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Log-concavity of the Fennessey-Larcombe-French Sequence

Arthur L.B. Yang and James J.Y. Zhao*

Abstract. We prove the log-concavity of the Fennessey-Larcombe-French sequence based on its three-term recurrence relation, which was recently conjectured by Zhao. The key ingredient of our approach is a sufficient condition for log-concavity of a sequence subject to certain three-term recurrence.

1. Introduction

The objective of this paper is to prove the log-concavity conjecture of the Fennessey-Larcombe-French sequence, which was posed by Zhao [17] in the study of log-balancedness of combinatorial sequences.

Let us begin with an overview of Zhao's conjecture. Recall that a sequence $\{a_k\}_{k\geq 0}$ is said to be log-concave if

$$a_k^2 \geq a_{k+1}a_{k-1}, \quad \text{for } k \geq 1,$$

and it is log-convex if

$$a_k^2 \leq a_{k+1}a_{k-1}, \quad \text{for } k \geq 1.$$

We say that $\{a_k\}_{k\geq 0}$ is log-balanced if the sequence itself is log-convex while $\left\{\frac{a_k}{k!}\right\}_{k\geq 0}$ is log-concave.

The Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$ can be given by the following three-term recurrence relation [9]

$$n(n+1)^2V_{n+1} = 8n(3n^2+5n+1)V_n - 128(n-1)(n+1)^2V_{n-1}, \quad \text{for } n \geq 1, \tag{1.1}$$

with the initial values $V_0 = 1$ and $V_1 = 8$. This sequence was introduced by Larcombe, French and Fennessey [8], in connection with a series expansion of the complete elliptic integral of the second kind, precisely,

$$\int_0^{\pi/2} \sqrt{1-c^2\sin^2\theta}\, d\theta = \frac{\pi\sqrt{1-c^2}}{2}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{1-c^2}}{16}\right)^n V_n.$$

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*Corresponding author.

The Fennessey-Larcombe-French sequence is closely related to the Catalan-Larcombe-French sequence, which was first studied by E. Catalan [1] and later examined and clarified by Larcombe and French [7]. Let $\{P_n\}_{n\geq 0}$ denote the Catalan-Larcombe-French sequence, and the following three-term recurrence relation holds:

$$(n+1)^2 P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2 P_{n-1}, \quad \text{for } n \geq 1,$$

with $P_0 = 1$ and $P_1 = 8$. As a counterpart of V_n, the numbers P_n appear as coefficients in the series expansion of the complete elliptic integral of the first kind, precisely,

$$\int_0^{\pi/2} \frac{1}{\sqrt{1-c^2\sin^2\theta}}\, d\theta = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{1-c^2}}{16}\right)^n P_n.$$

Many interesting properties have been found for the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence, and the reader may consult references [5–9,15].

Recently, there has arisen an interest in the study of the log-behavior of the Catalan-Larcombe-French sequence. For instance, Xia and Yao [14] obtained the log-convexity of the Catalan-Larcombe-French sequence, and confirmed a conjecture of Sun [12]. By using a log-balancedness criterion due to Došlić [4], Zhao [16] proved the log-balancedness of the Catalan-Larcombe-French sequence.

Zhao further studied the log-behavior of the Fennessey-Larcombe-French sequence, and obtained the following result.

Theorem 1.1. [17] *Both* $\{nV_n\}_{n\geq 1}$ *and* $\left\{\frac{V_n}{(n-1)!}\right\}_{n\geq 1}$ *are log-concave.*

She also made the following conjecture.

Conjecture 1.2. *The Fennessey-Larcombe-French sequence* $\{V_n\}_{n\geq 1}$ *is log-concave.*

Note that the Hadamard product of two log-concave sequences without internal zeros is still log-concave, see [11, Proposition 2]. Since both $\{n\}_{n\geq 1}$ and $\left\{\frac{1}{(n-1)!}\right\}_{n\geq 1}$ are log-concave, Conjecture 1.2 implies Theorem 1.1.

In this paper, we obtain a sufficient condition for proving the log-concavity of a sequence satisfying a three-term recurrence. Then we give an affirmative answer to Conjecture 1.2 by using this criterion. By further employing a result of Wang and Zhu [13, Theorem 2.1], we derive the monotonicity of the sequence $\left\{\sqrt[n]{V_{n+1}}\right\}_{n\geq 1}$ from the log-concavity of $\{V_n\}_{n\geq 1}$.

2. Log-concavity derived from three-term recurrence

The aim of this section is to prove the log-concavity of the Fennessey-Larcombe-French sequence based on its three-term recurrence relation.

We first give a sufficient condition for log-concavity of a positive sequence subject to certain three-term recurrence. It should be mentioned that the log-behavior of sequences satisfying three-term recurrences has been extensively studied, see Liu and Wang [10], Chen and Xia [3], Chen, Guo and Wang [2], and Wang and Zhu [13]. However, most of these studies have focused on the log-convexity of such sequences instead of their log-concavity. Our criterion for determining the log-concavity of a sequence satisfying a three-term recurrence is as follows.

Proposition 2.1. *Let* $\{S_n\}_{n\geq 0}$ *be a positive sequence satisfying the following recurrence relation:*

$$a(n)S_{n+1} + b(n)S_n + c(n)S_{n-1} = 0, \quad \text{for } n \geq 1, \tag{2.1}$$

where $a(n)$*,* $b(n)$ *and* $c(n)$ *are real functions of* n*. Suppose that there exists an integer* n_0 *such that for any* $n > n_0$*,*

(i) *it holds* $a(n) > 0$*, and*

(ii) *either* $b^2(n) < 4a(n)c(n)$ *or* $\frac{S_n}{S_{n-1}} \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}$.

Then the sequence $\{S_n\}_{n\geq n_0}$ *is log-concave, namely,* $S_n^2 \geq S_{n+1}S_{n-1}$ *for any* $n > n_0$*.*

Proof. Let $r(n) = S_n/S_{n-1}$. It suffices to show that $r(n) \geq r(n+1)$ for any $n > n_0$. On one hand, the conditions (i) and (ii) imply that

$$a(n)r^2(n) + b(n)r(n) + c(n) \geq 0, \quad \text{for } n > n_0.$$

Since $\{S_n\}_{n\geq 0}$ is a positive sequence, so is $\{r_n\}_{n\geq 1}$. Thus, the above inequality is equivalent to the following

$$a(n)r(n) + b(n) + \frac{c(n)}{r(n)} \geq 0, \quad \text{for } n > n_0. \tag{2.2}$$

On the other hand, dividing both sides of (2.1) by S_n, we obtain

$$a(n)r(n+1) + b(n) + \frac{c(n)}{r(n)} = 0. \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$a(n)r(n+1) \leq a(n)r(n), \quad \text{for } n > n_0.$$

By the condition (i), we have $r(n+1) \leq r(n)$ for any $n > n_0$. This completes the proof. □

We are now able to give the main result of this section, which offers an affirmative answer to Conjecture 1.2.

Theorem 2.2. *Let $\{V_n\}_{n\geq 0}$ be the Fennessey-Larcombe-French sequence given by* (1.1)*. Then, for any $n \geq 2$, we have $V_n^2 \geq V_{n-1}V_{n+1}$.*

Proof. By the recurrence relation (1.1), we have $V_1 = 8$, $V_2 = 144$, $V_3 = 2432$ and $V_4 = 40000$. It is easy to verify that $V_2^2 \geq V_1V_3$ and $V_3^2 \geq V_2V_4$.

We proceed to use Proposition 2.1 to prove that $V_n^2 > V_{n-1}V_{n+1}$ for $n > 3$, namely taking $n_0 = 3$. For the sequence $\{V_n\}_{n\geq 0}$, the corresponding polynomials $a(n)$, $b(n)$, $c(n)$ appearing in Proposition 2.1 are as follows:

$$\begin{aligned} a(n) &= n(n+1)^2, \\ b(n) &= -8n(3n^2+5n+1), \\ c(n) &= 128(n-1)(n+1)^2. \end{aligned}$$

It is clear that $a(n) > 0$ for any $n > 3$. By a routine computation, we get

$$b^2(n) - 4a(n)c(n) = 64(n^6 + 6n^5 + 15n^4 + 26n^3 + 25n^2 + 8n) > 0, \quad \text{for } n > 3.$$

It suffices to show that

$$\frac{V_n}{V_{n-1}} \geq \frac{-b(n) + \sqrt{b^2(n) - 4a(n)c(n)}}{2a(n)}, \quad \text{for } n > 3. \tag{2.4}$$

This inequality also implies the positivity of V_n since its right-hand side is positive for any $n > 3$. (Note that $b(n)$ is negative.) However, it is difficult to directly prove (2.4). The key idea of our proof is to find an intermediate function $h(n)$ such that

$$\frac{V_n}{V_{n-1}} \geq h(n) \geq \frac{-b(n) + \sqrt{b^2(n) - 4a(n)c(n)}}{2a(n)}, \quad \text{for } n > 3.$$

Let

$$h(n) = \frac{16(n^3 - n^2 + 1)}{n^3 - n^2}, \quad \text{for } n \geq 2, \tag{2.5}$$

and we shall show that this function fulfills our purpose. This will be done in two steps.

First, we need to prove that

$$h(n) - \frac{-b(n) + \sqrt{b^2(n) - 4a(n)c(n)}}{2a(n)} \geq 0, \quad \text{for } n > 3. \tag{2.6}$$

A straightforward computation shows that the quantity on the left-hand side is equal to

$$\frac{32(4n^6 + 7n^5 + n^4 + n^3 + 9n^2 + 8n + 2)}{(n^4 - n^2)(n+1)(n^5 + 2n^4 + n^2 + 8n + 4 + (n^2 - n)\sqrt{n^6 + 6n^5 + 15n^4 + 26n^3 + 25n^2 + 8n})},$$

which is clearly positive for $n > 3$.

Secondly, we need to prove that

$$\frac{V_n}{V_{n-1}} \geq h(n), \quad \text{for } n > 3. \tag{2.7}$$

For convenience, let $g(n) = V_n/V_{n-1}$. We use induction on n to prove that $g(n) \geq h(n)$ for $n > 3$. By the recurrence relation (1.1), we have

$$g(n+1) = \frac{8(3n^2+5n+1)}{(n+1)^2} - \frac{128(n-1)}{ng(n)}, \quad n \geq 1, \tag{2.8}$$

with the initial value $g(1) = 8$. It is clear that $g(3) = 152/9 = h(3)$ and $g(4) = 625/38 > 49/3 = h(4)$ by (2.5) and (2.8). Assume that $g(n) > h(n)$, and we proceed to show that $g(n+1) > h(n+1)$. Note that

$$\begin{aligned} g(n+1) - h(n+1) &= \frac{8(3n^2+5n+1)}{(n+1)^2} - \frac{128(n-1)}{ng(n)} - \frac{16(n^3+2n^2+n+1)}{n(n+1)^2} \\ &= \frac{8(n^3+n^2-n-2)}{n(n+1)^2} - \frac{128(n-1)}{ng(n)} \\ &= \frac{8(n^3+n^2-n-2)g(n) - 128(n-1)(n+1)^2}{n(n+1)^2 g(n)}. \end{aligned}$$

By the induction hypothesis, we have $g(n) > h(n) > 0$ and thus

$$\begin{aligned} g(n+1) - h(n+1) &> \frac{8(n^3+n^2-n-2)h(n) - 128(n-1)(n+1)^2}{n(n+1)^2 g(n)} \\ &= \frac{128(2n^2-n-2)}{n^3(n-1)(n+1)^2 g(n)} > 0. \end{aligned}$$

Combining (2.6) and (2.7), we obtain the inequality (2.4). This completes the proof. □

Wang and Zhu [13, Theorem 2.1] showed that if $\{z_n\}_{n\geq 0}$ is a log-concave sequence of positive integers with $z_0 > 1$, then $\left\{\sqrt[n]{z_n}\right\}_{n\geq 1}$ is strictly decreasing. Applying their criterion to the Fennessey-Larcombe-French sequence, we obtain immediately the following result.

Proposition 2.3. *The sequence* $\left\{\sqrt[n]{V_{n+1}}\right\}_{n\geq 1}$ *is strictly decreasing.*

Proof. Let $\{z_n\}_{n\geq 0}$ be the sequence given by $z_n = V_{n+1}$. It is clear that $z_0 = V_1 = 8 > 1$. Moreover, by Theorem 2.2, the sequence $\{z_n\}_{n\geq 0}$ is log-concave. Thus, $\left\{\sqrt[n]{z_n}\right\}_{n\geq 1}$ is strictly decreasing by [13, Theorem 2.1]. □

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Arthur L.B. Yang
Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China
E-mail address: yang@nankai.edu.cn

James J.Y. Zhao
Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China
E-mail address: jjyzhao@tju.edu.cn