infty} \left(\sum_{\boldsymbol{b} \in P(n)} \prod_{m=1}^{n} R(\Delta_f(m), h_m) z^{mh_m} \right) \\
&= \prod_{m=1}^{\infty} \left(\sum_{h_m=0}^{\infty} R(\Delta_f(m), h_m) z^{mh_m} \right) \\
&= \prod_{m=1}^{\infty} \left(\sum_{t_1=0}^{\infty} \sum_{t_2=0}^{\infty} \cdots \sum_{t_{\Delta_f(m)}=0}^{\infty} z^{m\left(t_1 + t_2 + \cdots + t_{\Delta_f(m)}\right)} \right) \\
&= \prod_{m=1}^{\infty} \left(\sum_{t_1=0}^{\infty} z^{t_1 m} \sum_{t_2=0}^{\infty} z^{t_2 m} \cdots \sum_{t_{\Delta_f(m)}=0}^{\infty} z^{t_{\Delta_f(m)} m} \right) \\
&= \prod_{m=1}^{\infty} (1 - z^m)^{-\Delta_f(m)}.
\end{aligned}$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

All constants c_i depend only on r. In this section $f(x) = \sqrt[r]{x}$. Then $q_f(n) = p_r(n)$. For any real number $r > 1$, let

$$g_r(n) = \# \left\{ k : \left\lfloor \sqrt[r]{k} \right\rfloor = n \right\}.$$

That is, $g_r(n) = \Delta_f(n)$.

Lemma 3.1. *Let r be a real number with $r > 1$. Then*

$$p_r(n) \geq p_r(n-1) + g_r(n) \ (n \geq 2), \quad p_r(n) \geq n + 1 \ (n \geq 1).$$

Proof. If $n-1=\lfloor\sqrt[r]{a_1}\rfloor+\cdots+\lfloor\sqrt[r]{a_k}\rfloor$ is an r-th root partition of $n-1$, then $n=\lfloor\sqrt[r]{a_0}\rfloor+\lfloor\sqrt[r]{a_1}\rfloor+\cdots+\lfloor\sqrt[r]{a_k}\rfloor$ $(a_0=1)$ is an r-th root partition of n. Since $n=\lfloor\sqrt[r]{b_1}\rfloor$ $(n^r\le b_1<(n+1)^r)$ are $g_r(n)$ r-th root partitions of n which can not be obtained from any r-th root partition of $n-1$, we have

$$p_r(n)\ge p_r(n-1)+g_r(n)\quad (n\ge 2).$$

It follows from $p_r(1)\ge 2$ and $g_r(n)\ge 1$ that

$$p_r(n)\ge p_r(n-1)+1\ge\cdots\ge p_r(1)+n-1\ge n+1.$$ □

Lemma 3.2. *Let $0<z<1$ and $r>1$. Then*

$$\frac{c_3 z}{(1-z)^r}\le\sum_{n=1}^{\infty}n^{r-1}z^n\le\frac{c_4 z}{(1-z)^r},$$

where c_3 and c_4 are two explicit positive constants.

Proof. It is known that the Gamma function has the following property

$$\Gamma(t)=\lim_{n\to\infty}\frac{n!n^t}{t(t+1)\cdots(t+n)}.$$

Hence

$$\begin{aligned}\Gamma(r)&=\lim_{n\to\infty}\frac{n!n^r}{r(r+1)\cdots(r+n)}\\&=\lim_{n\to\infty}\frac{(n-1)!n^{r-1}}{r(r+1)\cdots(r+n-2)}.\end{aligned}$$

Thus there exist two explicit positive constants c_3 and c_4 (depending only on r) such that

$$c_3\le\frac{(n-1)!n^{r-1}}{r(r+1)\cdots(r+n-2)}\le c_4.$$

That is,

$$c_3\frac{r(r+1)\cdots(r+n-2)}{(n-1)!}\le n^{r-1}\le c_4\frac{r(r+1)\cdots(r+n-2)}{(n-1)!}.$$

Here and later, we consider $r(r+1)\cdots(r+n-2)$ as 1 if $n=1$. Since

$$\begin{aligned}\frac{z}{(1-z)^r}&=z\sum_{n=0}^{\infty}\frac{(-r)(-r-1)\cdots(-r-n+1)}{n!}(-z)^n\\&=\sum_{n=0}^{\infty}\frac{r(r+1)\cdots(r+n-1)}{n!}z^{n+1}\\&=\sum_{n=1}^{\infty}\frac{r(r+1)\cdots(r+n-2)}{(n-1)!}z^n,\end{aligned}$$

it follows that

$$\frac{c_3 z}{(1-z)^r}\le\sum_{n=1}^{\infty}n^{r-1}z^n\le\frac{c_4 z}{(1-z)^r}.$$ □

Lemma 3.3. *Let r be a real number with $r > 1$. Then*

$$(r-1)n^{r-1} < g_r(n) < r2^r n^{r-1}.$$

Proof. Since

$$g_r(n) = \#\{k : n^r \le k < (n+1)^r\} = \#\{k : \lceil n^r \rceil \le k < \lceil (n+1)^r \rceil\},$$

it follows that

$$g_r(n) = \lceil (n+1)^r \rceil - \lceil n^r \rceil = (n+1)^r - n^r + \Delta,$$

where $|\Delta| < 1$ and $\lceil x \rceil$ denotes the least integer not less than x. By the Lagrange mean value theorem, we have

$$g_r(n) = r\xi^{r-1} + \Delta$$

for some real number $\xi \in (n, n+1)$. Then,

$$(r-1)n^{r-1} \le rn^{r-1} - 1 < g_r(n) < r(n+1)^{r-1} + 1 < 2r(n+1)^{r-1} \le r2^r n^{r-1}. \qquad \square$$

Lemma 3.4. *We have*

$$\frac{c_5 z}{(1-z)^r} < \log w(z) < \frac{c_6 z}{(1-z)^r}, \quad 0 < z < 1, \tag{3.1}$$

where c_5 and c_6 are two explicit positive constants.

Proof. It follows from Theorem 1.1 that (noting that $\Delta_f(n) = g_r(n)$ for $f(x) = \sqrt[r]{x}$)

$$w(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-g_r(n)}.$$

Thus

$$\log w(z) = -\sum_{n=1}^{\infty} g_r(n) \log(1 - z^n). \tag{3.2}$$

By Lemma 3.3 and (3.2), we have

$$\log w(z) > \sum_{n=1}^{\infty} g_r(n) z^n > (r-1) \sum_{n=1}^{\infty} n^{r-1} z^n, \tag{3.3}$$

$$\log w(z) < -r2^r \sum_{n=1}^{\infty} n^{r-1} \log(1 - z^n). \tag{3.4}$$

By Lemma 3.2 and (3.3), we obtain the lower bound of (3.1).

Now we prove the upper bound of (3.1) by (3.4). By Lemma 3.2,

$$\begin{aligned}
-\sum_{n=1}^{\infty} n^{r-1}\log(1-z^n) &= \sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{\infty}\frac{z^{kn}}{k}\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty} n^{r-1}z^{kn}\\
&< c_4\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k}{(1-z^k)^r}\\
&= \frac{c_4}{(1-z)^r}\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k}{(1+z+\cdots+z^{k-1})^r}\\
&< \frac{c_4 z}{(1-z)^r}\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^{k-1}}{1+z+\cdots+z^{k-1}}\\
&< \frac{c_4 z}{(1-z)^r}\sum_{k=1}^{\infty}\frac{1}{k^2}\\
&= \frac{\pi^2}{6}\frac{c_4 z}{(1-z)^r}.
\end{aligned}$$

Therefore, in combination with (3.4) we obtain the upper bound of (3.1). □

Proof of Theorem 1.2. For $f(x)=\sqrt[r]{x}$, we have $q_f(n)=p_r(n)$. Since

$$\log x > \frac{x-1}{x}$$

for any real number $x\in(0,1)$ and using Lemma 3.4, it follows from (1.3) that, for $0<z<1$, we have

$$\log(p_r(n)z^n) \le \log w(z) \le c_6\frac{z}{(1-z)^r} < \frac{c_6}{z^{r-1}(-\log z)^r}.$$

That is,

$$\log p_r(n) \le \frac{c_6}{z^{r-1}(-\log z)^r} - n\log z.$$

We choose $z_1=\exp(-n^{-1/(r+1)})$. Then $z_1\ge e^{-1}$ and

$$\log p_r(n) \le \frac{c_6}{e^{-r+1}(-\log z_1)^r} - n\log z_1 = c_2 n^{r/(r+1)}, \tag{3.5}$$

where $c_2=c_6e^{r-1}+1$. Thus, we have proved the upper bound in Theorem 1.2.

We will use this upper bound to give a lower bound of $\log p_r(n)$. For $0<z<1$, by Lemma 3.1 and (3.5), we have

$$\begin{aligned}
w(z) &= 1+\sum_{k=1}^{n-1}p_r(k)z^k+\sum_{k=n}^{\infty}p_r(k)z^k\\
&\le np_r(n)+\sum_{k=n}^{\infty}\exp(c_2k^{r/(r+1)})z^k.
\end{aligned}$$

We choose $z_2 = \exp\left(-(r+1)c_2 n^{-1/(r+1)}\right)$. Then

$$\exp\left(c_2 k^{r/(r+1)}\right) z_2^{k/(r+1)} = \exp\left(c_2\left(k^{r/(r+1)} - kn^{-1/(r+1)}\right)\right) \le 1, \quad k \ge n.$$

Thus, by Lemma 3.1 and $c_2 > 1$, we have

$$\begin{aligned} w(z_2) &\le np_r(n) + \sum_{k=n}^{\infty} z_2^{kr/(r+1)} = np_r(n) + \frac{z_2^{nr/(r+1)}}{1 - z_2^{r/(r+1)}} \\ &= np_r(n) + \frac{\exp\left(-c_2 r n^{r/(r+1)}\right)}{1 - \exp\left(-c_2 r n^{-1/(r+1)}\right)} \\ &= np_r(n) + \frac{\exp\left(-c_2 r\left(n^{r/(r+1)} - n^{-1/(r+1)}\right)\right)}{\exp\left(c_2 r n^{-1/(r+1)}\right) - 1} \\ &< np_r(n) + \frac{1}{c_2 r} n^{1/(r+1)} \exp\left(-c_2 r\left(n^{r/(r+1)} - n^{-1/(r+1)}\right)\right) \\ &< np_r(n) + \frac{1}{c_2 r} n < np_r(n) + n = (n+1)p_r(n) \le p_r(n)^2. \end{aligned}$$

Since $\log x < x - 1$ for any real number $x \in (0,1)$ and using Lemma 3.4, it follows that

$$\log p_r(n) \ge \frac{1}{2}\log w(z_2) \ge \frac{c_5 z_2}{2(1-z_2)^r} \ge \frac{c_5 z_2}{2(-\log z_2)^r} \ge c_1 n^{r/(r+1)},$$

where

$$c_1 = \frac{c_5 \exp(-(r+1)c_2)}{2(r+1)^r c_2^r}.$$

Thus, we have proved the lower bound in Theorem 1.2. This completes the proof. □

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