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An Evolutionary Property of the Bifurcation Curves for a Positone Problem with Cubic Nonlinearity

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Abstract. We study an evolutionary property of the bifurcation curves for a positone problem with cubic nonlinearity

$$\begin{cases} u''(x) + \lambda f(u) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \\ f(u) = -\varepsilon u^3 + \sigma u^2 + \tau u + \rho, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameters, $\varepsilon > 0$ is an evolution parameter, and σ, $\rho > 0$, $\tau \geq 0$ are constants. In addition, we improve lower and upper bounds of the critical bifurcation value $\widetilde{\varepsilon}$ of the problem.

1. Introduction

In this paper we mainly study an evolutionary property of the bifurcation curves for a positone problem with cubic nonlinearity

$$(1.1) \qquad \begin{cases} u''(x) + \lambda f(u) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \\ f(u) = -\varepsilon u^3 + \sigma u^2 + \tau u + \rho, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameters, $\varepsilon > 0$ is an evolution parameter, and $\sigma, \rho > 0$, $\tau \geq 0$ are constants. Problems about bifurcation curves have been widely studied by many authors, cf. [1–3, 5–7, 9, 10]. For any $\varepsilon > 0$, it is easy to see that there exist a positive number β_ε which is the unique positive zero of $f(u)$, and a positive number $\gamma = \sigma/(3\varepsilon) < \beta_\varepsilon$, which is the unique (positive) inflection point of $f(u)$, such that cubic polynomial f satisfies

(i) $f(0) = \rho > 0$ (positone), $f'(0) = \tau \geq 0$, $f(u) > 0$ on $(0, \beta_\varepsilon)$, and $f(\beta_\varepsilon) = 0$,

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(ii) $f(u)$ is strictly convex on $(0,\gamma)$ and is strictly concave on $(\gamma,\beta_\varepsilon)$. (So f is convex-concave on $(0,\beta_\varepsilon)$.)

For any $\varepsilon>0$, on the $(\lambda,\|u\|_\infty)$-plane, we study the evolution of bifurcation curves S_ε of positive solutions of (1.1), defined by

$$S_\varepsilon \equiv \{(\lambda,\|u_\lambda\|_\infty) : \lambda>0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that, on the $(\lambda,\|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped if S_ε is a continuous curve and there exist two positive numbers $\lambda_*<\lambda^*$ such that S_ε has *exactly two* turning points at some points $(\lambda^*,\|u_{\lambda^*}\|_\infty)$ and $(\lambda_*,\|u_{\lambda_*}\|_\infty)$, and

(i) $\lambda_*<\lambda^*$ and $\|u_{\lambda^*}\|_\infty<\|u_{\lambda_*}\|_\infty$,

(ii) at $(\lambda^*,\|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_ε turns to the *left*,

(iii) at $(\lambda_*,\|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_ε turns to the *right*.

See Figure 1.1(i) for example.

Hung and Wang [6, Theorem 2.1] recently proved the global bifurcation of bifurcation curves S_ε of (1.1) and gave lower and upper bounds of the critical bifurcation value $\widetilde{\varepsilon}$.

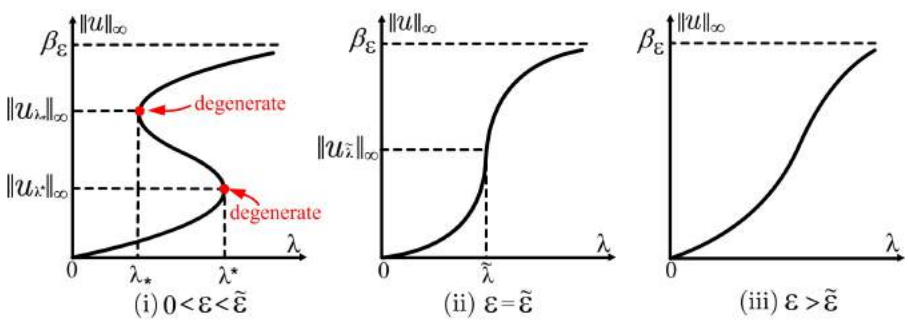

Figure 1.1: Global bifurcation of bifurcation curves S_ε of (1.1) with varying $\varepsilon>0$.

Theorem 1.1. *Consider* (1.1) *with varying* $\varepsilon>0$. *Then there exists a critical value* $\widetilde{\varepsilon}=\widetilde{\varepsilon}(\sigma,\rho,\tau)$ *satisfying*

$$\left(0.170\sqrt{\frac{\sigma^3}{\rho}}\approx\right)\sqrt{\frac{25\sigma^3}{864\rho}}<\widetilde{\varepsilon}<\sqrt{\frac{\sigma^3}{27\rho}}\left(\approx 0.192\sqrt{\frac{\sigma^3}{\rho}}\right) \tag{1.2}$$

such that the following assertions (i)–(iii) *holds:*

(i) *(See Figure* 1.1(i)*.) For* $0<\varepsilon<\widetilde{\varepsilon}$*, the bifurcation curve* S_ε *is* S*-shaped on the* $(\lambda,\|u\|_\infty)$*-plane. Let* $(\lambda^*,\|u_{\lambda^*}\|_\infty)$ *and* $(\lambda_*,\|u_{\lambda_*}\|_\infty)$ *be exactly two turning points of the bifurcation curve* S_a *satisfying* $\lambda_*<\lambda^*$ *and* $\|u_{\lambda^*}\|_\infty<\|u_{\lambda_*}\|_\infty$*. Then* u_{λ_*} *and* u_{λ^*} *are only two degenerate positive solutions of* (1.1).

(ii) *(See Figure* 1.1(ii).*) For $\varepsilon = \widetilde{\varepsilon}$, the bifurcation curve $S_{\widetilde{\varepsilon}}$ is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. Moreover,* (1.1) *has exactly one (cusp type) degenerate positive solution $u_{\widetilde{\lambda}}$.*

(iii) *(See Figure* 1.1(iii).*) For $\varepsilon > \widetilde{\varepsilon}$, the bifurcation curve S_ε is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. Moreover, all positive solutions u_λ of* (1.1) *are nondegenerate.*

The paper is organized as follows: Section 2 contains the main result (Theorem 2.1). Section 3 contains lemmas needed to prove the main result. Section 3 also contains the proofs of lemmas in this section except Lemma 3.9 and 3.10(ii), and assertions (3.30), (3.52) and (3.54). Note that the proofs of Lemmas 3.9 and 3.10(ii) and assertions (3.30), (3.52) and (3.54) are easy but tedious. Thus, we omit them in this paper and put them in [4]. Finally, Section 4 contains the proof of the main result.

2. Main result

The main result in this paper is Theorems 2.1. For $0 < \varepsilon < \widetilde{\varepsilon}$, let $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ be the exactly two turning points of the S-shaped bifurcation curve S_ε satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$. In Theorem 2.1, we show the variation of the values of $\|u_{\lambda^*}\|_\infty$ and $\|u_{\lambda_*}\|_\infty$ with varying parameter $\varepsilon \in (0, \widetilde{\varepsilon})$. In addition, we improve lower and upper bounds of the critical bifurcation value $\widetilde{\varepsilon}$ given in (1.2). Notice that, for $0 < \varepsilon < \widetilde{\varepsilon}$, the cubic polynomial $f(u) = -\varepsilon u^3 + \sigma u^2 + \tau u + \rho$ has a unique inflection point at $\gamma = \sigma/(3\varepsilon) < \beta_\varepsilon$, and there exist two positive numbers p_1, p_2 satisfying $p_1 < \gamma < p_2$, which are positive zeros of cubic polynomial

$$f(u) - uf'(u) = 2\varepsilon u^3 - \sigma u^2 + \rho. \tag{2.1}$$

(The numbers p_1 and p_2 both exist for $0 < \varepsilon < \widetilde{\varepsilon}$, see Lemma 3.1 stated below.) That is, both the y-intercepts of the tangent lines to the graph of cubic polynomial f at the points $(p_1, f(p_1))$ and $(p_2, f(p_2))$ equal 0. These three values γ, p_1 and p_2 play important roles in the variation of the values of $\|u_{\lambda^*}\|_\infty$ and $\|u_{\lambda_*}\|_\infty$ with varying parameter $\varepsilon \in (0, \widetilde{\varepsilon})$. Theorem 2.1 provides more complete structures on the global bifurcation curves S_ε of (1.1) with varying parameter $\varepsilon \in (0, \widetilde{\varepsilon})$, cf. Theorem 1.1.

Theorem 2.1. *(See Figures* 1.1(i)–(ii) *and* 2.1.*) Consider* (1.1) *with varying $\varepsilon \in (0, \widetilde{\varepsilon})$. Then there exist two positive numbers $\widehat{\varepsilon}$ and $\bar{\varepsilon}$ satisfying*

$$\widehat{\varepsilon} < \left(0.176\sqrt{\frac{\sigma^3}{\rho}} \approx\right) \sqrt{\frac{31\sigma^3}{1000\rho}} < \bar{\varepsilon} < \widetilde{\varepsilon} < \sqrt{\frac{83\sigma^3}{2500\rho}} \left(\approx 0.182\sqrt{\frac{\sigma^3}{\rho}}\right), \tag{2.2}$$

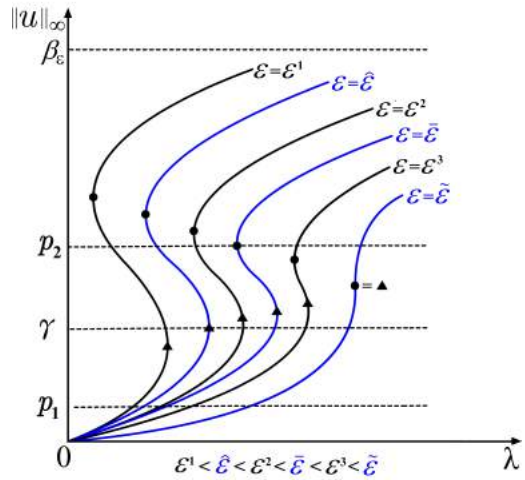

Figure 2.1: Evolutionary bifurcation curves S_ε with varying $\varepsilon \in (0, \widetilde{\varepsilon})$. The two notations $\bullet$ and $\blacktriangle$ denote the two turning points $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ and $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, respectively.

such that

$$p_1 < \|u_{\lambda^*}\|_\infty < \gamma < p_2 < \|u_{\lambda_*}\|_\infty \quad \textit{for } 0 < \varepsilon < \widehat{\varepsilon}, \tag{2.3}$$

$$p_1 < \gamma = \|u_{\lambda^*}\|_\infty < p_2 < \|u_{\lambda_*}\|_\infty \quad \textit{for } \varepsilon = \widehat{\varepsilon}, \tag{2.4}$$

$$p_1 < \gamma < \|u_{\lambda^*}\|_\infty < p_2 < \|u_{\lambda_*}\|_\infty \quad \textit{for } \widehat{\varepsilon} < \varepsilon < \overline{\varepsilon}, \tag{2.5}$$

$$p_1 < \gamma < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty = p_2 \quad \textit{for } \varepsilon = \overline{\varepsilon}, \tag{2.6}$$

$$p_1 < \gamma < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty < p_2 \quad \textit{for } \overline{\varepsilon} < \varepsilon < \widetilde{\varepsilon}, \tag{2.7}$$

$$p_1|_{\varepsilon=\widetilde{\varepsilon}} < \gamma|_{\varepsilon=\widetilde{\varepsilon}} < \lim_{\varepsilon\to\widetilde{\varepsilon}^+} \|u_{\lambda^*}\|_\infty = \lim_{\varepsilon\to\widetilde{\varepsilon}^+} \|u_{\lambda^*}\|_\infty = \left\|u_{\widetilde{\lambda}}\right\|_\infty < p_2|_{\varepsilon=\widetilde{\varepsilon}}, \tag{2.8}$$

where $u_{\widetilde{\lambda}}$ *is defined in Theorem* 1.1(ii).

3. Lemmas

To prove Theorem 2.1, we develop some new time-map techniques. The time-map formula which we apply to study (1.1) with $f(u) = -\varepsilon u^3 + \sigma u^2 + \tau u + \rho$ takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2}\, du \equiv T_\varepsilon(\alpha) \quad \text{for } 0 < \alpha < \beta_\varepsilon \text{ if } \varepsilon > 0, \tag{3.1}$$

where $F(u) \equiv \int_0^u f(t)\, dt = -\frac{1}{4}\varepsilon u^4 + \frac{1}{3}\sigma u^3 + \frac{1}{2}\tau u^2 + \rho u$ and β_ε is the unique positive zero of the cubic polynomial $f(u)$ for $\varepsilon > 0$; see Laetsch [8]. Note that it can be proved that $T_\varepsilon(\alpha)$ is a twice differentiable function of $\alpha \in (0, \beta_\varepsilon)$ for $\varepsilon > 0$, and is differentiable function of $\varepsilon \in (0, \infty)$ for $\alpha > 0$. The proofs are easy but tedious and hence we omit them. In addition, by (3.1) and Theorem 1.1, we note that (i) if $0 < \varepsilon < \widetilde{\varepsilon}$, $T_\varepsilon(\alpha)$ has exactly two critical points at $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$ where $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ are exactly two turning points of the S-shaped bifurcation curve S_ε; (ii) if $\varepsilon = \widetilde{\varepsilon}$, $T_\varepsilon(\alpha)$ has exactly one critical point at $\|u_{\widetilde{\lambda}}\|_\infty$ where $(\widetilde{\lambda}, \|u_{\widetilde{\lambda}}\|_\infty)$ is the unique turning point of the monotone bifurcation curve $S_{\widetilde{\varepsilon}}$; (iii) if $\varepsilon > \widetilde{\varepsilon}$, $T_\varepsilon(\alpha)$ is a strictly decreasing function on $(0, \infty)$. See Figure 1.1.

For the sake of convenience, we let

$$\varepsilon_1 \equiv \sqrt{\frac{7\sigma^3}{270\rho}},\ \varepsilon_2 \equiv \sqrt{\frac{25\sigma^3}{864\rho}},\ \varepsilon_3 \equiv \sqrt{\frac{31\sigma^3}{1000\rho}},\ \varepsilon_4 \equiv \sqrt{\frac{13\sigma^3}{400\rho}},\ \varepsilon_5 \equiv \sqrt{\frac{\sigma^3}{27\rho}}.$$

Clearly, $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < \varepsilon_5$ for any $\sigma, \rho > 0$. In addition, $\widetilde{\varepsilon} < \varepsilon_5 = \sqrt{\frac{\sigma^3}{27\rho}}$ by (1.2). For $T_\varepsilon(\alpha)$ in (3.1), we compute that

$$T'_\varepsilon(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{[F(\alpha) - F(u)]^{3/2}}\, du, \tag{3.2}$$

$$T''_\varepsilon(\alpha) = \frac{1}{2\sqrt{2}\alpha^2} \int_0^\alpha \frac{-\frac{3}{2}[\theta(\alpha) - \theta(u)][\alpha f(\alpha) - uf(u)] + [F(\alpha) - F(u)][\alpha\theta'(\alpha) - u\theta'(u)]}{[F(\alpha) - F(u)]^{5/2}}\, du, \tag{3.3}$$

$$\frac{\partial}{\partial\varepsilon} T'_\varepsilon(\alpha) = \frac{1}{96\sqrt{2}\varepsilon\alpha} \int_0^\alpha \frac{A_\alpha(3A_\alpha + 2B_\alpha + 12C_\alpha + 42D_\alpha)}{[F(\alpha) - F(u)]^{5/2}}\, du, \tag{3.4}$$

where $A_\alpha \equiv \varepsilon(\alpha^4 - u^4)$, $B_\alpha \equiv \sigma(\alpha^3 - u^3)$, $C_\alpha \equiv \tau(\alpha^2 - u^2)$, $D_\alpha \equiv \rho(\alpha - u)$ and

$$\theta(u) = 2F(u) - uf(u) = \frac{1}{6}u(3\varepsilon u^3 - 2\sigma u^2 + 6\rho), \tag{3.5}$$

cf. [5, (3.4) and p. 230] and [6, p. 1946]. By (3.3) and (3.4), we obtain that

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} T'_\varepsilon(w(\varepsilon)) &= T''_\varepsilon(w(\varepsilon), \varepsilon) w'(\varepsilon) + \frac{\partial}{\partial\varepsilon} T'_\varepsilon(\alpha)\big|_{\alpha = w(\varepsilon)} \\ &= \frac{1}{96\sqrt{2}\varepsilon w} \int_0^w \frac{1}{[F(w) - F(u)]^{5/2}} \\ &\qquad \times \big[3A_w^2 + 2A_w B_w + 12A_w C_w + 42A_w D_w \\ &\qquad\quad + \frac{4\varepsilon w'(\varepsilon)}{w(\varepsilon)} \big(30A_w D - 20B_w D_w - 12C_w D_w - 6D_w^2 \\ &\qquad\qquad - 4A_w B_w + 3A_w C_w + 3A_w^2 + 2B_w^2\big)\big]\, du, \end{aligned} \tag{3.6}$$

where $w = w(\varepsilon)$ is a differentiable function of $\varepsilon > 0$.

Lemma 3.1. *(See Figure 3.1.) Consider (1.1). Assume that $0 < \varepsilon < \varepsilon_5 = \sqrt{\frac{\sigma^3}{27\rho}}$. Then $\theta'(u) = f(u) - uf'(u) = 2\varepsilon u^3 - \sigma u^2 + \rho$ has exactly two positive zeros $p_1 < p_2$ satisfying*

$$p_1 = \frac{\sigma}{6\varepsilon}\left[1 + 2\sin\left(\frac{\phi}{3} - \frac{\pi}{6}\right)\right] < \gamma = \frac{\sigma}{3\varepsilon} < p_2 = \frac{\sigma}{6\varepsilon}\left[1 + 2\cos\left(\frac{\phi}{3}\right)\right] < \beta_\varepsilon, \tag{3.7}$$

where

$$\phi = \arccos\left(\frac{\sigma^3 - 54\varepsilon^2\rho}{\sigma^3}\right) \in (0, \pi). \tag{3.8}$$

Furthermore, $\theta(u)$ is a strictly increasing function on $(0, p_1) \cup (p_2, \beta_\varepsilon)$, and is a strictly decreasing function on (p_1, p_2).

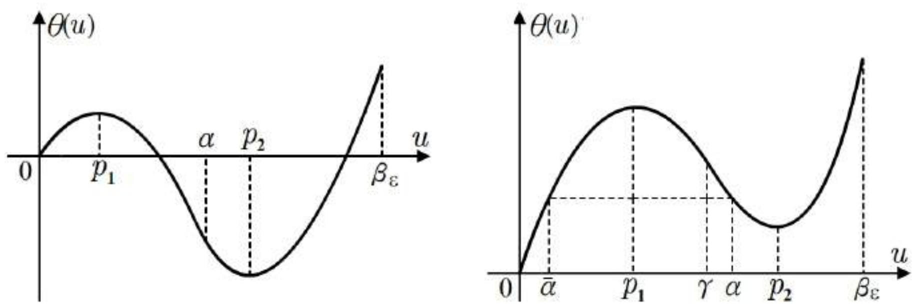

Figure 3.1: Graphs of $\theta(u)$ on $(0, \beta_\varepsilon)$. (i) $\theta(p_2) \leq 0$. (ii) $\theta(p_2) > 0$.

Proof. First $\theta'(u) = f(u) - uf'(u)$ by (3.5). By (2.1), we compute that

$$\frac{\partial}{\partial u}[\theta'(u)] = \frac{\partial}{\partial u}[f(u) - uf'(u)] = 2(3\varepsilon u - \sigma)u \begin{cases} < 0 & \text{if } 0 < u < \gamma, \\ = 0 & \text{if } u = \gamma = \sigma/(3\varepsilon), \\ > 0 & \text{if } u > \gamma. \end{cases}$$

It follows that $\theta'(u) = f(u) - uf'(u)$ is a strictly decreasing function of u on $(0, \gamma)$, and is a strictly increasing function of u on $[\gamma, \infty)$. We compute that, for $0 < \varepsilon < \varepsilon_5$,

$$\theta'(\gamma) = f(\gamma) - \gamma f'(\gamma) = \frac{\rho}{\varepsilon^2}\left(\varepsilon^2 - \frac{\sigma^3}{27\rho}\right) < \frac{\rho}{\varepsilon^2}\left(\varepsilon_5^2 - \frac{\sigma^3}{27\rho}\right) = 0.$$

It is easy to see that $\theta'(0) = f(0) > 0$ and $\theta'(\beta_\varepsilon) = -\beta_\varepsilon f'(\beta_\varepsilon) > 0$. Thus, for $0 < \varepsilon < \varepsilon_5$, $\theta'(u) = f(u) - uf'(u)$ has exactly two positive zeros $p_1 < p_2$ in $(0, \beta_\varepsilon)$. Since

$$\cos u = 4\cos^3\left(\frac{u}{3}\right) - 3\cos\left(\frac{u}{3}\right) \quad \text{and} \quad \sin u = 3\sin\left(\frac{u}{3}\right) - 4\sin^3\left(\frac{u}{3}\right), \tag{3.9}$$

and by (3.8), we compute that

$$\begin{aligned} &\theta'\left(\frac{\sigma}{6\varepsilon}\left[1 + 2\sin\left(\frac{\phi}{3} - \frac{\pi}{6}\right)\right]\right) \\ &= \frac{1}{54\varepsilon^2}\left\{54\varepsilon^2\rho - \sigma^3 + \sigma^3\left[4\sin^3\left(\frac{\phi}{3} - \frac{\pi}{6}\right) - 3\sin\left(\frac{\phi}{3} - \frac{\pi}{6}\right)\right]\right\} \\ &= \frac{1}{54\varepsilon^2}\left[54\varepsilon^2\rho - \sigma^3 - \sigma^3\sin\left(\phi - \frac{1}{2}\pi\right)\right] = \frac{1}{54\varepsilon^2}\left[54\varepsilon^2\rho - \sigma^3 + \sigma^3\cos\phi\right] \\ &= \frac{1}{54\varepsilon^2}\left[54\varepsilon^2\rho - \sigma^3 + \sigma^3\left(\frac{\sigma^3 - 54\rho\varepsilon^2}{\sigma^3}\right)\right] = 0 \end{aligned} \tag{3.10}$$

and

$$
\begin{aligned}
\theta'\left(\frac{\sigma}{6\varepsilon}\left[1+2\cos\left(\frac{\phi}{3}\right)\right]\right) &= \frac{1}{54\varepsilon^2}\left\{54\varepsilon^2\rho-\sigma^3+\sigma^3\left[4\cos^3\left(\frac{\phi}{3}\right)-3\cos\left(\frac{\phi}{3}\right)\right]\right\} \\
&= \frac{1}{54\varepsilon^2}\left[54\varepsilon^2\rho-\sigma^3+\sigma^3\cos\phi\right] \\
&= \frac{1}{54\varepsilon^2}\left[54\varepsilon^2\rho-\sigma^3+\sigma^3\left(\frac{\sigma^3-54\rho\varepsilon^2}{\sigma^3}\right)\right] = 0.
\end{aligned}
\tag{3.11}
$$

In addition, it is easy to see that $\phi\in(0,\pi)$ for $0<\varepsilon<\varepsilon_5$. We observe that, for $0<\phi<\pi$,

$$
\begin{aligned}
&\frac{\sigma}{6\varepsilon}\left[1+2\cos\left(\frac{\phi}{3}\right)\right]-\frac{\sigma}{6\varepsilon}\left[1+2\sin\left(\frac{\phi}{3}-\frac{\pi}{6}\right)\right] \\
&= \frac{\sqrt{3}\sigma}{6\varepsilon}\left[\sqrt{3}\cos\left(\frac{\phi}{3}\right)-\sin\left(\frac{\phi}{3}\right)\right] \\
&= \frac{\sqrt{3}\sigma}{3\varepsilon}\sin\left(\frac{\pi}{3}-\frac{\phi}{3}\right)>0.
\end{aligned}
\tag{3.12}
$$

So by (3.10)–(3.12), we see that (3.7) holds. Thus we obtain that $\theta(u)$ is a strictly increasing function on $(0,p_1)\cup(p_2,\beta_\varepsilon)$, and is a strictly decreasing function on (p_1,p_2). The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Consider* (1.1)*. Assume that* $0<\varepsilon<\varepsilon_5$*. Then the following assertions* (i) *and* (ii) *hold:*

(i)

$$
p_2^3=\frac{\sigma p_2^2-\rho}{2\varepsilon} \quad \textit{and} \quad N\equiv\frac{\varepsilon}{p_2}\left(\frac{\partial}{\partial\varepsilon}p_2\right)=-\frac{p_2}{3(p_2-\gamma)}<-1.
\tag{3.13}
$$

(ii) *(See Figure* 3.1(ii)*.) For* $\varepsilon_2\le\varepsilon<\varepsilon_5$*,* $\theta(p_2)>0$*. Moreover, for any* $\alpha\in(p_1,p_2]$*, there exists a unique number* $\overline{\alpha}\in(0,p_1)$ *such that* $\theta(\overline{\alpha})=\theta(\alpha)$*,* $\theta(u)<\theta(\alpha)$ *for* $0<u<\overline{\alpha}$*, and* $\theta(u)>\theta(\alpha)$ *for* $\overline{\alpha}<u<\alpha$*.*

Proof. By Lemma 3.1, we see that $2\varepsilon p_2^3-\sigma p_2^2+\rho=\theta'(p_2)=0$. It follows that $p_2^3=(\sigma p_2^2-\rho)/(2\varepsilon)$. Since $\theta'(p_2)=0$ and $0<\phi<\pi$, we compute and observe that

$$
\begin{aligned}
0=\frac{\partial\theta'(p_2)}{\partial\varepsilon} &= 2p_2^3+6\varepsilon p_2^2\left(\frac{\partial}{\partial\varepsilon}p_2\right)-2\sigma p_2\left(\frac{\partial}{\partial\varepsilon}p_2\right) \\
&= 6p_2^2(p_2-\gamma)\left[\frac{p_2}{3(p_2-\gamma)}+\frac{\varepsilon}{p_2}\left(\frac{\partial}{\partial\varepsilon}p_2\right)\right].
\end{aligned}
$$

So we obtain that $N=-\varepsilon p_2/(3\varepsilon p_2-\sigma)$. In addition, since $0<\phi<\pi$, and by Lemma 3.1, we see that

$$
N+1=\frac{2\varepsilon p_2-\sigma}{3\varepsilon(p_2-\gamma)}=\frac{2\sigma}{9\varepsilon(p_2-\gamma)}\left[\cos\left(\frac{\phi}{3}\right)-1\right]<1,
$$

which implies $N < -1$. Hence, assertion (i) holds.

We observe that the cubic polynomial $4t^3 - 3t + 9/16$ has exactly two positive zeros $(\sqrt{21} - 3)/8$ and $3/4$. By (3.9), we compute that

$$\left[4\cos^3\left(\frac{\phi}{3}\right) - 3\cos\left(\frac{\phi}{3}\right)\right]_{\varepsilon=\varepsilon_2} = \cos\phi\big|_{\varepsilon=\varepsilon_2} = -\frac{9}{16}.$$

Since $\cos(\phi/3) > \cos(\pi/3) = 1/2$ for $0 < \phi < \pi$, we observe that

$$\frac{1}{2} = \cos\left(\frac{\pi}{3}\right) < \cos\left(\frac{\phi}{3}\right) \le \cos\left(\frac{\phi}{3}\right)\Big|_{\varepsilon=\varepsilon_2} = \frac{3}{4} \quad \text{for } \varepsilon_2 \le \varepsilon < \varepsilon_5. \tag{3.14}$$

By (3.13), we see that $\rho = \sigma p_2^2 - 2\varepsilon p_2^3$. So

$$\begin{aligned}\theta(p_2) &= \frac{1}{6}p_2(3\varepsilon p_2^3 - 2\sigma p_2^2 + 6\rho) = \frac{1}{6}p_2\left[3\varepsilon p_2^3 - 2\sigma p_2^2 + 6(\sigma p_2^2 - 2\varepsilon p_2^3)\right] \\ &= \frac{3\varepsilon p_2^3}{2}\left(\frac{4\sigma}{9\varepsilon} - p_2\right) = \frac{\sigma p_2^3}{2}\left[\frac{5}{6} - \cos\left(\frac{\phi}{3}\right)\right] \quad \text{(by (3.7))} \\ &\ge \frac{\sigma p_2^3}{2}\left(\frac{5}{6} - \frac{3}{4}\right) = \frac{\sigma p_2^3}{24} > 0\end{aligned}$$

by (3.14). So by Lemma 3.1, for $\alpha \in (p_1, p_2]$, there exists a unique $\overline{\alpha} \in (0, p_1)$ such that $\theta(\overline{\alpha}) = \theta(\alpha)$, $\theta(u) < \theta(\alpha)$ for $0 < u < \overline{\alpha}$, and $\theta(u) > \theta(\alpha)$ for $\overline{\alpha} < u < \alpha$; see Figure 3.1. Hence, assertion (ii) holds. The proof of Lemma 3.2 is complete. □

For $\varepsilon_2 \le \varepsilon < \varepsilon_5$, by Lemma 3.2, there exist two numbers $\overline{\gamma}$, $\overline{p}_2 \in (0, p_1)$ such that $\theta(\overline{\gamma}) = \theta(\gamma)$ and $\theta(\overline{p}_2) = \theta(p_2)$. We write the formulas of $\overline{\gamma}$ and $\overline{p}_2$ in the following lemma.

Lemma 3.3. *Consider* (1.1). *Assume that* $\varepsilon_2 \le \varepsilon < \varepsilon_5$. *Then*

$$\overline{\gamma} = \frac{1 - 4\cos(\frac{y}{3} + \frac{\pi}{3})}{9}\frac{\sigma}{\varepsilon} \quad \textit{and} \quad \overline{p}_2 = \frac{-2\sqrt{18 - 8\cos^2(\frac{\phi}{3})}\cos(\frac{z}{3} + \frac{\pi}{3}) - 2\cos(\frac{\phi}{3}) + 3}{18}\frac{\sigma}{\varepsilon},$$

where ϕ *is defined in Lemma* 3.1,

$$y = \arccos\left(\frac{19\sigma^3 - 729\varepsilon^2\rho}{8\sigma^3}\right) \in (0, \pi) \quad \textit{and} \quad z = \arccos\left(\frac{24\cos(\frac{\phi}{3}) + 44\cos\phi}{\sqrt{2\left[9 - 4\cos^2(\frac{\phi}{3})\right]^3}}\right). \tag{3.15}$$

Proof. Since $\varepsilon_2 \le \varepsilon < \varepsilon_5$, we have that $0 < y < \pi$. We let

$$\Gamma_\gamma \equiv \frac{1 - 4\cos(\frac{y}{3} + \frac{\pi}{3})}{9}\frac{\sigma}{\varepsilon} \quad \text{and} \quad \Gamma_{p_2} \equiv \frac{-2\sqrt{18 - 8\cos^2(\frac{\phi}{3})}\cos(\frac{z}{3} + \frac{\pi}{3}) - 2\cos(\frac{\phi}{3}) + 3}{18}\frac{\sigma}{\varepsilon}.$$

To complete the proof, we divide the proof into two steps.

Step 1. We prove that $\overline{\gamma} = \Gamma_\gamma$. By Lemma 3.2(ii), it is sufficient to prove that $0 < \Gamma_\gamma < \gamma$ and $\theta(\Gamma_\gamma) = \theta(\overline{\gamma})$. We observe that $2\pi/3 < y < \pi$ because

$$\cos y \le \frac{19\sigma^3 - 729\varepsilon_2^2\rho}{8\sigma^3} = -\frac{67}{256} < -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right) \quad \text{for } \varepsilon_2 \le \varepsilon < \varepsilon_5$$

by (3.15). So we have that

$$-\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right) < \cos\left(\frac{y}{3} + \frac{\pi}{3}\right) < \cos\left(\frac{5\pi}{9}\right) < 0 \quad \text{for } \varepsilon_2 \le \varepsilon < \varepsilon_5.$$

It follows that

$$0 < \frac{\sigma}{9\varepsilon} < \Gamma_\gamma < \frac{1 - 4(\frac{-1}{2})}{9}\frac{\sigma}{\varepsilon} = \frac{\sigma}{3\varepsilon} = \gamma \quad \text{for } \varepsilon_2 \le \varepsilon < \varepsilon_5. \tag{3.16}$$

By (3.9) and (3.15), we compute and observe that

$$\begin{aligned}
&\theta(\Gamma_\gamma) - \theta(\gamma) \\
&= \frac{16\sigma^4}{6561\varepsilon^3}\left[2\cos\left(\frac{\pi}{3} + \frac{y}{3}\right) + 1\right]\left[\frac{19\sigma^3 - 729\varepsilon^2\rho}{8\sigma^3} + 4\cos^3\left(\frac{\pi}{3} + \frac{y}{3}\right) - 3\cos\left(\frac{\pi}{3} + \frac{y}{3}\right)\right] \\
&= \frac{16\sigma^4}{6561\varepsilon^3}\left[2\cos\left(\frac{\pi}{3} + \frac{y}{3}\right) + 1\right]\left(\frac{19\sigma^3 - 729\varepsilon^2\rho}{8\sigma^3} - \cos y\right) = 0.
\end{aligned}$$

So by (3.16) and Lemma 3.2(ii), we see that $\overline{\gamma} = \Gamma_\gamma$.

Step 2. We prove that $\overline{p}_2 = \Gamma_{p_2}$. By Lemma 3.2(ii), it is sufficient to prove that $0 < \Gamma_{p_2} < \overline{p}_2$ and $\theta(\Gamma_{p_2}) = \theta(\overline{p}_2)$. First, we assert that, for $\varepsilon_2 \le \varepsilon < \varepsilon_5$,

$$\frac{1}{2} < \Gamma_1 \equiv \cos\left(\frac{\phi}{3}\right) \le \frac{3}{4} \quad \text{and} \quad -\frac{1}{2} \le \Gamma_2 \equiv \cos\left(\frac{z}{3} + \frac{\pi}{3}\right) \le 0. \tag{3.17}$$

Indeed, by (3.14), the inequalities $1/2 < \Gamma_1 \le 3/4$ hold immediately for $\varepsilon_2 \le \varepsilon < \varepsilon_5$. By (3.8), we compute and observe that $\cos\phi \le \cos\phi\big|_{\varepsilon=\varepsilon_2} = -9/16$ for $\varepsilon_2 \le \varepsilon < \varepsilon_5$. So by (3.15),

(3.18)

$$-1 \le 4\cos^3\left(\frac{z}{3}\right) - 3\cos\left(\frac{z}{3}\right) = \cos z = \frac{24\Gamma_1 + 44\cos\phi}{\sqrt{2(9 - 4\Gamma_1^2)^3}} \le \frac{24(\frac{3}{4}) - 44(\frac{9}{16})}{\sqrt{2\left[9 - 4(\frac{1}{2})^2\right]^3}} = -\frac{27}{128}.$$

Clearly, the cubic polynomial $4t^3 - 3t - l$ where $l \in [-1, -27/128]$ has two positive zeros $\eta_1 \in (0, 1/2]$ and $\eta_2 \in [1/2, \sqrt{3}/2]$. Since $\cos(z/3) \ge \cos(\pi/3) = 1/2$, and by (3.18), we see that $1/2 \le \cos(z/3) \le \sqrt{3}/2$. This implies that $\pi/2 \le (z+\pi)/3 \le 2\pi/3$. So we obtain that $-1/2 \le \Gamma_2 \le 0$. Thus assertion (3.17) holds. By (3.17), we observe that

$$p_2 - \Gamma_{p_2} = 4\Gamma_1 + \sqrt{18 - 8\Gamma_1^2}\Gamma_2 > 4\left(\frac{1}{2}\right) + \sqrt{18 - 8\left(\frac{1}{2}\right)^2}\left(\frac{-1}{2}\right) = 0,$$

$$\Gamma_{p_2} = \frac{\left(-2\sqrt{18 - 8\Gamma_1^2}\Gamma_2 - 2\Gamma_1 + 3\right)\sigma}{18\varepsilon} \ge \frac{\sigma(-2\Gamma_1 + 3)}{18\varepsilon} > 0.$$

Thus, $0 < \Gamma_{p_2} < p_2$. In addition, by (3.8), (3.9) and (3.15), we observe that

$$4\Gamma_1^3 - 3\Gamma_1 = \cos\phi = \frac{\sigma^3 - 54\varepsilon^2\rho}{\sigma^3}, \tag{3.19}$$

$$4\Gamma_2^3 - 3\Gamma_2 = -\cos z = \frac{-4(6\sigma^3\Gamma_1 - 594\varepsilon^2\rho + 11\sigma^3)}{\sigma^3(9-4\Gamma_1^2)\sqrt{2(9-4\Gamma_1^2)}}. \tag{3.20}$$

By (3.19) and (3.20), we compute and find that

$$\begin{aligned}
\theta(\Gamma_{p_2}) - \theta(p_2) &= \frac{\sigma(4\Gamma_1 + \sqrt{18-8\Gamma_1^2}\Gamma_2)}{26244\varepsilon^3} \\
&\quad \times \left\{ -10\sigma^3(4\Gamma_1^3 - 3\Gamma_1) + 24\sigma^3\Gamma_1 - 2916\varepsilon^2\rho + 54\sigma^3 \right. \\
&\qquad \left. + \left[\sigma^3(9-4\Gamma_1^2)(4\Gamma_2^3 - 3\Gamma_2)\sqrt{2(9-4\Gamma_1^2)}\right] \right\} \\
&= \frac{\sigma(4\Gamma_1 + \sqrt{18-8\Gamma_1^2}\Gamma_2)}{26244\varepsilon^3} \\
&\quad \times \left[-10\sigma^3\left(\frac{\sigma^3 - 54\varepsilon^2\rho}{\sigma^3}\right) + 24\sigma^3\Gamma_1 - 2916\varepsilon^2\rho + 54\sigma^3 \right. \\
&\qquad \left. - 4\left(6\sigma^3\Gamma_1 - 594\varepsilon^2\rho + 11\sigma^3\right) \right] \\
&= 0.
\end{aligned}$$

So by Lemma 3.2(ii), we see that $\overline{p}_2 = \Gamma_{p_2}$. The proof of Lemma 3.3 is complete. □

For the sake of convenience, we let

$$K(\varepsilon) \equiv \frac{1}{6}\left[1 + 2\sin\left(\frac{\phi}{3} - \frac{\pi}{6}\right)\right], \quad L(\varepsilon) \equiv \frac{1}{6}\left[1 + 2\cos\left(\frac{\phi}{3}\right)\right],$$
$$R(\varepsilon) \equiv \frac{1}{18}\left[-2\sqrt{2}\sqrt{9 - 4\cos^2\left(\frac{\phi}{3}\right)}\cos\left(\frac{z}{3} + \frac{\pi}{3}\right) - 2\cos\left(\frac{\phi}{3}\right) + 3\right],$$

where ϕ and z are defined in Lemmas 3.1 and 3.3 respectively. By Lemmas 3.1 and 3.3, it is easy to see that $p_1 = K(\varepsilon)\sigma/\varepsilon$, $p_2 = L(\varepsilon)\sigma/\varepsilon$ and $\overline{p}_2 = R(\varepsilon)\sigma/\varepsilon$. We estimate the numbers $\overline{p}_2$, p_1 and p_2 in the following lemma.

Lemma 3.4. *Consider* (1.1). *Assume that* $\varepsilon_2 \le \varepsilon < \varepsilon_5$. *Then the following assertions* (i)–(iv) *hold:*

(i) $K(\varepsilon)$ *is a strictly increasing function of* ε *on* $[\varepsilon_2, \varepsilon_5)$.

(ii) $L(\varepsilon)$ *is a strictly decreasing function of* ε *on* $[\varepsilon_2, \varepsilon_5)$.

(iii) $R(\varepsilon)$ *is a strictly increasing function of* ε *on* $[\varepsilon_2, \varepsilon_5)$.

(iv) *For* $\varepsilon_2 \le \varepsilon \le \varepsilon_4$,

$$\frac{3\sigma}{25\varepsilon} < \overline{p}_2 < \frac{23\sigma}{125\varepsilon} < \frac{23\sigma}{100\varepsilon} < p_1 < \frac{261\sigma}{1000\varepsilon} < \frac{396\sigma}{1000\varepsilon} < p_2 < \frac{417\sigma}{1000\varepsilon}. \tag{3.21}$$

Proof. Since $0 < \phi/3 < \pi/3$ and by (3.8), we compute and observe that, for $\varepsilon_2 \le \varepsilon < \varepsilon_5$,

$$\frac{\partial K(\varepsilon)}{\partial \varepsilon} = \frac{2\sin(\frac{2\pi}{3} - \frac{\phi}{3})}{9\sqrt{\varepsilon_5^2 - \varepsilon^2}} > 0 \quad \text{and} \quad \frac{\partial L(\varepsilon)}{\partial \varepsilon} = -\frac{2\sin(\frac{\phi}{3})}{9\sqrt{\varepsilon_5^2 - \varepsilon^2}} < 0.$$

So assertions (i) and (ii) hold. We note that $\theta'(p_2) = 0$ by Lemma 3.1. Since $\theta(\overline{p}_2) - \theta(p_2) = 0$ and $0 < \overline{p}_2 < p_2$, and by (3.5), we compute and find that

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varepsilon}\left[\theta(\overline{p}_2) - \theta(p_2)\right] = \theta'(\overline{p}_2)\frac{\partial \overline{p}_2}{\partial \varepsilon} + \left.\frac{\partial \theta(u)}{\partial \varepsilon}\right|_{u=\overline{p}_2} - \left.\frac{\partial \theta(u)}{\partial \varepsilon}\right|_{u=p_2} \\ &= \theta'(\overline{p}_2)\frac{\partial \overline{p}_2}{\partial \varepsilon} + \frac{\overline{p}_2^4}{2} - \frac{p_2^4}{2} < \theta'(\overline{p}_2)\frac{\partial \overline{p}_2}{\partial \varepsilon}. \end{aligned} \tag{3.22}$$

Since $\theta'(\overline{p}_2) > 0$ by Lemma 3.1, and by (3.22), we see that $\partial \overline{p}_2/\partial \varepsilon > 0$ for $\varepsilon_2 \le \varepsilon < \varepsilon_5$. By Lemma 3.3, we further see that $R(\varepsilon) = \varepsilon \overline{p}_2/\sigma$ is a strictly increasing function of ε on $[\varepsilon_2, \varepsilon_5)$. So assertion (iii) holds. Finally, by assertions (i)–(iii), we compute that

$$\begin{aligned} &\left(\frac{0.232\sigma}{\varepsilon} \approx\right) \frac{K(\varepsilon_2)\sigma}{\varepsilon} \le p_1 = \frac{K(\varepsilon)\sigma}{\varepsilon} \le \frac{K(\varepsilon_4)\sigma}{\varepsilon} \left(\approx \frac{0.260\sigma}{\varepsilon}\right), \\ &\left(\frac{0.3967\sigma}{\varepsilon} \approx\right) \frac{L(\varepsilon_4)\sigma}{\varepsilon} \le p_2 = \frac{K(\varepsilon)\sigma}{\varepsilon} \le \frac{L(\varepsilon_2)\sigma}{\varepsilon} \left(\approx \frac{0.4166\sigma}{\varepsilon}\right), \\ &\left(\frac{0.1207\sigma}{\varepsilon} \approx\right) \frac{R(\varepsilon_2)\sigma}{\varepsilon} \le \overline{p}_2 = \frac{R(\varepsilon)\sigma}{\varepsilon} \le \frac{R(\varepsilon_4)\sigma}{\varepsilon} \left(\approx \frac{0.1830\sigma}{\varepsilon}\right). \end{aligned}$$

So (3.21) holds for $\varepsilon_2 \le \varepsilon \le \varepsilon_4$. That is, assertion (iv) holds. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Consider* (1.1). *Then* $T_\varepsilon'(\gamma) < 0$ *for* $0 < \varepsilon \le \varepsilon_1$ *and* $T_\varepsilon'(p_2) < 0$ *for* $0 < \varepsilon \le \varepsilon_2$.

Proof. Let

$$G(\alpha) \equiv \int_0^\alpha t\theta'(t)\,dt = \frac{2}{5}\varepsilon\alpha^5 - \frac{1}{4}\sigma\alpha^4 + \frac{1}{2}\rho\alpha^2.$$

Suppose that $\theta(\gamma) \le 0$, see Figure 3.1(i). We note that $\theta(u) > \theta(\gamma)$ for $0 < u < \gamma$. So by Lemma 3.1 and (3.2), $T_\varepsilon'(\gamma) < 0$. Suppose that $\theta(\gamma) > 0$, see Figure 3.1(ii). Since $\gamma = \sigma/(3\varepsilon)$, we see that, for $0 < \varepsilon \le \varepsilon_1$,

$$G(\gamma) = \frac{\gamma^2}{20}\left[8\varepsilon\left(\frac{\sigma}{3\varepsilon}\right)^3 - 5\sigma\left(\frac{\sigma}{3\varepsilon}\right)^2 + 10\rho\right] = \frac{\rho\gamma^2}{2\varepsilon^2}\left(\varepsilon^2 - \frac{7\sigma^3}{270\rho}\right) \le 0.$$

So by [5, (3.11)], we see that

$$T'_\varepsilon(\gamma) < \frac{1}{2\sqrt{2}\gamma[F(\gamma)-F(\overline{\gamma})]^{3/2}}\int_0^\gamma t\theta'(t)\,dt = \frac{G(\gamma)}{2\sqrt{2}\gamma[F(\gamma)-F(\overline{\gamma})]^{3/2}} \le 0.$$

Thus $T'_\varepsilon(\gamma) < 0$ for $0 < \varepsilon \le \varepsilon_1$. In addition, $T'_\varepsilon(p_2) < 0$ for $0 < \varepsilon \le \varepsilon_2$, see [6, the proof of Lemma 3.8]. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Consider* (1.1). *Then, for* $0 < u < \alpha < \beta_\varepsilon$,

$$A_\alpha < \frac{4\varepsilon\alpha}{3\sigma}B_\alpha,\ B_\alpha < \frac{3\sigma}{2\varepsilon(\alpha+u)}A_\alpha,\ D_\alpha > \frac{\rho}{4\varepsilon\alpha^3}A_\alpha,\ D_\alpha > \frac{\rho}{3\sigma\alpha^2}B_\alpha.$$

Proof. We compute and find that, for $0 < u < \alpha < \beta_\varepsilon$,

$$\begin{aligned}
\frac{4\varepsilon\alpha}{3\sigma}B_\alpha - A_\alpha &= \frac{\varepsilon}{3}(\alpha-u)^2(\alpha^2+2u\alpha+3u^2) > 0,\\
\frac{3\sigma}{2\varepsilon(\alpha+u)}A_\alpha - B_\alpha &= \frac{1}{2}\sigma(\alpha-u)^3 > 0,\\
D_\alpha - \frac{\rho}{4\varepsilon\alpha^3}A_\alpha &= \frac{\rho}{4\alpha^3}(3\alpha^2+2u\alpha+u^2)(\alpha-u)^2 > 0,\\
D_\alpha - \frac{\rho}{3\sigma\alpha^2}B_\alpha &= \frac{\rho}{3\alpha^2}(2\alpha+u)(\alpha-u)^2 > 0.
\end{aligned}$$

The proof of Lemma 3.6 is complete. □

Lemma 3.7. *Consider* (1.1). *Then there exist* $\widehat{\varepsilon} \in (\varepsilon_1, \varepsilon_5)$ *and* $\overline{\varepsilon} \in (\varepsilon_2, \varepsilon_5)$ *such that*

$$T'_\varepsilon(\gamma)\begin{cases} <0 & \text{for } 0<\varepsilon<\widehat{\varepsilon},\\ =0 & \text{for } \varepsilon=\widehat{\varepsilon},\\ >0 & \text{for } \widehat{\varepsilon}<\varepsilon<\varepsilon_5\end{cases} \quad \text{and} \quad T'_\varepsilon(p_2)\begin{cases} <0 & \text{for } 0<\varepsilon<\overline{\varepsilon},\\ =0 & \text{for } \varepsilon=\overline{\varepsilon},\\ >0 & \text{for } \overline{\varepsilon}<\varepsilon<\varepsilon_5.\end{cases} \tag{3.23}$$

Remark 3.8. We shall prove $\widehat{\varepsilon} < \overline{\varepsilon}$ in Lemma 3.10.

Proof of Lemma 3.7. We divide the proof into five steps.

Step 1. We prove the first inequality of (3.23). By (3.6), we have that

$$\frac{\partial}{\partial\varepsilon}T'_\varepsilon(\gamma) = \frac{1}{96\sqrt{2}\varepsilon\gamma}\int_0^\gamma \frac{P_1(u)}{[F(\gamma)-F(u)]^{5/2}}\,du, \tag{3.24}$$

where $P_1(u) \equiv -9A_\gamma^2 + 18A_\gamma B_\gamma - 78A_\gamma D_\gamma - 8B_\gamma^2 + 80B_\gamma D_\gamma + 24D_\gamma^2 + 48C_\gamma D_\gamma$. By Lemma 3.6, we obtain that, for $0 < u < \gamma$,

$$A_\gamma < \frac{4\varepsilon\gamma}{3\sigma}B_\gamma < \frac{4}{9}B_\gamma \quad \text{and} \quad D_\gamma > \frac{3\varepsilon^2\rho}{\sigma^3}B_\gamma > \frac{3\varepsilon_1^2\rho}{\sigma^3}B_\gamma = \frac{7}{90}B_\gamma. \tag{3.25}$$

Since $C_\gamma \geq 0$ for $0 < u < \gamma$, and by (3.25), we observe that, for $0 < u < \gamma$,

$$\begin{aligned}
P_1(u) &> -9A_\gamma\left(\frac{4}{9}B_\gamma\right) + 18A_\gamma B_\gamma - 78A_\gamma D_\gamma - 8B_\gamma^2 + 80B_\gamma D_\gamma + 24D_\gamma\left(\frac{7}{90}B_\gamma\right) \\
&= 14A_\gamma B_\gamma - 78A_\gamma D_\gamma - 8B_\gamma^2 + \frac{1228}{15}B_\gamma D_\gamma \\
&> 14A_\gamma B_\gamma - 78\left(\frac{4}{9}B_\gamma\right)D_\gamma - 8B_\gamma^2 + \frac{1228}{15}B_\gamma D_\gamma \\
&= B_\gamma\left(14A_\gamma - 8B_\gamma + \frac{236}{5}D_\gamma\right) \\
&> B_\gamma\left[14A_\gamma - 8B_\gamma + \frac{236}{5}\left(\frac{7}{90}B_\gamma\right)\right] = B_\gamma\left(14A_\gamma - \frac{974}{225}B_\gamma\right) \\
&= \frac{2B_\gamma}{2025\varepsilon^2}(\gamma - u)(14175\varepsilon^3u^3 + 342\varepsilon^2\sigma u^2 + 114\varepsilon\sigma^2 u + 38\sigma^3) > 0.
\end{aligned}$$

So by (3.24), $\frac{\partial}{\partial\varepsilon}T'_\varepsilon(\gamma) > 0$ for $\varepsilon_1 < \varepsilon < \varepsilon_5$. By Lemma 3.5 and Theorem 1.1(iii), we see that $T'_{\varepsilon_1}(\gamma) < 0$ and $T'_{\varepsilon_5}(\gamma) > 0$. Then there exists $\widehat{\varepsilon} \in (\varepsilon_1, \varepsilon_5)$ such that the first inequality of (3.23) holds.

Step 2. We prove that $V(u) > 0$ where

$$\begin{aligned}
V(u) \equiv{}& 6(20N+7)A_{p_2}D_{p_2} + 3(1+4N)A_{p_2}^2 + 2(1-8N)A_{p_2}B_{p_2} \\
&- 24ND_{p_2}^2 + 8B_{p_2}N(B_{p_2} - 10D_{p_2}),
\end{aligned} \tag{3.26}$$

and N is defined in (3.13). Let $u^* \in (0, p_2)$ be given. To prove that $V(u^*) > 0$, we discuss two cases: $(B_{p_2} - 10D_{p_2})\big|_{u=u^*} \leq 0$ and $(B_{p_2} - 10D_{p_2})\big|_{u=u^*} > 0$.

Case 1. Assume that $(B_{p_2} - 10D_{p_2})\big|_{u=u^*} \leq 0$. By (3.13) and Lemma 3.6, we observe that, for $u = u^*$,

$$\begin{aligned}
V(u^*) &> 6(20N+7)A_{p_2}D_{p_2} + 3(1+4N)A_{p_2}^2 + 2(1-8N)A_{p_2}B_{p_2} \\
&\quad - 24ND_{p_2}\left(\frac{\rho}{4\varepsilon p_2^3}A_{p_2}\right) + 8N(B_{p_2} - 10D_{p_2})\left(\frac{3\sigma}{4\varepsilon p_2}A_{p_2}\right) \\
&= A_{p_2}\left[\left(120N + 42 - \frac{60\sigma N}{\varepsilon p_2} - \frac{6\rho N}{\varepsilon p_2^3}\right)D_{p_2} + 3(1+4N)A_{p_2}\right. \\
&\qquad \left. + \left(2 - 16N + \frac{6\sigma N}{\varepsilon p_2}\right)B_{p_2}\right].
\end{aligned} \tag{3.27}$$

In addition, since $\sigma p_2^2 = 2\varepsilon p_2^3 + \rho$ and $N < 0$ by (3.13), we compute and observe that

$$\begin{aligned}
120N + 42 - \frac{60\sigma N}{\varepsilon p_2} - \frac{6\rho N}{\varepsilon p_2^3} &= \frac{6}{\varepsilon p_2^3}\left[(7\varepsilon + 20N\varepsilon)p_2^3 - 10N(\sigma p_2^2) - N\rho\right] \\
&= \frac{6}{\varepsilon p_2^3}(7\varepsilon p_2^3 - 11N\rho) > 0.
\end{aligned} \tag{3.28}$$

By (3.13), (3.27), (3.28) and Lemma 3.6, we observe that, for $u = u^*$,

$$\begin{aligned}\frac{V(u^*)}{A_{p_2}} &> \left(120N + 42 - \frac{60\sigma N}{\varepsilon p_2} - \frac{6N\rho}{\varepsilon p_2^3}\right)\left(\frac{\rho}{3\sigma p_2^2}B_{p_2}\right) + 3(1+4N)\left(\frac{4\varepsilon p_2}{3\sigma}B_{p_2}\right)\\ &\quad + \left(2 - 16N + \frac{6\sigma N}{\varepsilon p_2}\right)B_{p_2}\\ &= \frac{2(\rho + 4\sigma p_2^2 - \varepsilon p_2^3)}{3\varepsilon\sigma p_2^4(p_2-\gamma)}B_{p_2}\theta'(p_2) = 0.\end{aligned}$$

It follows that $V(u^*) > 0$ because $A_{p_2} > 0$ for $u = u^* \in (0, p_2)$.

Case 2. Assume that $(B_{p_2} - 10D_{p_2})\big|_{u=u^*} > 0$. Since

$$B_{p_2} < \frac{3\sigma}{2\varepsilon(p_2+u)}A_{p_2} \quad \text{and} \quad \frac{\rho}{4\varepsilon p_2^3}A_{p_2} < D_{p_2} \quad \text{for } 0 < u < p_2$$

by Lemma 3.6, we observe that, for $u = u^*$,

$$\begin{aligned}V(u^*) &> 6(20N+7)A_{p_2}D_{p_2} + 3(1+4N)A_{p_2}^2 + 2(1-8N)A_{p_2}B_{p_2}\\ &\quad - 24ND_{p_2}\left(\frac{\rho}{4\varepsilon p_2^3}A_{p_2}\right) + 8N(B_{p_2} - 10D_{p_2})\left[\frac{3\sigma}{2\varepsilon(p_2+u^*)}A_{p_2}\right]\\ &= A_{p_2}U(u^*),\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}U(u) \equiv &\left[120N + 42 - \frac{6\rho N}{\varepsilon p_2^3} - \frac{120\sigma N}{\varepsilon(p_2+u)}\right]D_{p_2} + 3(1+4N)A_{p_2}\\ &+ \left[2 - 16N + \frac{12\sigma N}{\varepsilon(p_2+u)}\right]B_{p_2}.\end{aligned}$$

We assert that

$$U(u) > 0 \quad \text{for } u \in (0, p_2). \tag{3.30}$$

Indeed, by Lemmas 3.2(i) and 3.4(iv), we apply elementary analytic techniques to prove that $\partial^5 U(u)/\partial u^5 < 0$ for $u \in (0, p_2)$, $\partial^4 U(u)/\partial u^4\big|_{u=p_2} > 0$, $\partial^3 U(u)/\partial u^3\big|_{u=p_2} < 0$, $\partial^2 U(u)/\partial u^2\big|_{u=p_2} > 0$, and $\partial U(u)/\partial u\big|_{u=p_2} < 0$. So $U(u)$ is a strictly decreasing function of u on $(0, p_2)$. Clearly, $U(p_2) = 0$. Thus, assertion (3.30) holds. The complete proofs are easy but rather lengthy, and hence we put them in [4]. So by (3.29) and (3.30), we obtain that $V(u^*) > \left(A_{p_2}\big|_{u=u^*}\right)U(u^*) > 0$.

Thus, in either Cases 1 or 2, we obtain that $V(u^*) > 0$, which implies that $V(u) > 0$ for $u \in (0, p_2)$.

Step 3. We prove the second inequality of (3.23). We recall the function $V(u)$ defined in (3.26) and the number N defined in (3.13). By (3.6), we have that

$$\frac{\partial}{\partial\varepsilon}T_\varepsilon'(p_2) = \frac{1}{96\sqrt{2}\varepsilon p_2}\int_0^{p_2} \frac{P_2(u)}{[F(p_2) - F(u)]^{5/2}}\,du, \tag{3.31}$$

where $P_2(u) \equiv V(u) + 12C_{p_2}[(1+N)A_{p_2} - 4ND_{p_2}]$. By Step 2, we see that $V(u) \geq 0$ for $0 < u < p_2$. By (3.13), Lemmas 3.2 and 3.6, we observe that, for $0 < u < p_2$,

$$\begin{aligned}
P_2(u) &\geq 12C_{p_2}[(1+N)A_{p_2} - 4ND_{p_2}] > 12C_{p_2}\left[(1+N)A_{p_2} - 4N\left(\frac{\rho}{4\varepsilon p_2^3}A_{p_2}\right)\right] \\
&= \frac{12A_{p_2}C_{p_2}}{\varepsilon p_2^3}\left[(1+N)\varepsilon p_2^3 - N\rho\right] \\
&= \frac{12A_{p_2}C_{p_2}}{\varepsilon p_2^3}\left[\left(1 - \frac{p_2}{3p_2 - 3\gamma}\right)\varepsilon\left(\frac{\sigma p_2^2 - \rho}{2\varepsilon}\right) + \left(\frac{p_2}{3p_2 - 3\gamma}\right)\rho\right] \\
&= \frac{2A_{p_2}C_{p_2}\sigma}{\varepsilon^2 p_2^3(p_2 - \gamma)}(2\varepsilon p_2^3 - \sigma p_2^2 + \rho) = \frac{2A_{p_2}C_{p_2}\sigma}{\varepsilon^2 p_2^3(p_2 - \gamma)}\theta'(p_2) = 0.
\end{aligned}$$

So $\frac{\partial}{\partial\varepsilon}T'_\varepsilon(p_2) > 0$ by (3.31). Thus $T''_\varepsilon(p_2)$ is a strictly increasing function of ε on $(\varepsilon_2, \varepsilon_5)$. By Lemma 3.5 and Theorem 1.1(iii), we see that $T'_{\varepsilon_2}(p_2) < 0$ and $T'_{\varepsilon_5}(p_2) > 0$. Then the second inequality of (3.23) holds. The proof of Lemma 3.7 is complete. $\square$

Next, in Lemmas 3.10 and 3.11 stated below, we prove that $T'_{\varepsilon_3}(\gamma) > 0$, $T'_{\varepsilon_3}(p_2) < 0$, $T'_{\varepsilon_4}(p_2) > 0$, and $T''_\varepsilon(p_2) > 0$ for some $\varepsilon \in [\varepsilon_2, \varepsilon_4]$ satisfying $T'_\varepsilon(p_2) = 0$. First of all, we observe that

(3.32)
$$\frac{\theta(\alpha) - \theta(u)}{[F(\alpha) - F(u)]^{3/2}} = H_1(u,\alpha)H_2(u,\alpha) \quad \text{and} \quad \frac{-u\theta'(u)}{[F(p_2) - F(u)]^{3/2}} = H_3(u,p_2)H_4(u,p_2),$$

where

$$H_1(u,\alpha) \equiv \frac{(\alpha - u)^{3/2}}{6[F(\alpha) - F(u)]^{3/2}}, \qquad H_2(u,\alpha) \equiv \frac{6[\theta(\alpha) - \theta(u)]}{(\alpha - u)^{3/2}},$$
$$H_3(u,\alpha) \equiv \frac{u(p_2 - u)^{3/2}}{[F(p_2) - F(u)]^{3/2}}, \qquad H_4(u,\alpha) \equiv \frac{-\theta'(u)}{(p_2 - u)^{3/2}}.$$

Clearly, $H_1(u,\alpha) > 0$ and $H_3(u,\alpha) > 0$ for $0 < u < \alpha$. Then we compute that

$$\int H_2(u,\alpha)\,du = \sqrt{\alpha - u}\,I_1(u,\alpha) \quad \text{and} \quad \int H_4(u,\alpha)\,du = \frac{I_2(u,\alpha)}{\sqrt{\alpha - u}}, \tag{3.33}$$

where

$$I_1(u,\alpha) \equiv \frac{2}{35}\left[-15\varepsilon u^3 - (39\varepsilon\alpha - 14\sigma)u^2 - (87\varepsilon\alpha^2 - 42\sigma\alpha)u - 279\varepsilon\alpha^3 + 154\alpha^2\sigma - 210\rho\right],$$
$$I_2(u,\alpha) \equiv \frac{2}{15}\left[6au^3 + (12a\alpha - 5b)u^2 + (48a\alpha^2 - 20b\alpha)u - 96a\alpha^3 + 40b\alpha^2 - 15d\right].$$

To prove Lemmas 3.10 and 3.11, in the following lemma, we further study some properties of $H_1(u,\alpha)$, $H_3(u,\alpha)$, $I_1(u,\alpha)$ and $I_2(u,\alpha)$.

Lemma 3.9. *Consider* (1.1). *For $\varepsilon_2 \leq \varepsilon < \varepsilon_5$, the following assertions* (i)–(vii) *hold:*

(i) *For* $0 \le \alpha \le \frac{21\sigma}{50\varepsilon}$, $H_1(u,\alpha)$ *is a strictly decreasing function of* u *on* $\left[0, \frac{21\sigma}{50\varepsilon}\right]$.

(ii) *For* $0 \le u \le \frac{21\sigma}{50\varepsilon}$, $H_1(u,\alpha)$ *is a strictly decreasing function of* α *on* $\left[\frac{\sigma}{3\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

(iii) *For* $0 \le \alpha \le \frac{21\sigma}{50\varepsilon}$, $I_1(u,\alpha)$ *is a strictly increasing function of* u *on* $(0, \widehat{u}(\alpha))$ *and is strictly decreasing function of* u *on* $(\widehat{u}(\alpha), \infty)$, *where*

$$\widehat{u}(\alpha) \equiv \frac{1}{45\varepsilon}\left[14\sigma - 39\varepsilon\alpha + \sqrt{14(-171\varepsilon^2\alpha^2 + 57\varepsilon\sigma\alpha + 14\sigma^2)}\right] > 0.$$

Furthermore, $\widehat{u}(\alpha)$ *is a strictly decreasing function of* α *on* $\left(\frac{\sigma}{6\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$, *and* $\widehat{u}(\gamma) = \gamma$.

(iv) *For* $u \ge 0$, $I_1(u,\alpha)$ *is a strictly decreasing function of* α *on* $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

(v) *For* $\frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}$, $H_3(u,\alpha)$ *is a strictly decreasing function of* u *on* $\left[0, \frac{27\sigma}{100\varepsilon}\right]$.

(vi) *For* $0 \le u \le \frac{27\sigma}{100\varepsilon}$, $H_3(u,\alpha)$ *is a strictly increasing function of* α *on* $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

(vii) $I_2(0,\alpha)$ *is a negative and strictly decreasing functions of* α *on* $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

The proof of Lemma 3.9 is easy but rather lengthy, and hence it is given in [4]

Assume that $\varepsilon_2 \le \varepsilon < \varepsilon_5$ and $w \in \{\gamma, p_2\}$. By Lemma 3.2, there exists $\overline{w} \in (0, p_1)$ such that $\theta(\overline{w}) = \theta(w)$. Let $\{\alpha_i\}_{i=0}^n$ and $\{\beta_i\}_{i=0}^n$ be uniform partitions of $[0, \overline{w}]$ and $[\overline{w}, w]$, respectively. By Lemma 3.4, we see that $w \in \{\gamma, p_2\} \subset \left[\frac{\sigma}{3\varepsilon}, \frac{21\sigma}{50\varepsilon}\right)$. By (3.2) and (3.33), we observe that

(3.34)

$$\begin{aligned}
T_\varepsilon'(w) &= \frac{1}{2\sqrt{2}w}\sum_{i=0}^{n-1}\left(\int_{\alpha_i}^{\alpha_{i+1}} H_1(u,w)H_2(u,w)\,du + \int_{\beta_i}^{\beta_{i+1}} H_1(u,w)H_2(u,w)\,du\right) \\
&\ge \frac{1}{2\sqrt{2}w}\sum_{i=0}^{n-1}\left(H_1(\alpha_{i+1},w)\int_{\alpha_i}^{\alpha_{i+1}} H_2(u,w)\,du + H_1(\beta_i,w)\int_{\beta_i}^{\beta_{i+1}} H_2(u,w)\,du\right) \\
&= \frac{1}{2\sqrt{2}w}\sum_{i=0}^{n-1}\Big[H_1(\alpha_{i+1},\beta_n)\left(\sqrt{\beta_n-\alpha_{i+1}}I_1(\alpha_{i+1},\beta_n) - \sqrt{\beta_n-\alpha_i}I_1(\alpha_i,\beta_n)\right) \\
&\qquad + H_1(\beta_i,\beta_n)\left(\sqrt{\beta_n-\beta_{i+1}}I_1(\beta_{i+1},\beta_n) - \sqrt{\beta_n-\beta_i}I_1(\beta_i,\beta_n)\right)\Big].
\end{aligned}$$

Similarly,

(3.35)

$$\begin{aligned}
T_\varepsilon'(w) \le \frac{1}{2\sqrt{2}w}\sum_{i=0}^{n-1}\Big[&H_1(\alpha_i,\beta_n)\left(\sqrt{\beta_n-\alpha_{i+1}}I_1(\alpha_{i+1},\beta_n) - \sqrt{\beta_n-\alpha_i}I_1(\alpha_i,\beta_n)\right) \\
&+ H_1(\beta_{i+1},\beta_n)\left(\sqrt{\beta_n-\beta_{i+1}}I_1(\beta_{i+1},\beta_n) - \sqrt{\beta_n-\beta_i}I_1(\beta_i,\beta_n)\right)\Big].
\end{aligned}$$

We note that if α_i and β_i are given for $i = 0, 1, \ldots, n$, the sums

$$\sum_{i=0}^{n-1} \Big[H_1(\alpha_{i+1}, \beta_n) \left(\sqrt{\beta_n - \alpha_{i+1}} I_1(\alpha_{i+1}, \beta_n) - \sqrt{\beta_n - \alpha_i} I_1(\alpha_i, \beta_n) \right) + H_1(\beta_i, \beta_n) \left(\sqrt{\beta_n - \beta_{i+1}} I_1(\beta_{i+1}, \beta_n) - \sqrt{\beta_n - \beta_i} I_1(\beta_i, \beta_n) \right) \Big], \tag{3.36}$$

and

$$\sum_{i=0}^{n-1} \Big[H_1(\alpha_i, \beta_n) \left(\sqrt{\beta_n - \alpha_{i+1}} I_1(\alpha_{i+1}, \beta_n) - \sqrt{\beta_n - \alpha_i} I_1(\alpha_i, \beta_n) \right) + H_1(\beta_{i+1}, \beta_n) \left(\sqrt{\beta_n - \beta_{i+1}} I_1(\beta_{i+1}, \beta_n) - \sqrt{\beta_n - \beta_i} I_1(\beta_i, \beta_n) \right) \Big] \tag{3.37}$$

can be computed but difficult because the numbers α_i and β_i for $i = 0, 1, \ldots, n$ may be complex, see Lemmas 3.1 and 3.3. We choose suitable numbers n, α_i^*, α_{i*}, β_i^*, β_{i*}, satisfying

$$0 \le \alpha_i^* \le \alpha_i \le \alpha_{i*} < \frac{21\sigma}{50\varepsilon}, \quad \text{and} \quad 0 \le \beta_i^* \le \beta_i \le \beta_{i*} < \frac{21\sigma}{50\varepsilon} \quad \text{for } i = 0, 1, \ldots, n,$$

such that by Lemma 3.9(i)–(iv), it is easy to compute and obtain the upper and lower bounds of $H_1(\alpha_i, \beta_n)$, $H_1(\beta_i, \beta_n)$, $I_1(\alpha_i, \beta_n)$, and $I_1(\beta_i, \beta_n)$. Then we apply these upper and lower bounds to determinate that the sum (3.36) is positive or the sum (3.37) is negative. So by (3.34) and (3.35), we see that $T_\varepsilon'(w) > 0$ or $T_\varepsilon'(w) < 0$. Therefore, we have the following Lemma 3.10.

Lemma 3.10. *Consider* (1.1). *The following assertions* (i)–(iii) *holds.*

(i) *If* $\varepsilon = \varepsilon_4$, *then* $T_{\varepsilon_4}'(p_2) > 0$.

(ii) *If* $\varepsilon = \varepsilon_3$, *then* $T_{\varepsilon_3}'(\gamma) > 0$ *and* $T_{\varepsilon_3}'(p_2) < 0$.

(iii) $\widehat{\varepsilon} < \overline{\varepsilon} < \widetilde{\varepsilon}$.

Proof. We prove the assertion (i). Since the proof of assertion (ii) is similar and rather lengthy, we put the proof in [4] and omit it here. Assertion (iii) follows immediately by Theorem 1.1, Lemma 3.7 and assertion (ii). We note that $\sigma/(3\varepsilon_4) = \gamma < p_2 < 21\sigma/(50\varepsilon_4)$ by Lemmas 3.1 and 3.4(iv). Let $\{\alpha_i\}_{i=0}^2$ and $\{\beta_i\}_{i=0}^2$ be uniform partitions of $[0, \overline{p}_2]$ and $[\overline{p}_2, p_2]$, respectively. This implies that

$$\alpha_0 = 0,\ \alpha_1 = \frac{\overline{p}_2}{2},\ \alpha_2 = \overline{p}_2 = \beta_0,\ \beta_1 = \frac{\overline{p}_2 + p_2}{2},\ \beta_2 = p_2.$$

By Lemma 3.4 and direct computations, we have that

$$\frac{\sigma}{6\varepsilon} < \frac{396\sigma}{1000\varepsilon_4} \equiv p_{2*} < p_2 = \frac{L(\varepsilon_4)\sigma}{\varepsilon_4} \left(\approx \frac{0.3967\sigma}{\varepsilon_4} \right) < p_2^* \equiv \frac{397\sigma}{1000\varepsilon_4} < \frac{21\sigma}{50\varepsilon_4}, \tag{3.38}$$

$$\frac{183\sigma}{1000\varepsilon_4} \equiv \widetilde{p}_{2*} < \overline{p}_2 = \frac{R(\varepsilon_4)\sigma}{\varepsilon_4} \left(\approx \frac{0.1830\sigma}{\varepsilon_4} \right) < \widetilde{p}_2^* \equiv \frac{184\sigma}{1000\varepsilon_4} < \frac{21\sigma}{50\varepsilon_4}, \tag{3.39}$$

from which it follows that, for $i = 0, 1, 2$,

$$0 \leq \frac{i\widetilde{p}_{2*}}{2} \equiv \alpha_{i*} \leq \alpha_i \leq \alpha_i^* \equiv \frac{i\widetilde{p}_2^*}{2} < \frac{21\sigma}{50\varepsilon_4}, \tag{3.40}$$

$$0 < \frac{ip_{2*} + (1-i)\widetilde{p}_{2*}}{2} \equiv \beta_{i*} \leq \beta_i \leq \beta_i^* \equiv \frac{ip_2^* + (1-i)\widetilde{p}_2^*}{2} < \frac{21\sigma}{50\varepsilon_4}. \tag{3.41}$$

So by Lemma 3.9(iii),

$$\alpha_1 < \alpha_1^* < \widehat{u}(p_2^*) < \widehat{u}(p_2) \leq \widehat{u}(p_{2*}) < \beta_{1*} < \beta_1 < \beta_1^* \tag{3.42}$$

because we compute and find that

$$\widehat{u}(p_2) \geq \widehat{u}(p_2^*) = \frac{\sigma}{45\varepsilon_4}\left(\sqrt{\frac{67745027}{500000}} - \frac{1483}{1000}\right) > \frac{23\sigma}{250\varepsilon_4} = \alpha_1^*,$$
$$\widehat{u}(p_2) \leq \widehat{u}(p_{2*}) = \frac{\sigma}{45\varepsilon_4}\left(\sqrt{\frac{4268453}{31250}} - \frac{361}{250}\right) < \frac{579\sigma}{2000\varepsilon_4} = \beta_{1*}.$$

By Lemma 3.9, (3.34), (3.36), (3.38)–(3.42), we assert that

$$\begin{aligned} 2\sqrt{2}p_2 T'_{\varepsilon_4}(p_2) \geq{}& H_1(\alpha_1^*, \beta_2^*)\left[\sqrt{\beta_{2*} - \alpha_1^*}I_1(\alpha_1^*, \beta_2^*) - \sqrt{\beta_{2*}}I_1(0, \beta_{2*})\right] \\ &+ H_1(\alpha_{2*}, \beta_{2*})\left[\sqrt{\beta_{2*} - \beta_1^*}I_1(\beta_1^*, \beta_2^*) - \sqrt{\beta_2^* - \alpha_{1*}}I_1(\alpha_1^*, \beta_{2*})\right] \\ &- H_1(\beta_{1*}, \beta_{2*})\sqrt{\beta_2^* - \beta_{1*}}I_1(\beta_{1*}, \beta_{2*}). \end{aligned} \tag{3.43}$$

The proof of assertion (3.43) can be seen below. Assume that $k \equiv \tau/\sqrt{\sigma\rho}$. Clearly, $k \geq 0$. We compute that

$$H_1(\alpha_1^*, \beta_2^*)\left[\sqrt{\beta_{2*} - \alpha_1^*}I_1(\alpha_1^*, \beta_2^*) - \sqrt{\beta_{2*}}I_1(0, \beta_{2*})\right] = \frac{1}{(\sigma\rho)^{\frac{1}{4}}}a_1(k), \tag{3.44}$$

$$H_1(\alpha_{2*}, \beta_{2*})\left[\sqrt{\beta_{2*} - \beta_1^*}I_1(\beta_1^*, \beta_2^*) - \sqrt{\beta_2^* - \alpha_{1*}}I_1(\alpha_1^*, \beta_{2*})\right] = \frac{-1}{(\sigma\rho)^{\frac{1}{4}}}a_2(k), \tag{3.45}$$

$$H_1(\beta_{1*}, \beta_{2*})\sqrt{\beta_2^* - \beta_{1*}}I_1(\beta_{1*}, \beta_{2*}) = \frac{1}{(\sigma\rho)^{\frac{1}{4}}}a_3(k), \tag{3.46}$$

where

$$a_1(k) \equiv \frac{1040\sqrt{15}(1893263429\sqrt{19} + 11939328\sqrt{11})}{7(956758909\sqrt{13} + 1907100000k)^{3/2}},$$
$$a_2(k) \equiv \frac{2\sqrt{2}(26009511424\sqrt{47} - 2857000131\sqrt{2743})}{273(5695037\sqrt{13} + 11580000k)^{3/2}},$$
$$a_3(k) \equiv \frac{841661983552\sqrt{215}}{7(701986729\sqrt{13} + 142584000k)^{3/2}}.$$

We further compute that, for $k \geq 0$,

$$\left[\frac{11}{40}a_1(k)\right]^{2/3} - [a_2(k)]^{2/3} \approx \frac{5.3 \times 10^{13}k + 6.4 \times 10^{13}}{(956758909\sqrt{13} + 19071 \times 10^5 k)(5695037\sqrt{13} + 11580000k)} (> 0),$$

$$\left[\frac{29}{40}a_1(k)\right]^{2/3} - [a_3(k)]^{2/3} \approx \frac{1.4 \times 10^{15}k + 1.6 \times 10^{16}}{(305914073\sqrt{13} + 60964800k)(3476171\sqrt{13} + 706875k)} (> 0),$$

from which it follows that by (3.43)–(3.46), for $k \geq 0$,

$$T'_{\varepsilon_4}(p_2) \geq \frac{a_1(k) - a_2(k) - a_3(k)}{2\sqrt{2}p_2(\sigma\rho)^{\frac{1}{4}}} = \frac{\left(\frac{11}{40}a_1(k) - a_2(k)\right) + \left(\frac{29}{40}a_1(k) - a_3(k)\right)}{2\sqrt{2}p_2(\sigma\rho)^{\frac{1}{4}}} > 0.$$

So $T'_{\varepsilon_4}(p_2) > 0$ for $\varepsilon = \varepsilon_4$.

Finally, we prove our assertion (3.43). Since $H(u, p_2) > 0$ for $0 < u < \overline{p}_2$ and $H(u, p_2) < 0$ for $\overline{p}_2 < u < p_2$ by (3.32), we observe that

$$\int_0^{\alpha_1} H_2(u, w)\, du = \sqrt{\beta_2 - \alpha_1} I_1(\alpha_1, \beta_2) - \sqrt{\beta_2} I_1(0, \beta_2) > 0. \tag{3.47}$$

By Lemma 3.9(iii)–(iv) and (3.38)–(3.42), we observe that

$$I_1(0, \beta_2) \leq I_1(0, \beta_{2*}) = -\frac{62184\rho}{35546875} < 0, \tag{3.48}$$

$$I_1(\alpha_1^*, \beta_{2*}) \geq I_1(\alpha_1, \beta_2) \geq I_1(\alpha_{1*}, \beta_2^*) = \frac{1893263429\rho}{455 \times 10^7} > 0, \tag{3.49}$$

$$I_1(\beta_{1*}, \beta_{2*}) \geq I_1(\beta_1, \beta_2) \geq I_1(\beta_1^*, \beta_2^*) = \frac{2857000131\rho}{455 \times 10^7} > 0. \tag{3.50}$$

By (3.49) and (3.50), we further observe that

$$\begin{aligned}
&\sqrt{\beta_{2*} - \beta_1^*} I_1(\beta_1^*, \beta_2^*) - \sqrt{\beta_2^* - \alpha_{1*}} I_1(\alpha_1^*, \beta_{2*}) \\
&\leq \sqrt{\beta_2 - \beta_1} I_1(\beta_1, \beta_2) - \sqrt{\beta_2 - \alpha_1} I_1(\alpha_1, \beta_2) \\
&\leq \sqrt{\beta_2^* - \beta_{1*}} I_1(\beta_{1*}, \beta_{2*}) - \sqrt{\beta_{2*} - \alpha_1^*} I_1(\alpha_{1*}, \beta_2^*) \\
&= \rho(13)^{3/4} \left(\frac{\rho}{\sigma}\right)^{1/4} \left[\frac{3034838883}{591500000000}\sqrt{215} - \frac{1893263429}{147875000000}\sqrt{38}\right] < 0.
\end{aligned} \tag{3.51}$$

So by (3.34) and (3.47)–(3.51), we observe that

$$\begin{aligned}
2\sqrt{2}p_2 T'_\varepsilon(p_2) &\geq H_1(\alpha_1, \beta_2) \left[\sqrt{\beta_2 - \alpha_1} I_1(\alpha_1, \beta_2) - \sqrt{\beta_2} I_1(0, \beta_2)\right] \\
&\quad + H_1(\alpha_2, \beta) \left[\sqrt{\beta_2 - \beta_1} I_1(\beta_1, \beta_2) - \sqrt{\beta_2 - \alpha_1} I_1(\alpha_1, \beta_2)\right] \\
&\quad - H_1(\beta_1, \beta) \sqrt{\beta_2 - \beta_1} I_1(\beta, \beta) \\
&\geq H_1(\alpha_1^*, \beta_2^*) \left[\sqrt{\beta_{2*} - \alpha_1^*} I_1(\alpha_1^*, \beta_2^*) - \sqrt{\beta_{2*}} I_1(0, \beta_{2*})\right] \\
&\quad + H_1(\alpha_{2*}, \beta_{2*}) \left[\sqrt{\beta_{2*} - \beta_1^*} I_1(\beta_1^*, \beta_2^*) - \sqrt{\beta_2^* - \alpha_{1*}} I_1(\alpha_1^*, \beta_{2*})\right] \\
&\quad - H_1(\beta_{1*}, \beta_{2*}) \sqrt{\beta_2^* - \beta_{1*}} I_1(\beta_{1*}, \beta_{2*}).
\end{aligned}$$

So assertion (3.43) holds. The proof of Lemma 3.10 is complete. □

Lemma 3.11. *Consider* (1.1) *with* $\varepsilon = \overline{\varepsilon} \in (\varepsilon_2, \varepsilon_5)$ *defined in Lemma* 3.7. *Then* $T''_{\overline{\varepsilon}}(p_2) > 0$.

Proof. By Lemma 3.10(i), $\overline{\varepsilon} \in (\varepsilon_2, \varepsilon_4)$. By elementary analysis and Lemma 3.4(iv), we assert that for $\varepsilon_2 < \varepsilon < \varepsilon_4$ and $0 < u < p_2$,

$$\frac{183}{100}[F(p_2) - F(u)] < p_2 f(p_2) - uf(u) < \frac{21}{10}[F(p_2) - F(u)]. \tag{3.52}$$

Since the proof of assertion (3.52) is easy but rather lengthy, and hence we put the proof in [4] and omit it here. By Lemma 3.1, we see that $\theta(p_2) - \theta(u) > 0$ for $0 < u < \overline{p}_2$ and $\theta(p_2) - \theta(u) < 0$ for $\overline{p}_2 < u < p_2$. Since $T''_{\overline{\varepsilon}}(p_2) = 0$, and by Lemma 3.1, (3.3) and (3.52), we have that

$$\begin{aligned}
&2\sqrt{2}p_2^2 T''_{\overline{\varepsilon}}(p_2) \\
&= -\frac{3}{2}\int_0^{p_2} \frac{[\theta(p_2) - \theta(u)][p_2 f(p_2) - uf(u)]}{[F(p_2) - F(u)]^{5/2}}\,du + \int_0^{p_2} \frac{p_2\theta'(p_2) - u\theta'(u)}{[F(p_2) - F(u)]^{3/2}}\,du \\
&> -\frac{3}{2}\left(\frac{21}{10}\right)\int_0^{\overline{p}_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du - \frac{3}{2}\left(\frac{183}{100}\right)\int_{\overline{p}_2}^{p_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du \\
&\quad + \int_0^{p_2} \frac{p_2\theta'(p_2) - u\theta'(u)}{[F(p_2) - F(u)]^{3/2}}\,du \quad \text{(by (3.52))} \\
&= -\frac{63}{20}\left\{\int_0^{\overline{p}_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du - \int_{\overline{p}_2}^{p_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du\right\} \\
&\quad - \left(\frac{549}{200} - \frac{63}{20}\right)\int_{\overline{p}_2}^{p_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du + \int_0^{p_2} \frac{p_2\theta'(p_2) - u\theta'(u)}{[F(p_2) - F(u)]^{3/2}}\,du \\
&= -\frac{63}{20}T''_{\overline{\varepsilon}}(p_2) + \frac{81}{200}\int_{\overline{p}_2}^{p_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du + \int_0^{p_2} \frac{p_2\theta'(p_2) - u\theta'(u)}{[F(p_2) - F(u)]^{3/2}}\,du \\
&= \frac{81}{200}\int_{\overline{p}_2}^{p_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2) - F(u)]^{3/2}}\,du.
\end{aligned} \tag{3.53}$$

We assert that, for $\varepsilon_2 < \varepsilon < \varepsilon_4$,

$$\begin{aligned}
&\frac{81}{200}\int_{\overline{p}_2}^{p_2} \frac{\theta(p_2) - \theta(u)}{[F(p_2) - F(u)]^{3/2}}\,du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2) - F(u)]^{3/2}}\,du \\
&> -\frac{81}{200}H_1(\widetilde{p}_{2*}, p_{2*})\sqrt{p_2^* - \widetilde{p}_{2*}}\,I_1(\widetilde{p}_2^*, p_{2*}) - H_3(p_{1_*}, p_2^*)\frac{I_2(0, p_{2*})}{\sqrt{p_2^*}} > 0
\end{aligned} \tag{3.54}$$

for some positive numbers p_{1_*}, $\widetilde{p}_{2*}$, $\widetilde{p}_2^*$, p_{2*} and p_2^* satisfying $p_{1_*} \le p_1$ and

$$\frac{3\sigma}{25\varepsilon} \le \widetilde{p}_{2*} < \overline{p}_2 < \widetilde{p}_2^* \le \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} \le p_{2*} < p_2 < p_2^* \le \frac{417\sigma}{1000\varepsilon}.$$

Since the proof of assertion (3.54) is rather lengthy and is similar to the proof of assertion (i) of Lemma 3.10, we put the proof in [4] and omit it here. So by (3.53) and (3.54), we see that $T''_{\overline{\varepsilon}}(p_2) > 0$. The proof of Lemma 3.11 is complete. □

4. Proof of the main result

By Theorem 1.1, for $0 < \varepsilon < \widetilde{\varepsilon}$, the S-shaped bifurcation curve S_ε has exactly two turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$, see Figure 1.1(i). Thus by (3.1), for $0 < \varepsilon < \widetilde{\varepsilon}$, $T_\varepsilon(\alpha)$ has exactly two (positive) critical points at $\alpha_\varepsilon^- \, (\equiv \|u_{\lambda^*}\|_\infty) < \alpha_\varepsilon^+ \, (\equiv \|u_{\lambda_*}\|_\infty)$.

Proof of Theorem 2.1. Let $0 < \varepsilon < \widetilde{\varepsilon}$. Since $\theta(p_1) - \theta(u) > 0$ for $0 < u < p_1$ by Lemma 3.1, and by (3.2), we see that $T_\varepsilon'(p_1) > 0$ for $\varepsilon > 0$. By Lemma 3.10(iii), $\widehat{\varepsilon} < \overline{\varepsilon} < \widetilde{\varepsilon}$. Assume that $\widehat{\varepsilon} < \varepsilon < \widetilde{\varepsilon}$. Since $T_\varepsilon'(\gamma) > 0$ by Lemma 3.7, we see that either $\gamma < \alpha_\varepsilon^-$ or $\alpha_\varepsilon^+ < \gamma$. By [8, Lemma 3.2], we obtain that either $T_\varepsilon(\alpha)$ is a strictly increasing function of α on $(0, \gamma]$, or $T_\varepsilon(\alpha)$ is a strictly increasing and then strictly decreasing function of α on $(0, \gamma]$. So we further obtain that $\gamma < \alpha_\varepsilon^-$. It follows that

$$p_1 < \alpha_\varepsilon^- < \gamma < \alpha_\varepsilon^+ \quad \text{for } 0 < \varepsilon < \widehat{\varepsilon}, \tag{4.1}$$

$$p_1 < \gamma = \alpha_\varepsilon^- < \alpha_\varepsilon^+ \quad \text{for } \varepsilon = \widehat{\varepsilon}, \tag{4.2}$$

$$p_1 < \gamma < \alpha_\varepsilon^- < \alpha_\varepsilon^+ \quad \text{for } \widehat{\varepsilon} < \varepsilon < \widetilde{\varepsilon}. \tag{4.3}$$

By Lemma 3.7, it is easy to see that $\alpha_\varepsilon^- < p_2 < \alpha_\varepsilon^+$ for $0 < \varepsilon < \overline{\varepsilon}$. For $\varepsilon = \overline{\varepsilon}$, we see that either $p_2 = \alpha_\varepsilon^-$ or $p_2 = \alpha_\varepsilon^+$. By Lemma 3.11, we see that $p_2 = \alpha_\varepsilon^+$. We note that α_ε^- and α_ε^+ are continuous functions of ε on $(0, \widetilde{\varepsilon})$ by [6, Remark 2.2]. By Lemma 3.1, it is easy to see that p_2 is a continuous function of ε on $(0, \widetilde{\varepsilon})$. Since $T_\varepsilon'(p_2) > 0$ for $\overline{\varepsilon} < \varepsilon < \widetilde{\varepsilon}$ by Lemma 3.7, we observe that $\alpha_\varepsilon^- < \alpha_\varepsilon^+ < p_2$ for $\overline{\varepsilon} < \varepsilon < \widetilde{\varepsilon}$. So we have that

$$p_1 < \alpha_\varepsilon^- < p_2 < \alpha_\varepsilon^+ \quad \text{for } 0 < \varepsilon < \overline{\varepsilon}, \tag{4.4}$$

$$p_1 < \alpha_\varepsilon^- < \alpha_\varepsilon^+ = p_2 \quad \text{for } \varepsilon = \overline{\varepsilon}, \tag{4.5}$$

$$p_1 < \alpha_\varepsilon^- < \alpha_\varepsilon^+ < p_2 \quad \text{for } \overline{\varepsilon} < \varepsilon < \widetilde{\varepsilon}. \tag{4.6}$$

Thus, by (4.1)–(4.6), inequalities (2.3)–(2.7) hold. By [6, Remark 2.2], we see that $\lim_{\varepsilon \to \widetilde{\varepsilon}^+} \|u_{\lambda^*}\|_\infty = \lim_{\varepsilon \to \widetilde{\varepsilon}^+} \|u_{\lambda^*}\|_\infty = \left\|u_{\widetilde{\lambda}}\right\|_\infty$. Assume that $p_2 = \left\|u_{\widetilde{\lambda}}\right\|_\infty$ for $\varepsilon = \widetilde{\varepsilon}$. By Theorem 1.1(iii), we see that $T_{\widetilde{\varepsilon}}'(p_2) = T_{\widetilde{\varepsilon}}'(\left\|u_{\widetilde{\lambda}}\right\|_\infty) = 0$ for $\varepsilon = \widetilde{\varepsilon}$, which is a contradiction to Lemma 3.7. So $p_2 > \left\|u_{\widetilde{\lambda}}\right\|_\infty$ for $\varepsilon = \widetilde{\varepsilon}$. Similarly, by Lemma 3.7, we observe that $\gamma < \left\|u_{\widetilde{\lambda}}\right\|_\infty$ for $\varepsilon = \widetilde{\varepsilon}$. In addition, by Lemma 3.4, $K(\widetilde{\varepsilon}) < K(\varepsilon_5) = 1/3$ for $\varepsilon_2 \leq \widetilde{\varepsilon} < \varepsilon_5$. This implies that $p_1 = K(\widetilde{\varepsilon})\sigma/\widetilde{\varepsilon} < \sigma/(3\widetilde{\varepsilon}) = \gamma$ for $\varepsilon = \widetilde{\varepsilon}$. Thus, (2.8) holds.

Finally, we prove $\widetilde{\varepsilon} < \sqrt{\frac{83\sigma^3}{2500\rho}}$. By (2.7) and (2.8), it is sufficient to prove that $T_\varepsilon'(\alpha) > 0$ for $0 < \alpha \leq p_2$ and $\varepsilon = \sqrt{\frac{83\sigma^3}{2500\rho}}$. By Lemmas 3.1 and 3.4, we compute that

$$p_2 = \frac{\sigma}{\varepsilon} L\left(\sqrt{\frac{332\sigma^3}{10000\rho}}\right) \left(\approx \frac{0.391\sigma}{\varepsilon}\right) < \frac{21\sigma}{50\varepsilon} \quad \text{if } \varepsilon = \sqrt{\frac{83\sigma^3}{2500\rho}}.$$

By Lemma 3.9(i), $H_1(u,\alpha)$ is a strictly decreasing function of u on $[0,\alpha]$ for $0<\alpha\le p_2$. So by (3.2), (3.32) and Lemmas 3.1 and 3.2, we observe that, for $0<\alpha\le p_2$ and $\varepsilon=\sqrt{\frac{83\sigma^3}{2500\rho}}$,

$$\begin{aligned}
T'_\varepsilon(\alpha) &= \frac{1}{2\sqrt{2}\alpha^2}\int_0^\alpha H_1(u,\alpha)H_2(u,\alpha)\,du \\
&\ge \frac{1}{2\sqrt{2}\alpha^2}\left[H_1(\overline{\alpha},\alpha)\int_0^{\overline{\alpha}} H_2(u,\alpha)\,du + H_1(\overline{\alpha},\alpha)\int_{\overline{\alpha}}^\alpha H_2(u,\alpha)\,du\right] \\
&= \frac{H_1(\overline{\alpha},\alpha)}{2\sqrt{2}\alpha^2}\int_0^\alpha H_2(u,\alpha)\,du \\
&= \frac{H_1(\overline{\alpha},\alpha)}{35\sqrt{2}\alpha^{3/2}}\left[279\varepsilon\alpha^3-154\alpha^2\sigma+210\rho\right] \\
&\ge \frac{H_1(\overline{\alpha},\alpha)}{35\sqrt{2}\alpha^{3/2}}\left[279\varepsilon\alpha^3-154\alpha^2\sigma+210\rho\right]_{\alpha=\frac{308\sigma}{837\varepsilon}} \\
&= \frac{1573043\sqrt{2}\rho H_1(\overline{\alpha},\alpha)}{174441681\alpha^{3/2}} > 0
\end{aligned}$$

since

$$\frac{\partial}{\partial\alpha}(279\varepsilon\alpha^3-154\alpha^2\sigma+210\rho)=\alpha(837\varepsilon\alpha-308\sigma)\begin{cases} <0 & \text{if } 0<\alpha<\frac{308\sigma}{837\varepsilon}, \\ =0 & \text{if } \alpha=\frac{308\sigma}{837\varepsilon}, \\ >0 & \text{if } \alpha>\frac{308\sigma}{837\varepsilon}. \end{cases}$$

Hence, $\widetilde{\varepsilon}<\sqrt{\frac{83\sigma^3}{2500\rho}}$, which implies that (2.2) holds. This completes the proof of Theorem 2.1. □

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