On a Third Order Flow of Convex Closed Plane Curves

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Abstract. We study a curve flow for convex closed plane curves. It is described by a third order linear equation for the radius of curvature of the evolving curve. It is shown that under the flow the evolving curve stays convex, bounds fixed area, length, and has fixed center. However, its curvature may blow up in finite time.

If the curvature of this flow does not blow up before time 2π, then the flow will exist smoothly for all time and is periodic in time with period 2π. In particular, the flow does not have a limiting curve unless the initial curve is a circle.

1. Introduction: the flow and the equation

Many previous works of the curve flows investigated parabolic type geometric equations and concerned the asymptotic behavior of the evolving curves, for example, the famous curve shortening flow (see [7,9,12], etc.), the non-local flows (see [4,8,13,15,16], etc.) and other relative works (see the references in the book [6]). In 1991, Goldstein-Petrich [11] and in 1992, Nakayama-Segur-Wadati [17] introduced a different type of flow, which is an initial value problem of the form:

$$(1.1) \qquad \begin{cases} \dfrac{\partial X}{\partial t}(\varphi, t) = -\dfrac{\partial k}{\partial s}(\varphi, t)\mathbf{N}_{\text{in}}(\varphi, t) & \text{in } S^1 \times (0, T) \\ X(\varphi, 0) = X_0(\varphi) & \text{on } S^1. \end{cases}$$

Here $X(\varphi, t) = (x(\varphi, t), y(\varphi, t)) \colon S^1 \times [0, T) \to \mathbb{R}^2$ is a family of time-dependent smooth counterclockwise oriented simple closed curves, with curvature $k(\varphi, t)$, and $\partial k(\varphi, t)/\partial s$ is the derivative of curvature with respect to its arc length parameter s. Here $\mathbf{N}_{\text{in}}(\varphi, t)$ is the *inward* unit normal of $X(\varphi, t)$ so that the frame $\{\mathbf{T}(\varphi, t), \mathbf{N}_{\text{in}}(\varphi, t)\}$ gives a positive orientation of $\mathbb{R}^2$ for all (φ, t).

It is known that the flow (1.1) preserves both the enclosed area and length of the evolving curve $X(\varphi, t)$. Hence each $X(\cdot, t)$ has the same enclosed area $A(0)$ and length

$L(0)$ as the initial curve $X_0(\cdot)$. By adding a suitable tangential component to the flow (1.1), which will not affect the geometry of the flow and can allow us to use, instead of φ, the arc length parameter s as variable, (1.1) can be reduced to the following modified KdV equation of the curvature

$$(1.2) \qquad \frac{\partial k}{\partial t} + \frac{\partial^3 k}{\partial s^3} + \frac{3}{2} k^2 \frac{\partial k}{\partial s} = 0, \quad \text{where } k = k(s,t), \ (s,t) \in L(0) \times (0,T)$$

with a periodic initial value $k_0(s)$. See the book [6] for details. One can obtain many interesting results of the equation (1.2) by the theory of m-KdV equations (see [1,3,5]).

Inspired by this m-KdV flow (1.1), we consider a flow for **convex** closed plane curves[1] of the form:

$$(1.3) \qquad \begin{cases} \dfrac{\partial X}{\partial t}(\varphi,t) = \left[\dfrac{\partial}{\partial s} \left(\dfrac{1}{2} \dfrac{1}{k^2(\varphi,t)} \right) \right] \mathbf{N}_{\mathrm{in}}(\varphi,t) & \text{in } S^1 \times (0,T), \\ X(\varphi,0) = X_0(\varphi) & \text{on } S^1, \end{cases}$$

where $X_0(\varphi)$, $\varphi \in S^1$, is a given initial smooth convex closed curve with enclosed area $A(0)$ and length $L(0)$.

The flow (1.3) has a smooth solution $X(\varphi,t) \in \mathbb{R}^2$ defined on $S^1 \times [0,T)$ for some short time $T > 0$, where each $X(\cdot,t)$ represents a convex closed curve. Moreover, since the coefficient in front of the normal vector $\mathbf{N}_{\mathrm{in}}$ is also a derivative, similar to (1.1), this flow is both area-preserving and length-preserving.

Remark 1.1. To see that the flow (1.3) does have a smooth convex solution (by "*convex solution*" we mean that each $X(\cdot,t)$ is a convex closed curve) $X(\varphi,t)$ for some short time, one can solve the linear equation (1.6) first, and then use the radius of curvature $\rho(\theta,t)$ to construct a family of convex closed curves satisfying the flow (1.3). This is a standard method.

In terms of the outward normal angle $\theta \in S^1$, the *support function* $u(\theta,t)$ (see below for its definition) of $X(\varphi,t)$ will satisfy the third order *linear* equation

$$(1.4) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(\theta,t) = -\dfrac{\partial^3 u}{\partial \theta^3}(\theta,t) - \dfrac{\partial u}{\partial \theta}(\theta,t), & (\theta,t) \in S^1 \times (0,T), \\ u(\theta,0) = u_0(\theta), & \theta \in S^1, \end{cases}$$

where $u_0(\theta)$ is the support function of the initial curve $X_0(\varphi)$. Moreover, by the identity between $u(\theta,t)$ and the positive curvature $k(\theta,t)$ of $X(\cdot,t)$

$$(1.5) \qquad \frac{1}{k(\theta,t)} = u_{\theta\theta}(\theta,t) + u(\theta,t), \quad (\theta,t) \in S^1 \times (0,T),$$

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[1]From now on, for a convex closed plane curve X_0, we always assume that it is embedded, counterclockwise oriented, and has positive curvature everywhere.

the radius of curvature $\rho(\theta, t) := 1/k(\theta, t)$ will also satisfy the third order *linear* equation

(1.6)
$$\begin{cases} \dfrac{\partial \rho}{\partial t}(\theta, t) = -\dfrac{\partial^3 \rho}{\partial \theta^3}(\theta, t) - \dfrac{\partial \rho}{\partial \theta}(\theta, t), & (\theta, t) \in S^1 \times (0, T), \\ \rho(\theta, 0) = \rho_0(\theta) > 0, & \theta \in S^1, \end{cases}$$

where $\rho_0(\theta) = 1/k_0(\theta)$ is the radius of curvature of the initial curve $X_0(\varphi)$. Note that $\rho_0(\theta)$ satisfies the *closing condition*

(1.7)
$$\int_0^{2\pi} \rho_0(\theta) \cos \theta \, d\theta = \int_0^{2\pi} \rho_0(\theta) \sin \theta \, d\theta = 0,$$

due to the fact that X_0 is a *closed* curve.

To derive (1.4), we note the following: since at each time $t \in [0, T)$ the curve $X(\varphi, t)$ is convex, one can also use its outward normal angle $\theta \in S^1$ to parametrize it (more precisely, $\varphi \longleftrightarrow \theta$ is a change of variables). Thus one can write $X(\varphi, t)$ as $X(\varphi(\theta, t), t) := \widetilde{X}(\theta, t)$, $(\theta, t) \in S^1 \times [0, T)$, where at the point $\widetilde{X}(\theta, t)$ its outward normal vector is given by $\mathbf{N}_{\text{out}}(\theta) = (\cos \theta, \sin \theta)$. Recall that the *support function* $u(\theta, t)$ of $\widetilde{X}(\theta, t)$ is defined by

$$u(\theta, t) := \left\langle \widetilde{X}(\theta, t), \mathbf{N}_{\text{out}}(\theta) \right\rangle = \left\langle \widetilde{X}(\theta, t), (\cos \theta, \sin \theta) \right\rangle,$$

which gives

$$\frac{\partial u}{\partial t}(\theta, t) = \frac{\partial}{\partial t} \left\langle \widetilde{X}(\theta, t), \mathbf{N}_{out}(\theta) \right\rangle = \left\langle \frac{\partial X}{\partial \varphi}(\varphi, t) \frac{\partial \varphi}{\partial t}(\theta, t) + \frac{\partial X}{\partial t}(\varphi, t), \mathbf{N}_{\text{out}}(\theta) \right\rangle.$$

Since $\frac{\partial X}{\partial \varphi}(\varphi, t)$ is perpendicular to $\mathbf{N}_{\text{out}}(\theta)$, and by the relation $\frac{\partial}{\partial s} = k \frac{\partial}{\partial \theta}$, which is valid only for convex curves, the above becomes

(1.8)
$$\frac{\partial u}{\partial t}(\theta, t) = \left\langle \frac{\partial X}{\partial t}(\varphi, t), \mathbf{N}_{\text{out}}(\theta) \right\rangle = \left\langle \left[\frac{\partial}{\partial s} \left(\frac{1}{2} \frac{1}{k^2(\varphi, t)} \right) \right] \mathbf{N}_{\text{in}}(\varphi, t), \mathbf{N}_{\text{out}}(\theta) \right\rangle$$
$$= -\frac{\partial}{\partial s} \left(\frac{1}{2} \frac{1}{k^2(\varphi, t)} \right) = -\frac{\partial}{\partial \theta} \left(\frac{1}{k(\theta, t)} \right) = -\frac{\partial^3 u}{\partial \theta^3}(\theta, t) - \frac{\partial u}{\partial \theta}(\theta, t),$$

which verifies both (1.4) and (1.6).

Similar to the curve shortening flow [9] and other curve flows of convex closed curves, the flow problem (1.3) is equivalent to the third order *linear* equation problem (1.4) (or (1.6)). More precisely, given an initial smooth convex closed curve $X_0(\varphi)$ with support function $u_0(\theta)$, if the flow (1.3) has a smooth convex solution $X(\varphi, t)$ defined on $S^1 \times [0, T)$, then its support function $u(\theta, t)$ will satisfy (1.4) on $S^1 \times [0, T)$, which has been explained in the above. Similarly, its radius of curvature $\rho(\theta, t)$ will also satisfy (1.6) on $S^1 \times [0, T)$.

Conversely, if $\rho(\theta, t)$ satisfies (1.6) on $S^1 \times [0, T)$ and remains *positive* on $S^1 \times [0, T)$, then one can easily check that the closing condition (1.7) is automatically preserved, i.e.,

(1.9)
$$\int_0^{2\pi} \rho(\theta, t) \cos \theta \, d\theta = \int_0^{2\pi} \rho(\theta, t) \sin \theta \, d\theta = 0, \quad \forall \, t \in [0, T),$$

and so we can use $\rho(\theta, t) > 0$ to construct a family of convex closed curves $X(\cdot, t)$ (with curvature given by $k(\theta, t) = 1/\rho(\theta, t)$), which satisfies the flow (1.3) on $S^1 \times [0, T)$. As this has been quite well-known, we will not go into details on this.

Also, if $u(\theta, t)$ satisfies (1.4) on $S^1 \times [0, T)$ with

$$(1.10) \qquad\qquad u_{\theta\theta}(\theta, t) + u(\theta, t) > 0 \quad \text{on } S^1 \times [0, T),$$

then the curve defined by

$$(1.11) \quad \widetilde{X}(\theta, t) := u(\theta, t)(\cos\theta, \sin\theta) + u_\theta(\theta, t)(-\sin\theta, \cos\theta) \in \mathbb{R}^2, \quad (\theta, t) \in S^1 \times [0, T),$$

is a convex closed curve with positive curvature $k(\theta, t) = 1/(u_{\theta\theta}(\theta, t) + u(\theta, t))$, and after suitable reparametrization, it will satisfy the flow (1.3). Again, we will not go into details on this.

In view of the above equivalence, from now on, when we consider the flow (1.3) for an initial convex closed curve $X_0(\varphi)$ with support function $u_0(\theta)$ and radius of curvature $\rho_0(\theta)$, we can just focus on the initial value problem (1.4) or the initial value problem (1.6), with the understanding that the solutions $u(\theta, t)$ and $\rho(\theta, t)$ represent respectively the support function and the radius of curvature of the evolving convex curve $X(\cdot, t)$ at the point with outward normal angle $\theta \in S^1$. They can describe the geometry of $X(\cdot, t)$ completely.

Remark 1.2. Be careful about the distinction: the solution $\rho(\theta, t)$ to the linear equation (1.6) is always defined on $S^1 \times [0, \infty)$. If $0 < T_{\max} \leq \infty$ is the first time such that $\rho_{\min}(t) := \min_{\theta \in S^1} \rho(\theta, t)$ is zero at $T_{\max}$, then one can construct the solution $X(\varphi, t)$ to the flow (1.3) only on the domain $S^1 \times [0, T_{\max})$ where $\rho(\theta, t)$ is positive. The maximum curvature $k_{\max}(t) = 1/\rho_{\min}(t)$ of $X(\cdot, t)$ blows up at time $T_{\max} > 0$.

The main theorem in this paper is to prove the following general behavior of the flow (1.3):

Theorem 1.3 (The behavior of the flow (1.3))**.** *Given an initial smooth convex closed curve $X_0(\varphi)$, $\varphi \in S^1$, and evolve it under the flow (1.3). Then the flow has a unique smooth convex solution $X(\varphi, t)$ defined on a maximal domain $S^1 \times [0, T_{\max})$, where $T_{\max}$ is the first curvature blow-up time of $X(\cdot, t)$, and we have either $0 < T_{\max} < 2\pi$ or $T_{\max} = \infty$. Moreover, each $X(\cdot, t)$, $t \in [0, T_{\max})$, remains uniformly convex, and has the same enclosed area, length, and center as the initial curve X_0. In addition, its radius of curvature $\rho(\theta, t)$ and support function $u(\theta, t)$ satisfy the estimates in Lemma 2.3 and Corollary 2.4 on $S^1 \times [0, T_{\max})$. If $T_{\max} = \infty$, then the solution $X(\varphi, t)$ is also periodic in time with period 2π. In such a case, the flow cannot have a limiting curve unless the initial curve X_0 is a circle.*

Proof. Theorem 1.3 can be proved using Lemma 2.1, Lemma 2.3, Corollary 2.4, Lemma 2.5, Lemma 2.6, and Corollary 2.8 in Section 2.1 and Section 2.2. □

2. Estimates on the support function and the radius of curvature

2.1. Quantities independent of time

Given an initial convex closed curve $X_0(\varphi)$ with $u_0(\theta)$ and $\rho_0(\theta)$, the solutions $u(\theta, t)$ to (1.4) and $\rho(\theta, t)$ to (1.6) are both defined on $S^1 \times [0, \infty)$. As long as $X(\cdot, t)$ is a smooth convex closed curve (see Remark 1.2), its enclosed area $A(t)$, length $L(t)$, and position center $P(t)$ can be expressed as

$$(2.1) \qquad A(t) = \frac{1}{2} \int_0^{2\pi} u(\theta, t) \rho(\theta, t)\, d\theta, \quad L(t) = \int_0^{2\pi} u(\theta, t)\, d\theta$$

and

$$(2.2) \qquad P(t) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{X}(\theta, t)\, d\theta = \frac{1}{\pi} \int_0^{2\pi} u(\theta, t)(\cos\theta, \sin\theta)\, d\theta.$$

Using the evolution equations for u and ρ, one can easily prove the following:

Lemma 2.1 (Preserving area, length and center)**.** *The enclosed area $A(t)$, length $L(t)$, and position center $P(t)$ of $X(\cdot, t)$ are all independent of time.*

Remark 2.2. By the above lemma, the isoperimetric ratio of $L^2(t)/(4\pi A(t))$ of $X(\cdot, t)$ will not improve. Hence $X(\cdot, t)$ will not become circular. This is because the flow (1.3) is not parabolic.

As the equations (1.4) and (1.6) are not parabolic, the useful maximum principle cannot be utilized. Instead, we do integral estimates. We have:

Lemma 2.3 (Conservation law)**.** *Let $m \in \mathbb{N} \cup \{0\}$ be a non-negative integer. We have*

$$(2.3) \qquad \frac{d}{dt} \int_0^{2\pi} \left(\frac{\partial^m \rho}{\partial \theta^m} \right)^2 (\theta, t)\, d\theta = 0, \quad \forall\, t \in [0, \infty)$$

and

$$(2.4) \qquad \frac{d}{dt} \int_0^{2\pi} \left(\frac{\partial^m \rho}{\partial t^m} \right)^2 (\theta, t)\, d\theta = 0, \quad \forall\, t \in [0, \infty).$$

The same result holds if we replace $\rho(\theta, t)$ by $u(\theta, t)$.

Proof. Note that each $\rho(\theta, t)$ is periodic in θ with period 2π. For $t \in [0, \infty)$, we have

$$\frac{d}{dt} \left[\frac{1}{2} \int_0^{2\pi} \left(\frac{\partial^m \rho}{\partial \theta^m} \right)^2 d\theta \right] = - \int_0^{2\pi} \left(\frac{\partial^m \rho}{\partial \theta^m} \right) \left(\frac{\partial^{m+3} \rho}{\partial \theta^{m+3}} + \frac{\partial^{m+1} \rho}{\partial \theta^{m+1}} \right) d\theta$$

$$= \int_0^{2\pi} \frac{\partial}{\partial \theta} \left[\frac{1}{2} \left(\frac{\partial^{m+1} \rho}{\partial \theta^{m+1}} \right)^2 - \frac{1}{2} \left(\frac{\partial^m \rho}{\partial \theta^m} \right)^2 \right] d\theta = 0.$$

This proves the first identity. For the second, we have

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^{2\pi}\left(\frac{\partial^m\rho}{\partial t^m}\right)^2 d\theta\right] = \int_0^{2\pi}\frac{\partial}{\partial\theta}\left[\frac{1}{2}\left(\frac{\partial^m(\rho_\theta)}{\partial t^m}\right)^2 - \frac{1}{2}\left(\frac{\partial^m\rho}{\partial t^m}\right)^2\right]d\theta = 0.$$

The proof is done.   □

There is an elementary Sobolev-type inequality saying that for any function $f(x)$ defined on interval $[0, L]$, if there exists a constant C such that $\|f\|_2 \leq C$ and $\left\|\frac{df}{dx}\right\|_2 \leq C$, then we have

$$(2.5) \qquad\qquad \|f\|_\infty \leq \left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)C.$$

Here $\|\cdot\|_2$ is the L^2 norm and $\|\cdot\|_\infty$ is the L^∞ norm for functions on $[0, L]$.

With the help of (2.5) and Lemma 2.3, we clearly have:

Corollary 2.4 (Uniform bound on space-time derivatives). *Let $\rho(\theta, t)\colon S^1 \times [0, \infty) \to \mathbb{R}$ be the smooth solution to (1.6). Then for any $m \in \mathbb{N} \cup \{0\}$ there exists a positive constant $C(m, \rho_0)$ depending only on m and ρ_0 such that*

$$(2.6) \qquad\qquad \sup_{S^1\times[0,\infty)}\left|\frac{\partial^m\rho}{\partial\theta^m}(\theta, t)\right| \leq C(m, \rho_0).$$

The same result holds if we replace $\frac{\partial^m\rho}{\partial\theta^m}(\theta, t)$ by $\frac{\partial^m\rho}{\partial t^m}(\theta, t)$. Also, the same result holds if we replace $\rho(\theta, t)$ by $u(\theta, t)$.

When $m = 0$ in (2.6), we obtain a uniform upper bound of $|\rho(\theta, t)| = \left|k^{-1}(\theta, t)\right|$. From this, we infer the following:

Lemma 2.5 (Positive lower bound of the curvature; preserving the convexity). *Under the flow (1.3), as long as the curvature does not blow up, there is a positive constant C depending only on X_0 such that the curvature k has C as its positive lower bound.*

2.2. Fourier series expansion and the periodic behavior of the flow (1.3)

As equation (1.6) is linear, one can use *Fourier series expansion* to express the solution explicitly and study the behavior of the flow. See [10, 14] also. By (1.9), the equation $\rho_t = -\rho_{\theta\theta\theta} - \rho_\theta$ and the estimate (2.6) on ρ, it is easy to check that the Fourier series expansion of $\rho(\theta, t)$ is given by

$$(2.7) \qquad \rho(\theta, t) = \frac{L(0)}{2\pi} + \sum_{n=2}^{\infty}\{a_n(t)\cos(n\theta) + b_n(t)\sin(n\theta)\}, \quad (\theta, t) \in S^1 \times [0, \infty),$$

where

$$(2.8) \quad \begin{cases} a_n(t) = a_n(0)\cos((n^3 - n)t) + b_n(0)\sin((n^3 - n)t), \\ b_n(t) = b_n(0)\cos((n^3 - n)t) - a_n(0)\sin((n^3 - n)t), \quad \forall t \in [0, \infty). \end{cases}$$

In the above, $L(0)$ is the length of X_0 and $a_n(0)$, $b_n(0)$ are the Fourier series coefficients of $\rho_0(\theta)$. For convenience of later discussion, we can also write (2.7) as

$$(2.9) \quad \rho(\theta, t) = \frac{L(0)}{2\pi} + \sum_{n=2}^{\infty} \left\{ a_n(0)\cos\left[n\theta + (n^3 - n)t\right] + b_n(0)\sin\left[n\theta + (n^3 - n)t\right] \right\}$$

for all $(\theta, t) \in S^1 \times [0, \infty)$.

By (2.9), we see that the behavior of $\rho(\theta, t)$ depending on the space variable θ is roughly the same as the behavior of it depending on the time variable t. In particular, we have the following useful property:

Lemma 2.6 (Periodicity of the solution). *The solution $\rho(\theta, t)$ to (1.6) on $S^1 \times [0, \infty)$ is periodic in both space and time with period 2π. The same result holds for $u(\theta, t)$.*

Remark 2.7. Formula (2.9) also explains why we have the conservation law as in Lemma 2.3.

By Lemma 2.6, we have:

Corollary 2.8. *If the curvature of the flow (1.3) does not blow up before time $t = 2\pi$, then the flow is smoothly defined on $S^1 \times [0, \infty)$. Moreover, it is periodic in both space and time with period 2π.*

Proof. If the curvature does not blow up before time $t = 2\pi$, then $\rho(\theta, t) > 0$ on $S^1 \times [0, 2\pi)$. Since it is periodic in both space and time with period 2π, we also have $\rho(\theta, 2\pi) = \rho_0(\theta) > 0$ on S^1. Hence $\rho(\theta, t) > 0$ on $S^1 \times [0, \infty)$ and the conclusion follows. $\square$

The following additional properties are all easy to verify:

Lemma 2.9. *We have the following:*

(1) *The solution $\rho(\theta, t)$ to the initial value problem (1.6) is actually defined on $S^1 \times (-\infty, \infty)$. This means that the flow (1.3) can be evolved backward in time (until its curvature blows up).*

(2) *If $\rho(\theta, t)$ is a solution to (1.6), then $\widetilde{\rho}(\theta, t) := \rho(-\theta, -t)$ is also a solution to the same equation with initial condition $\rho_0(-\theta)$, $\theta \in S^1$. In particular, if the initial condition $\rho_0(\theta)$ satisfies the property $\rho_0(\theta) = \rho_0(-\theta)$ (even function) for all $\theta \in S^1$, then we have $\rho(\theta, t) = \rho(-\theta, -t)$ on $S^1 \times (-\infty, \infty)$.*

(3) *The only self-similar solutions to the flow (1.3) are circles.*

2.3. Parallel curves and the blow-up of the curvature

The Fourier series expansion allows us to find special solutions of the equation (1.6). Moreover, one can prove the formation of a singularity under the flow (1.3), i.e., the curvature blows up in finite time.

Lemma 2.10 (The formation of a singularity)**.** *There exists a convex closed curve X_0 such that under the flow* (1.3), *its curvature blows up at time $T_{\max} \in (0, 2\pi)$.*

Proof. There are indeed many blow-up examples. We choose a 2π-periodic function $\rho_0(\theta)$ given by

$$(2.10) \qquad\qquad \rho_0(\theta) = 10 - 3.3\cos(2\theta) - 8\sin(3\theta), \quad \theta \in S^1.$$

The function satisfies the closing condition (1.7) and has a *positive* minimum value $0.1429\ldots$ over S^1, occurred at $\theta = 0.4510\ldots$ (we use Matlab 8.0 to compute it). Hence it is positive everywhere and one can find a convex closed curve X_0 whose radius of curvature, in terms of its outward normal angle θ, is given by the above $\rho_0(\theta)$. The solution $\rho(\theta, t)$ satisfying (1.6) with (2.10) as the initial data is given by

$$(2.11) \qquad \rho(\theta, t) = 10 - 3.3\cos(2\theta + 6t) - 8\sin(3\theta + 24t), \quad (\theta, t) \in S^1 \times [0, \infty).$$

Since we now have two degree of freedom, it is easy to find (θ, t) with $\rho(\theta, t) < 0$. We choose (θ, t) satisfying $2\theta + 6t = 4\pi$, $3\theta + 24t = 6\pi + \pi/2$, and get $t = 0.10472 > 0$, $\theta = 2\pi - 3t$. Therefore, the minimum value of $\rho(\theta, t)$ over $S^1 \times [0, \infty)$ is given by $-1.3 < 0$. The curvature of the flow does blow up in finite time (see Figure 2.1). $\qquad\square$

Remark 2.11. At the first time $T_{\max} > 0$ such that $\rho(\theta_0, T_{\max}) = 0$ for some $\theta_0 \in S^1$ (here $\rho(\theta, t)$ is given by (2.11)), by the three equations $\rho(\theta_0, T_{\max}) = 0$, $\rho_\theta(\theta_0, T_{\max}) = 0$, $\rho_t(\theta_0, T_{\max}) \leq 0$, we can easily see that $\rho_t(\theta_0, T_{\max}) < 0$. Thus the behavior of $\rho(\theta_0, t)$ for $t \to T_{\max}$ is given by $\lambda(T_{\max} - t)$, where $\lambda = -\rho_t(\theta_0, T_{\max}) > 0$. This says that the blow-up rate of $k_{\max}(t)$, as $t \to T_{\max}$, is at least $C(T_{\max} - t)^{-1}$ for some positive constant C. See Lemma 2.15 also.

If the curvature blows up, then its integral also blows up. That is:

Lemma 2.12. *Assume the curvature of the flow* (1.3) *blows up at time $T_{\max} \in (0, 2\pi)$, then we have*

$$(2.12) \qquad\qquad\qquad \int_0^{2\pi} k(\theta, t)\, d\theta \to \infty \quad \text{as } t \to T_{\max}.$$

Proof. For any $(p, t), (q, t) \in S^1 \times [0, T_{\max})$, by (2.6), we have

$$
\begin{aligned}
\log \rho(q, t) - \log \rho(p, t) &= \int_p^q \left(\frac{1}{\rho} \frac{\partial \rho}{\partial \theta} \right) (\theta, t) \, d\theta \\
&\leq \int_0^{2\pi} \frac{1}{\rho(\theta, t)} \left| \frac{\partial \rho}{\partial \theta}(\theta, t) \right| d\theta \leq C \int_0^{2\pi} k(\theta, t) \, d\theta,
\end{aligned}
$$
(2.13)

where C is a positive constant independent of time. In particular, we get

$$
\log \rho_{\max}(t) - \log \rho_{\min}(t) \leq C \int_0^{2\pi} k(\theta, t) \, d\theta, \quad t \in [0, T_{\max}),
$$

where by $\rho_{\min}(t) \leq \frac{L(0)}{2\pi} \leq \rho_{\max}(t)$ for all time, we have

$$
\log \frac{L(0)}{2\pi} - \log \rho_{\min}(t) \leq C \int_0^{2\pi} k(\theta, t) \, d\theta, \quad t \in [0, T_{\max}).
$$

The result follows since we have $\rho_{\min}(t) \to 0$ as $t \to T_{\max}$. $\qquad \square$

As for the gradient of the curvature, we have:

Lemma 2.13. *Assume the curvature of the flow* (1.3) *blows up at time* $T_{\max} \in (0, 2\pi)$, *then we have*

$$
\sup_{S^1 \times [0, T_{\max})} \left| \frac{k_\theta(\theta, t)}{k(\theta, t)} \right| = \infty.
$$
(2.14)

Remark 2.14. On the other hand, by (2.6), we have

$$
\left| \frac{k_\theta(\theta, t)}{k^2(\theta, t)} \right| = |\rho_\theta(\theta, t)| \leq C, \quad \forall (\theta, t) \in S^1 \times [0, T_{\max})
$$
(2.15)

for some constant C independent of time.

Proof. Assume that $\sup_{S^1 \times [0, T_{\max})} \left| \frac{k_\theta(\theta, t)}{k(\theta, t)} \right| = C < \infty$. Then, similar to (2.13), we will have

$$
\log \left(\frac{k_{\max}(t)}{k_{\min}(t)} \right) = \log k_{\max}(t) - \log k_{\min}(t) \leq C,
$$

which implies $k_{\min}(t) \to \infty$ too, as $t \to T_{\max}$. But this is impossible since the flow is both area-preserving and length-preserving. $\qquad \square$

Another method to ensure singularity is to use the idea of *parallel curves* (see the book [2, p. 47]). Basically, it works out only for *linear equations*. We see that if ρ satisfies the equation $\rho_t = -\rho_{\theta\theta\theta} - \rho_\theta$, so is $\widetilde{\rho} = \rho \pm c$ for any constant c. Moreover, $\widetilde{\rho}$ also satisfies the closing condition (1.7). Hence for each fixed time t, it represents the radius of curvature of a convex closed curve if $\widetilde{\rho}(\theta, t) > 0$ on S^1.

We recall that if a convex closed curve X_0 has radius of curvature $\rho_0(\theta) > 0$, then its *inward* parallel curve $\widetilde{X}_0$, with parallel distance $c > 0$, has radius of curvature $\widetilde{\rho}_0(\theta) = \rho_0(\theta) - c$ for all $\theta \in S^1$. Here we are assuming that $c > 0$ is not too big so that we still have $\widetilde{\rho}_0(\theta) > 0$ on S^1 and then $\widetilde{X}_0$ is still a smooth curve. With this, we can state the following:

Lemma 2.15 (The formation of a singularity using parallel curves). *Let X_0 be a smooth convex closed curve and evolve it under the flow* (1.3). *Assume that the maximum curvature $k_{\max}(t)$ of the evolving curve $X(\cdot, t)$ is* strictly increasing *on some time interval $[0, T_1]$, $T_1 > 0$, with $0 < k_{\max}(T_1) < \infty$. If we let $\widetilde{X}_0$ be the convex closed curve which is an* inward *parallel curve of X_0 with parallel distance*

$$(2.16) \qquad c = \frac{1}{k_{\max}(T_1)} > 0$$

and consider the flow (1.3) *with initial curve $\widetilde{X}_0$, then the solution $\widetilde{X}(\cdot, t)$ is smooth, convex, enclosed by $X(\cdot, t)$ for all $t \in [0, T_1)$, and will develop a singularity at time T_1, i.e., $\widetilde{k}_{\max}(T_1) = \infty$. Moreover, the blow-up rate of $\widetilde{k}_{\max}(t)$, as $t \to T_1$, is no less than $k_{\max}^2(T_1) \left[\partial_t k(\theta_1, T_1)(T_1 - t) \right]^{-1}$, where $k(\theta_1, T_1) = k_{\max}(T_1)$.*

Remark 2.16. In the above lemma, we can assume that $\partial_t k(\theta_1, T_1)$ is a positive finite number. Otherwise, we may shrink the interval $[0, T_1]$ a little bit to ensure it.

Proof. Let $\rho(\theta, t)$ be the radius of curvature of $X(\cdot, t)$ on $S^1 \times [0, T_1]$ By assumption, $\rho_{\min}(t) = 1/k_{\max}(t)$ is strictly decreasing on $[0, T_1]$ with $\rho_{\min}(T_1) > 0$. We know that the flow solution $\widetilde{X}(\cdot, t)$ has radius of curvature $\widetilde{\rho}$ given by

$$(2.17) \qquad \widetilde{\rho}(\theta, t) = \rho(\theta, t) - c = \rho(\theta, t) - \rho_{\min}(T_1), \quad \forall (\theta, t) \in S^1 \times [0, T_1).$$

Moreover, we have $\widetilde{\rho}(\theta, t) > 0$ on $S^1 \times [0, T_1)$ and $\widetilde{\rho}(\theta_1, T_1) = 0$. This says that $\widetilde{X}(\cdot, t)$ will develop a singularity at time T_1, i.e., $\widetilde{k}_{\max}(T_1) = \infty$. Finally, by (2.17) we have

$$\widetilde{k}_{\max}(t) \geq \widetilde{k}(\theta_1, t) = k(\theta_1, t) k(\theta_1, T_1) \left(\frac{k(\theta_1, T_1) - k(\theta_1, t)}{T_1 - t} \right)^{-1} \frac{1}{T_1 - t}, \quad t \in [0, T_1).$$

The proof is done. $\qquad \square$

2.4. Compare with the tangential component flow (rotational flow)

Recall that the position center $P(t)$ of $X(\cdot, t)$ is independent of time (see Lemma 2.1) and is given by

$$P(0) = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta)(\cos\theta, \sin\theta) \, d\theta,$$

where $u_0(\theta)$ is the initial support function. Without loss of generality, we may assume that $P(0) = (0,0)$. By this, similar to (2.9), the Fourier series expansion of the support function $u(\theta, t)$ is

$$(2.18) \quad u(\theta, t) = \frac{L(0)}{2\pi} + \sum_{n=2}^{\infty} \left\{ A_n(0) \cos\left[n\theta + (n^3 - n)t\right] + B_n(0) \sin\left[n\theta + (n^3 - n)t\right] \right\}$$

for all $(\theta, t) \in S^1 \times [0, \infty)$. Here $L(0)$ is the length of X_0 and $A_n(0)$, $B_n(0)$ are the Fourier series coefficients of $u_0(\theta)$.

It is easy to see that, without the 3rd-order term $-u_{\theta\theta\theta}$, the whole flow is just a *rotation*. That is, if the flow is such that its equation of the support function is $u_t = -u_\theta$ (with initial condition $u_0(\theta)$), then by (2.18) its Fourier series becomes

$$(2.19) \quad u(\theta, t) = \frac{L(0)}{2\pi} + \sum_{n=2}^{\infty} \left\{ A_n(0) \cos(n(\theta - t)) + B_n(0) \sin(n(\theta - t)) \right\},$$

which gives the identity $u(\theta, t) = u_0(\theta - t)$ for all (θ, t). Now the position vectors $\widetilde{X}_0(\theta)$ and $\widetilde{X}(\theta, t)$ corresponding to $u_0(\theta)$ and $u(\theta, t)$ are given by (see (1.11))

$$\widetilde{X}_0(\theta) = u_0(\theta)(\cos\theta, \sin\theta) + u_0'(\theta)(-\sin\theta, \cos\theta)$$

and

$$(2.20) \quad \begin{aligned} \widetilde{X}(\theta, t) &= u(\theta, t)(\cos\theta, \sin\theta) + u_\theta(\theta, t)(-\sin\theta, \cos\theta) \\ &= u_0(\theta - t)(\cos\theta, \sin\theta) + u_0'(\theta - t)(-\sin\theta, \cos\theta) \\ &= M(-t)\widetilde{X}_0(\theta - t) \end{aligned}$$

for all $(\theta, t) \in S^1 \times [0, \infty)$, where $M(-t)$ is the matrix given by (2.23) below. Thus for fixed time $t > 0$, the whole curve $\widetilde{X}(\cdot, t)$ is a counterclockwise rotation of $\widetilde{X}_0(\cdot)$ by angle t. The point $\widetilde{X}(\theta, t)$ on $\widetilde{X}(\cdot, t)$ comes from the rotation of the point $\widetilde{X}_0(\theta - t)$ on $\widetilde{X}_0(\cdot)$ by angle t.

We leave you to check that if we look at the "*tangential component flow (rotational flow)*" given by

$$(2.21) \quad \begin{cases} \dfrac{\partial X}{\partial t}(\varphi, t) = \langle X(\varphi, t), \mathbf{T}(\varphi, t) \rangle \, \mathbf{N}_{\text{in}}(\varphi, t), \\ X(\varphi, 0) = X_0(\varphi), \quad \varphi \in S^1, \end{cases}$$

where $X_0 \colon S^1 \to \mathbb{R}^2$ is a smooth convex closed curve, then its support function $u(\theta, t)$, in terms of its outward normal angle θ, will satisfy the equation $u_t(\theta, t) = -u_\theta(\theta, t)$. The effect of the flow (2.21) is described by the above (2.20).

Finally, to see the effect of the whole equation $u_t = -u_{\theta\theta\theta} - u_\theta$ on the initial data $u_0(\theta)$, one can write (2.18) as

$$(2.22) \qquad u(\theta,t) = \frac{L(0)}{2\pi} + \sum_{n=2}^{\infty} \left\langle M(n^3 t) \begin{pmatrix} A_n(0) \\ B_n(0) \end{pmatrix}, M(nt) \begin{pmatrix} \cos(n\theta) \\ \sin(n\theta) \end{pmatrix} \right\rangle,$$

where M is the clockwise rotation matrix given by

$$(2.23) \qquad M(\xi) = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}, \quad \xi \in \mathbb{R}.$$

The matrix $M(n^3 t)$ in (2.22) is due to the term $-u_{\theta\theta\theta}$ and the matrix $M(nt)$ in (2.22) is due to the term $-u_\theta$. Since one can also move $M(n^3 t)$ to the right-hand side of the inner product (or move $M(nt)$ to the left-hand side of the inner product), roughly speaking, both terms $-u_{\theta\theta\theta}$ and $-u_\theta$ have the *same* effect on the solution $u(\theta,t)$. This is unlike the usual phenomena in parabolic equations.

2.5. Condition to ensure no singularity

If the initial curve X_0 is circular enough (in the sense described by the inequality (2.24) below), then the flow (1.3) will not produce any singularity and remains smooth for all time. A simple criterion is the following:

Lemma 2.17. *Let X_0 be a smooth convex closed curve satisfying the condition*

$$(2.24) \qquad \sqrt{2\pi} \left(\int_0^{2\pi} \left(\frac{d\rho_0}{d\theta} \right)^2 (\theta)\, d\theta \right)^{1/2} < \frac{L(0)}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} \rho_0(\theta)\, d\theta.$$

Then under the flow (1.3), the evolving curve $X(\cdot,t)$ will remain convex and smooth for all time.

Proof. For $(p,t), (q,t) \in S^1 \times [0,\infty)$, by (2.3) and the assumption, we have

$$\rho(q,t) - \rho(p,t) = \int_p^q \frac{\partial\rho}{\partial\theta}(\theta,t)\, d\theta \leq \int_0^{2\pi} \left| \frac{\partial\rho}{\partial\theta}(\theta,t) \right| d\theta \leq \sqrt{2\pi} \left(\int_0^{2\pi} \left(\frac{\partial\rho}{\partial\theta} \right)^2 (\theta,t)\, d\theta \right)^{1/2}$$

$$= \sqrt{2\pi} \left(\int_0^{2\pi} \left(\frac{d\rho_0}{d\theta} \right)^2 (\theta)\, d\theta \right)^{1/2} < \frac{L(0)}{2\pi},$$

which implies $\rho_{\max}(t) - \rho_{\min}(t) < L(0)/2\pi$. From it we obtain

$$\rho_{\min}(t) > \rho_{\max}(t) - \frac{L(0)}{2\pi} = \rho_{\max}(t) - \frac{1}{2\pi} \int_0^{2\pi} \rho(\theta,t)\, d\theta \geq 0, \quad \forall t \in [0,\infty),$$

i.e., $\rho_{\min}(t) = 1/k_{\max}(t) > 0$ for all time. This says that the curvature will not blow up in finite time and the flow (1.3) is smoothly defined on $S^1 \times [0,\infty)$. $\qquad\square$

2.6. Pictures

Below, we shall give several pictures to illustrate the behavior of the flow, including the case of blowing up and the case of long time existence.
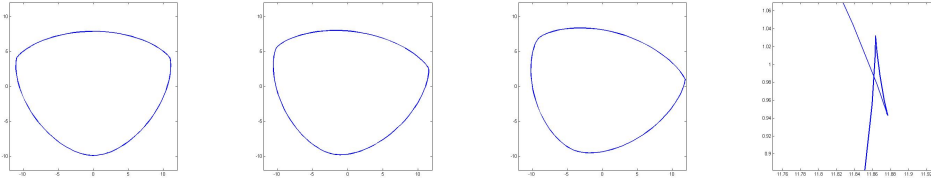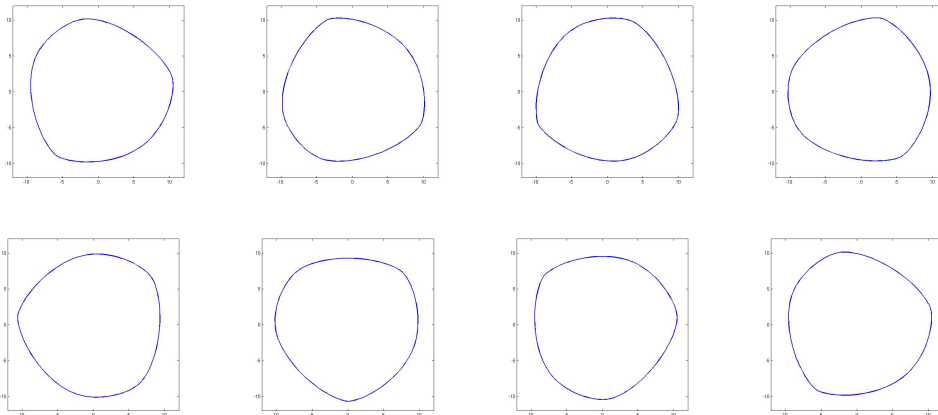

Figure 2.1: The Blow-up Case

Figure 2.2: The Case of Long Time Existence

In Figure 2.1, the flow develops a singularity after some short time $T_{\max}$, where $0 < T_{\max} < 2\pi$. The initial curve is smooth. In the second picture, the evolving curve looks like to be going to have a singularity (but is still smooth) for $t = 0.02$. The third picture presents a blow-up curve for $t = 0.04$. An enlarged picture of the singularity is given in the last picture.

In Figure 2.2, the flow is periodic and remains smooth all the time. Let us choose an initial curve with its support function given by $p = 10 + 0.5\cos(3\theta) + 0.2\sin(5\theta)$. Since the support function of the evolving curve is $p = 10 + 0.5\cos(3\theta + 24t) + 0.2\sin(5\theta + 120t)$, this long time existence flow has a time period equal to $\frac{1}{12}\pi$. Figure 2.2 presents the evolving process with time interval equal to $\frac{1}{84}\pi$ between each picture.

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