TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 4, pp. 909–919, August 2016
DOI: 10.11650/tjm.20.2016.6323
This paper is available online at http://journal.tms.org.tw

The Dual Log-Brunn-Minkowski Inequalities

Wei Wang* and Lijuan Liu

Abstract. In this article, we establish the dual log-Brunn-Minkowski inequality and the dual log-Minkowski inequality. Moreover, the equivalence between the dual log-Brunn-Minkowski inequality and the dual log-Minkowski inequality is demonstrated.

1. Introduction

As a cornerstone of the Brunn-Minkowski theory, the classical Brunn-Minkowski inequality provides a beautiful and powerful apparatus for conquering all sorts of geometrical problems involving metric quantities such as volumes, surface area, and mean width (see [3]).

The classical Brunn-Minkowski inequality states that for convex bodies K and L in Euclidean n-space, $\mathbb{R}^n$, the volume of the bodies and their Minkowski sum $K + L = \{x + y : x \in K, y \in L\}$, are related by

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K and L are homothetic.

The Brunn-Minkowski inequality exposes the crucial log-concavity property of the volume functional because the Brunn-Minkowski inequality has an equivalent formulations: for $0 \leq \lambda \leq 1$,

$$V((1-\lambda)K + \lambda L) \geq V(K)^{1-\lambda} V(L)^{\lambda}, \tag{1.1}$$

and for $0 < \lambda < 1$, there is equality if and only if K and L are translates.

In the early 1960s, Fiery [2] defined the Minkowski-Fiery L_p-combinations (or simply L_p-Minkowski combinations) of convex bodies. If K and L be two convex bodies that

Received May 24, 2015; Accepted September 8, 2015.
Communicated by Duy-Minh Nhieu.
2010 *Mathematics Subject Classification.* 52A40, 53A15.
Key words and phrases. Star body, Radial function, Log radial sum, Dual Brunn-Minkowski inequality, Dual Minkowski inequality.
A Project Supported by Scientific Research Fund of Hunan Provincial Education Department (11C0542) and the National Natural Science Foundations of China (NO. 11371239 and 11471206)
*Corresponding author.

contain the origin in their interiors, $p \geq 1$, and $0 \leq \lambda \leq 1$, then the L_p-Minkowski combinations, $(1-\lambda)\cdot K +_p \lambda \cdot L$, is defined by

$$(1.2)\quad (1-\lambda)\cdot K +_p \lambda\cdot L = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x\cdot u \leq ((1-\lambda)h_K(u)^p + (\lambda)h_L(u)^p)^{\frac{1}{p}}\right\},$$

where $x \cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$, and h_K is the support function of K.

Fiery also established the L_p-Brunn-Minkowski inequality. If $p > 1$, then

$$V((1-\lambda)K +_p \lambda L) \geq V(K)^{1-\lambda}V(L)^{\lambda},$$

with equality for $0 < \lambda < 1$ if and only if $K = L$.

Note that definition (1.2) makes sense for all $p > 0$. The function $((1-\lambda)h_K^p + \lambda h_L^p)^{\frac{1}{p}}$ is convex if $p \geq 1$. Whereas the function $((1-\lambda)h_K^p + \lambda h_L^p)^{\frac{1}{p}}$ is not convex if $0 < p < 1$. The limit of $((1-\lambda)h_K^p + \lambda h_L^p)^{\frac{1}{p}}$ is $h_K^{1-\lambda}h_L^{\lambda}$, as $p \to 0^+$.

Recently, Böröczky et al. [1] defined the log Minkowski combination of convex bodies. Let K and L be two convex bodies that contain the origin in their interiors, and $0 \leq \lambda \leq 1$, then the log Minkowski combination, $(1-\lambda)\cdot K +_0 \lambda \cdot L$, is defined by

$$(1.3)\quad (1-\lambda)\cdot K +_0 \lambda\cdot L = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x\cdot u \leq h_K(u)^{1-\lambda}h_L(u)^{\lambda}\right\}.$$

It is obviously that the convex body $(1-\lambda)\cdot K +_0 \lambda\cdot L$ is the Wolff shape of the function $h_K(u)^{1-\lambda}h_L(u)^{\lambda}$.

Böröczky et al. [1] established the log-Brunn-Minkowski inequality and log-Minkowski inequality for origin-symmetric convex bodies in the plane as follows.

Theorem 1.1. *If K and L are two origin-symmetric convex bodies in $\mathbb{R}^2$, then for $0 \leq \lambda \leq 1$,*

$$(1.4)\quad V((1-\lambda)\cdot K +_0 \lambda\cdot L) \geq V(K)^{1-\lambda}V(L)^{\lambda}.$$

When $0 < \lambda < 1$, equality in the inequality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Theorem 1.2. *If K and L are two origin-symmetric convex bodies in $\mathbb{R}^2$, then*

$$(1.5)\quad \int_{S^1} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_K(u) \geq \frac{1}{2}\log\frac{V(L)}{V(K)},$$

with equality if and only if, K and L are dilates or K and L are parallelograms with parallel sides. Here $\overline{V}_K$ is the cone-volume probability measure of K.

Unfortunately, the log-Brunn-Minkowski inequality (1.4) cannot hold for all convex bodies (e.g., an origin-centered cube and one of its translates). From the arithmetic-geometric mean inequality, it is easily seen that the log-Brunn-Minkowski inequality (1.4) is stronger than the Brunn-Minkowski inequality (1.1) for origin-symmetric convex bodies. For $n \geq 3$, Böröczky et al. [1] conjectured that there exists the log-Brunn-Minkowski inequality and log-Minkowski inequality for origin-symmetric convex bodies in $\mathbb{R}^n$, and showed that these two inequalities are equivalent.

Conjecture 1.3. *If K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ ($n \geq 3$), then for $0 \leq \lambda \leq 1$,*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.6}$$

Conjecture 1.4. *If K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ ($n \geq 3$), then*

$$\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_K(u) \geq \frac{1}{n} \log \frac{V(L)}{V(K)}. \tag{1.7}$$

The dual Brunn-Minkowski theory, was introduced by Lutwak [7] in the 1970s, helped achieving a major breakthrough in solving the Busemann-Petty problem in 1990s. In contrast to the Brunn-Minkowski theory, in the dual theory, convex bodies are replaced by star bodies, Minkowski sum replaced by radial sum, and mixed volumes are replaced by dual mixed volumes.

Let K and L be two star bodies about the origin in $\mathbb{R}^n$. For $p \neq 0$ and $0 \leq \lambda \leq 1$, Gardner [4] defined the L_p radial sum $(1-\lambda)\cdot K \widetilde{+}_p \lambda \cdot L$ by

$$\rho_{(1-\lambda)\cdot K \widetilde{+}_p \lambda \cdot L}(u)^p = (1-\lambda)\rho_K(u)^p + \lambda \rho_L(u)^p, \quad \forall\, u \in S^{n-1}.$$

Note that

$$\begin{aligned}
\lim_{p \to 0} \log \rho_{(1-\lambda)\cdot K \widetilde{+}_p \lambda \cdot L} &= \lim_{p \to 0} \frac{\log(1-\lambda)\rho_K^p + \lambda \rho_L^p}{p} \\
&= \lim_{p \to 0} \frac{(1-\lambda)\rho_K^p \log \rho_K + \lambda \rho_L^p \log \rho_L}{(1-\lambda)\rho_K^p + \lambda \rho_L^p} \\
&= (1-\lambda)\log \rho_K + \lambda \log \rho_L.
\end{aligned}$$

Let K and L be two star bodies in $\mathbb{R}^n$ and $0 \leq \lambda \leq 1$, then the log radial sum, $(1-\lambda)\cdot K \widetilde{+}_0 \lambda \cdot L$, is defined by

$$\rho_{(1-\lambda)\cdot K \widetilde{+}_0 \lambda \cdot L}(u) = \rho_K(u)^{1-\lambda} \rho_L(u)^{\lambda}, \quad \forall\, u \in S^{n-1}. \tag{1.8}$$

In particular, if $\lambda = 0$, then $(1-\lambda)\cdot K \widetilde{+}_0 \lambda \cdot L = K$. If $\lambda = 1$, then $(1-\lambda)\cdot K \widetilde{+}_0 \lambda \cdot L = L$.

The main purpose of this paper is to establish the dual forms of the log-Brunn-Minkowski inequality (1.6) and the log-Minkowski inequality (1.7) as follows.

Theorem 1.5. *If K and L are two star bodies in $\mathbb{R}^n$, then for $0 \le \lambda \le 1$,*

$$V((1-\lambda)\cdot K \widetilde{+}_0 \lambda \cdot L) \le V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.9}$$

When $0 < \lambda < 1$, equality in the inequality holds if and only if K and L are dilates.

Theorem 1.6. *If K and L are two star bodies in $\mathbb{R}^n$, then*

$$\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right) d\widetilde{V}_K(u) \le \frac{1}{n}\log\frac{V(L)}{V(K)}, \tag{1.10}$$

with equality if and only if K and L are dilates. Here $\widetilde{V}_K$ is the dual cone-volume probability measure of K (see Section 3 for a precise definition).

2. Notation and background material

For general reference for the theory of convex (star) bodies the reader may wish to consult the books of Gardner [4], Gruber [5], and Schneider [9].

The unit ball and its surface in $\mathbb{R}^n$ are denoted by B and S^{n-1}, respectively. We write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$. The radial function $\rho_K\colon S^{n-1} \to [0,\infty)$ of a compact star-shaped about the origin, $K \in \mathbb{R}^n$, is defined, for $u \in S^{n-1}$, by

$$\rho_K(u) = \max\{\lambda \ge 0 : \lambda u \in K\}. \tag{2.1}$$

If $\rho_K(\cdot)$ is positive and continuous, then K is called a star body about the origin. The set of star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathcal{S}^n$. Obviously, for $K, L \in \mathcal{S}^n$,

$$K \subseteq L \iff \rho_K(u) \le \rho_L(u), \quad \forall\, u \in S^{n-1}. \tag{2.2}$$

If $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$, then we say star bodies K and L are dilates. If $s > 0$, we have

$$\rho_{sK}(u) = s\rho_K(u), \quad \text{for all } u \in S^{n-1}. \tag{2.3}$$

If $\phi \in \mathrm{GL}(n)$, we have

$$\rho_{\phi K}(u) = \rho_K(\phi^{-1}u), \quad \text{for all } u \in S^{n-1}. \tag{2.4}$$

The radial Hausdorff metric between the star bodies K and L is

$$\widetilde{\delta}(K,L) = \max_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\widetilde{\delta}(K_i, K) \to 0, \quad \text{as } i \to \infty.$$

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of radial function $\rho(K_i, \cdot)$ converges uniformly to $\rho(K, \cdot)$.

Let K and L be are two star bodies about the origin in $\mathbb{R}^n$ and $0 \le \lambda \le 1$. The radial sum $(1-\lambda)K\widetilde{+}\lambda L$ was defined by (see [7])

$$\rho_{(1-\lambda)K\widetilde{+}\lambda L}(u) = (1-\lambda)\rho_K(u) + \lambda\rho_L(u), \quad \forall\, u \in S^{n-1}. \tag{2.5}$$

The dual quermassintegral $\widetilde{W}_i(K)$ has the following integral representation (see [8]):

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i}\, dS(u), \tag{2.6}$$

where S is the Lebesgue measure on S^{n-1}. In particular, $\widetilde{W}_0(K) = V(K)$. The dual mixed quermassintegral $\widetilde{W}_i(K, L)$ has the following integral representation (see [8]):

$$\widetilde{W}_i(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i-1}\rho_L(u)\, dS(u). \tag{2.7}$$

By using the Minkowski's integral inequality, we can obtain the dual Minkowski inequality for dual mixed quermassintegrals: If $K, L \in \mathcal{S}_0^n$, and $0 \le i < n-1$, then

$$\widetilde{W}_i(K, L)^{n-i} \le \widetilde{W}_i(K)^{n-i-1}\widetilde{W}_i(L), \tag{2.8}$$

equality holds if and only if K and L are dilates.

Suppose that μ is a probability measure on a space X and $g\colon X \to I \subset \mathbb{R}$ is a μ-intergrable function, where I is a possibly infinite interval. Jessen's inequality states that if $\phi\colon X \to I \subset \mathbb{R}$ is a concave function, then

$$\int_X \phi(g(x))\, d\mu(x) \le \phi\left(\int_X g(x)\, d\mu(x)\right). \tag{2.9}$$

If ϕ is strictly concave, equality holds if and only if $g(x)$ is a constant for μ-almost all $x \in X$ (see [6]).

3. Main results

Lemma 3.1. *Let $K, L \in \mathcal{S}^n$ and $0 \le \lambda \le 1$, then $(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L \in \mathcal{S}^n$.*

Proof. Since $\rho_K(\cdot)$ and $\rho_L(\cdot)$ are positive and continuous on S^{n-1}, the function $\rho_K(\cdot)^{1-\lambda}\cdot\rho_L(\cdot)^\lambda$ is positive and continuous on S^{n-1}. □

Lemma 3.2. *Let $K, L \in \mathcal{S}^n$ and $0 \le \lambda \le 1$. Then for $A \in \mathrm{GL}(n)$,*

$$A((1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L) = (1-\lambda)\cdot AK\widetilde{+}_0\lambda\cdot AL.$$

Proof. For $u \in S^{n-1}$, by (1.8) and (2.4), we have

$$\begin{aligned}\rho_{(1-\lambda)\cdot AK\widetilde{+}_0\lambda\cdot AL}(u) &= \rho_{AK}(u)^{1-\lambda}\rho_{AL}(u)^{\lambda}\\ &= \rho_K(A^{-1}u)^{1-\lambda}\rho_L(A^{-1}u)^{\lambda}\\ &= \rho_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(A^{-1}u)\\ &= \rho_{A((1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L)}(u).\end{aligned}$$

□

Lemma 3.3. *Let $K_i, L_i \in \mathcal{S}^n$ and $0 \le \lambda \le 1$. If $K_i \to K \in \mathcal{S}^n$, $L_i \to L \in \mathcal{S}^n$, as $i \to \infty$, then*

$$(1-\lambda)\cdot K_i\widetilde{+}_0\lambda\cdot L_i \to (1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L, \quad \text{as } i\to\infty.$$

Proof. For $u \in S^{n-1}$, by the continuity of the power function, we have

$$\begin{aligned}\lim_{i\to\infty}\rho_{(1-\lambda)\cdot K_i\widetilde{+}_0\lambda\cdot L_i}(u) &= \lim_{i\to\infty}\rho_{K_i}(u)^{1-\lambda}\rho_{L_i}(u)^{\lambda}\\ &= \rho_K(u)^{1-\lambda}\rho_L(u)^{\lambda}\\ &= \rho_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(u).\end{aligned}$$

□

Lemma 3.4. *Let $K, L \in \mathcal{S}^n$ and $0 \le \lambda_i \le 1$. If $\lambda_i \to \lambda \in [0,1]$, as $i \to \infty$, then*

$$(1-\lambda_i)\cdot K\widetilde{+}_0\lambda_i\cdot L \to (1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L, \quad \text{as } i\to\infty.$$

Proof. For $u \in S^{n-1}$, by the continuity of the exponential function, we have

$$\begin{aligned}\lim_{i\to\infty}\rho_{(1-\lambda_i)\cdot K\widetilde{+}_0\lambda_i\cdot L}(u) &= \lim_{i\to\infty}\rho_K(u)\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{\lambda_i}\\ &= \rho_K(u)\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{\lambda}\\ &= \rho_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(u).\end{aligned}$$

□

Lemma 3.5. *Let $K, L \in \mathcal{S}^n$ and $0 \le \lambda \le 1$, then*

$$(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L \subseteq (1-\lambda)\cdot K\widetilde{+}\lambda\cdot L,$$

with equality if and only if $K = L$.

Proof. For $u \in S^{n-1}$, by the arithmetic-geometric mean inequality and (2.5), we have

$$\begin{aligned}\rho_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(u) &= \rho_K(u)^{1-\lambda}\rho_L(u)^{\lambda}\\ &\le (1-\lambda)\rho_K(u) + \lambda\rho_L(u)\\ &= \rho_{(1-\lambda)\cdot K\widetilde{+}\lambda\cdot L}(u).\end{aligned} \tag{3.1}$$

From the equality conditions of the arithmetic-geometric mean inequality, equality in inequality (3.1) holds if and only if $K = L$. Combining (3.1) and (2.2), we obtain the desired result. □

In fact, we will prove the following dual log-Brunn-Minkowski inequality which is more general than Theorem 1.5.

Theorem 3.6. *If K and L are two star bodies in $\mathbb{R}^n$ and $0 \le \lambda \le 1$, then for $0 \le i < n$,*

$$\widetilde{W}_i((1-\lambda) \cdot K \widetilde{+}_0 \lambda \cdot L) \le \widetilde{W}_i(K)^{1-\lambda} \widetilde{W}_i(L)^{\lambda}. \tag{3.2}$$

When $0 < \lambda < 1$, equality in the inequality holds if and only if K and L are dilates.

Proof. By (2.6) and Hölder's inequality, we obtain that

$$\begin{aligned} \widetilde{W}_i((1-\lambda) \cdot K \widetilde{+}_0 \lambda \cdot L) &= \int_{S^{n-1}} \rho_{(1-\lambda)\cdot K \widetilde{+} \lambda \cdot L}(u)^{n-i}\, dS(u) \\ &= \int_{S^{n-1}} \left(\rho_K(u)^{1-\lambda} \rho_L(u)^{\lambda}\right)^{n-i} dS(u) \\ &\le \left(\int_{S^{n-1}} \rho_K(u)^{n-i}\, dS(u)\right)^{1-\lambda} \left(\int_{S^{n-1}} \rho_L(u)^{n-i}\, dS(u)\right)^{\lambda} \\ &= \widetilde{W}_i(K)^{1-\lambda} \widetilde{W}_i(L)^{\lambda}. \end{aligned} \tag{3.3}$$

When $0 < \lambda < 1$, by the equality conditions of Hölder's inequality, equality in (3.3) holds if and only if K and L are dilates. □

For $K \in \mathcal{S}^n$, we write the measure $\widetilde{V}_{i,K}(\cdot) = \frac{\rho_K^{n-i}(\cdot)\, dS(\cdot)}{n\widetilde{W}_i(K)}$. Since

$$\frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \rho_K^{n-i}(u)\, dS(u) = 1, \tag{3.4}$$

we say that the measure $\widetilde{V}_{i,K}(\cdot)$ is a dual mixed cone-volume probability measure of K on S^{n-1}. If $i = 0$, the measure $\widetilde{V}_{0,K}(\cdot)$ will be denoted by the dual cone-volume probability measure, and it will be written simply as $\widetilde{V}_K(\cdot)$.

Theorem 3.7. *If K and L are two star bodies in $\mathbb{R}^n$, and $0 \le i < n$, then*

$$\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right) d\widetilde{V}_{i,K} \le \frac{1}{n-i} \log \frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}, \tag{3.5}$$

with equality if and only if K and L are dilates.

Proof. By (2.9), (2.8), and the fact that the logarithmic function $\log(\cdot)$ is concave and increasing on $(0,\infty)$, we obtain

$$
\begin{aligned}
\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right) d\widetilde{V}_{i,K} &= \frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^{n-i}(u)\, dS(u) \\
&\le \log\left(\frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)} \rho_K^{n-i}(u)\, dS(u)\right) \\
&= \log\left(\frac{\widetilde{W}_i(K,L)}{\widetilde{W}_i(K)}\right) \\
&\le \log\left(\frac{\widetilde{W}_i(K)^{\frac{n-i-1}{n-i}} \widetilde{W}_i(L)^{\frac{1}{n-i}}}{\widetilde{W}_i(K)}\right) \\
&= \log\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{\frac{1}{n-i}}.
\end{aligned} \tag{3.6}
$$

This gives the desired inequality. Since $\log(\cdot)$ is strictly increasing, from the equality conditions of the dual Minkowski inequality (2.8), we have that equality in (3.6) holds if and only if K and L are dilates. □

Remark 3.8. The case $i = 0$ of Theorem 3.6 and Theorem 3.7 are Theorem 1.5 and Theorem 1.6, respectively.

Lemma 3.9. *Let $K, L \in \mathcal{S}^n$. Then*

$$
\lim_{\lambda\to 0^+} \frac{\rho_{(1-\lambda)\cdot K \widetilde{+}_0 \lambda\cdot L}(u) - \rho_K(u)}{\lambda} = \rho_K(u) \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right),
$$

uniformly for all $u \in S^{n-1}$.

Proof. For $u \in S^{n-1}$, we have

$$
\begin{aligned}
\lim_{\lambda\to 0^+} \frac{\rho_{(1-\lambda)\cdot K \widetilde{+}_0 \lambda\cdot L}(u) - \rho_K(u)}{\lambda} &= \rho_K(u) \lim_{\lambda\to 0^+} \frac{\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{\lambda} - 1}{\lambda} \\
&= \rho_K(u) \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right).
\end{aligned}
$$

Then the pointwise limit has been proved. Moreover, the convergence is uniform for any $u \in S^{n-1}$. □

Lemma 3.10. *Let $K, L \in \mathcal{S}^n$. For $i = 0, 1, \ldots, n-1$, then*

$$
\lim_{\lambda\to 0^+} \frac{\widetilde{W}_i((1-\lambda)\cdot K \widetilde{+}_0 \lambda\cdot L) - \widetilde{W}_i(K)}{\lambda} = \frac{n-i}{n} \int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^{n-i}(u)\, dS(u).
$$

Proof. By (2.6) and Lemma 3.9, it follows that

$$\begin{aligned}
&\lim_{\lambda\to 0^+} \frac{\widetilde{W}_i((1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L)-\widetilde{W}_i(K)}{\lambda} \\
&= \lim_{\lambda\to 0^+} \frac{1}{n}\int_{S^{n-1}} \frac{\rho^{n-i}_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(u)-\rho_K^{n-i}(u)}{\lambda}\,dS(u)) \\
&= \frac{1}{n}\int_{S^{n-1}} \lim_{\lambda\to 0^+} \frac{\rho^{n-i}_{(1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L}(u)-\rho_K^{n-i}(u)}{\lambda}\,dS(u) \\
&= \frac{n-i}{n}\int_{S^{n-1}} \rho_K^{n-i-1}(u)\lim_{\lambda\to 0^+}\frac{\rho_{K\widetilde{+}_0\lambda\cdot L}(u)-\rho_K(u)}{\lambda}\,dS(u) \\
&= \frac{n-i}{n}\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,dS(u).
\end{aligned}$$

□

Theorem 3.11. *Let K and L be two star bodies in $\mathbb{R}^n$. Then the dual log-Brunn-Minkowski inequality and the dual log-Minkowski inequality are equivalent.*

Proof. Let $Q_\lambda = (1-\lambda)\cdot K\widetilde{+}_0\lambda\cdot L$, it is obviously that $Q_0 = K$, $Q_1 = L$. We will first suppose that we have the dual log-Minkowski inequality (3.5). For $0<\lambda<1$, by (3.5), we have

$$\begin{aligned}
0 &= \int_{S^{n-1}} \log\left(\frac{\rho_K(u)^{1-\lambda}\rho_L(u)^\lambda}{\rho_{Q_\lambda}(u)}\right) d\widetilde{V}_{i,Q_\lambda} \\
&= (1-\lambda)\int_{S^{n-1}} \log\left(\frac{\rho_K(u)}{\rho_{Q_\lambda}(u)}\right) d\widetilde{V}_{i,Q_\lambda} + \lambda\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_{Q_\lambda}(u)}\right) d\widetilde{V}_{i,Q_\lambda} \\
&\le \frac{1-\lambda}{n-i}\log\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(Q_\lambda)} + \frac{\lambda}{n-i}\log\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(Q_\lambda)} \\
&= \frac{1}{n-i}\log\frac{\widetilde{W}_i(K)^{1-\lambda}\widetilde{W}_i(L)^\lambda}{\widetilde{W}_i(Q_\lambda)}.
\end{aligned}$$

This gives the dual log-Brunn-Minkowski inequality (3.2). From the equality conditions of the dual log-Minkowski inequality (3.5), we have that equality in (3.2) holds if and only if K and L are dilates.

Suppose now that the dual log-Brunn-Minkowski inequality (3.2) holds. We define the function $f\colon [0,1]\to(0,\infty)$ by $f(\lambda)=\widetilde{W}_i(Q_\lambda)$.

For given $\sigma,\tau\in[0,1]$, if $\alpha\in[0,1]$ and $\alpha=(1-\lambda)\sigma+\lambda\tau$, we have

$$\begin{aligned}
\rho_{(1-\lambda)\cdot K_\sigma\widetilde{+}_0\lambda\cdot K_\tau} &= \rho_{K_\sigma}(u)^{1-\lambda}\rho_{K_\tau}(u)^\lambda \\
&= \left(\rho_K(u)^{1-\sigma}\rho_L(u)^\sigma\right)^{1-\lambda}\left(\rho_K(u)^{1-\tau}\rho_L(u)^\tau\right)^\lambda \\
&= \rho_K(u)^{1-[(1-\lambda)\sigma+\lambda\tau]}\rho_L(u)^{(1-\lambda)\sigma+\lambda\tau} \\
&= \rho_K(u)^{1-\alpha}\rho_L(u)^\alpha \\
&= \rho_{Q_\alpha}.
\end{aligned}$$

Thus,

$$(1-\lambda)\cdot K_\sigma \widetilde{+}_0 \lambda\cdot K_\tau = (1-\alpha)\cdot K \widetilde{+}_0 \alpha\cdot L. \tag{3.7}$$

From (3.7) and (3.2), we have

$$\begin{aligned} f(\alpha) = f((1-\lambda)\sigma+\lambda\tau) &= \widetilde{W}_i((1-\alpha)\cdot K\widetilde{+}_0\alpha\cdot L) \\ &= \widetilde{W}_i((1-\lambda)\cdot K_\sigma\widetilde{+}_0\lambda\cdot K_\tau) \\ &\le \widetilde{W}_i(K_\sigma)^{1-\lambda}\widetilde{W}_i(K_\tau)^{\lambda} \\ &= f(\sigma)^{1-\lambda}f(\tau)^{\lambda}, \end{aligned}$$

which is the desired log convexity of f. Equivalently, the function $\lambda \mapsto \log \widetilde{W}_i(Q_\lambda)$ is a convex function, and thus

$$\frac{1}{\widetilde{W}_i(Q_0)}\frac{d\widetilde{W}_i(Q_\lambda)}{d\lambda}\bigg|_{\lambda=0} \le \log\widetilde{W}_i(Q_1) - \log\widetilde{W}_i(Q_0) = \log\widetilde{W}_i(L) - \log\widetilde{W}_i(K). \tag{3.8}$$

Combining (3.8) and Lemma 3.10, we obtain the dual log-Minkowski inequality. From the equality conditions of the dual log-Brunn-Minkowski inequality (3.2), we have that equality in (3.5) holds if and only if K and L are dilates. □

References

[1] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math. **231** (2012), no. 3-4, 1974–1997. http://dx.doi.org/10.1016/j.aim.2012.07.015

[2] W. J. Firey, *p-means of convex bodies*, Math. Scand. **10** (1962), 17–24.

[3] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 355–405. http://dx.doi.org/10.1090/s0273-0979-02-00941-2

[4] ______, *Geometric Tomography*, Second edition, Encyclopedia of Mathematics and its Applications, **58**, Cambridge University Press, Cambridge, 2006. http://dx.doi.org/10.1017/cbo9781107341029

[5] P. M. Gruber, *Convex and Discrete Geometry*, Grundlehren der Mathematischen Wissenschaften, **336**, Springer, Berlin, 2007. http://dx.doi.org/10.1007/978-3-540-71133-9

[6] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Reprint of the 1952 edition, Cambridge mathematical Library, Cambridge University Press, Cambridge, 1988.

[7] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), no. 2, 531–538. http://dx.doi.org/10.2140/pjm.1975.58.531

[8] ———, *The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), no. 1, 131–150.

[9] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications, **44**, Cambridge University Press, Cambridge, 1993. http://dx.doi.org/10.1017/cbo9780511526282

Wei Wang
Department of Mathematics, Tongji University, Siping Road, Shanghai, 200092, P. R. China
and
School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, 411201, P. R. China
E-mail address: wwang@hnust.edu.cn

Lijuan Liu
School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, 411201, P. R. China
E-mail address: ljliu@shu.edu.cn