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The Inverse p-maxian Problem on Trees with Variable Edge Lengths

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Abstract. We concern the problem of modifying the edge lengths of a tree in minimum total cost so that the prespecified p vertices become the p-maxian with respect to the new edge lengths. This problem is called the inverse p-maxian problem on trees. Gassner proposed in 2008 an efficient combinatorial algorithm to solve the inverse 1-maxian problem on trees. For the case $p \geq 2$, we claim that the problem can be reduced to $O(p^2)$ many inverse 2-maxian problems. We then develop algorithms to solve the inverse 2-maxian problem under various objective functions. The problem under l_1-norm can be formulated as a linear program and thus can be solved in polynomial time. Particularly, if the underlying tree is a star, the problem can be solved in linear time. We also develop $O(n \log n)$ algorithms to solve the problems under Chebyshev norm and bottleneck Hamming distance, where n is the number of vertices of the tree.

1. Introduction

Location theory plays an important role in Operations Research due to its numerous applications. Here, we want to find optimal locations of new facilities. Location theory was intensively investigated, see Kariv and Hakimi [19, 20], Hamacher [17], Eiselt [12]. Recently, a new approach of location theory, the so-called inverse location problem, has been focused and become an interesting research topic. In the inverse setting, we aim to modify the parameters in minimum cost so that the prespecified locations become optimal in the perturbed problem. *Inverse median* and *inverse center* problems are the popular topics in the field of inverse location theory.

For inverse 1-median problems, Burkard et al. [9] solved the inverse 1-median problem on trees and the inverse 1-median problem on the plane with Manhattan norm in $O(n \log n)$ time. Then Galavii [13] proposed a linear time algorithm for the inverse 1-median problem on trees. Also, Nguyen [23] generalized this problem to the inverse 1-median problem on block graphs and solved it in $O(n \log n)$ time by a binary search algorithm. Nguyen and Chi [27] further considered the inverse 1-median problem under uncertain costs and

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developed an $O(n^2 \log n)$ algorithm to represent the total cost under all confidence levels. Burkard et al. [7] investigated and solved the inverse Fermat-Weber problem in $O(n \log n)$ time if the given points are not colinear. Otherwise, the problem can be formulated as a convex program. For the inverse 1-median problem on a cycle, Burkard et al. [10] developed an $O(n^2)$ algorithm based on the concavity of the corresponding linear programmming constraints. Additionally, the inverse p-median problem on networks with variable edge lengths is NP-hard, see Bonab et al. [5]. However, the inverse 2-median problem on a tree can be solved in polynomial time. More particularly, if the underlying tree is a star, the corresponding problem is solvable in linear time. Sepasian and Rahbarnia [29] investigated the inverse 1-median problem on trees with both vertex weights and edge lengths variations. They proposed an $O(n \log n)$ algorithm to solve that problem. While most recently papers concerned the inverse 1-median problem under linear cost functions, Guan and Zhang [16] solved the inverse 1-median problem on trees under Chebyshev norm and Hamming distance by a binary search algorithm in linear time.

Cai et al. [11] were the first who showed that although the 1-center problem on directed networks can be solved in polynomial time, the inverse problem is NP-hard. Hence, it is interesting to study some special situations which are polynomially solvable. Alizadeh and Burkard [1] developed a combinatorial algorithm with complexity of $O(n^2)$ to solve the inverse 1-center problem on unweighted trees with variable edge lengths, provided that the edge lengths remain positive throughout the modification. Dropping this condition, the problem can be solved in $O(n^2\boldsymbol{c})$ time where $\boldsymbol{c}$ is the compressed depth of the tree. For the corresponding uniform-cost problem, Alizadeh and Burkard [2] devised improved algorithms with running time $O(n \log n)$ and $O(\boldsymbol{c}n \log n)$. Then Alizadeh et al. [4] applied the AVL-tree structure to develop an $O(n \log n)$ algorithm for solving the inverse 1-center problem on trees with edge length augmentation. Especially, the uniform-cost problem can be solved in linear time. Based on the method to solve the reverse 1-center problem on weighted trees (see [22]), Nguyen [21] solved the uniform-cost inverse 1-center problem on a weighted trees not required topology change in quadratic time and in $O(n^2\boldsymbol{c})$ time if $\boldsymbol{c}$ is the compressed depth of the tree. For the inverse 1-center problem on a simple generalization of tree graphs, the so-called cactus graphs, Nguyen ang Chassein [25] showed the NP-hardness. If the modification of vertex weights is taken into account, the inverse 1-center problem on trees can be solved in $O(n^2)$ time, see Nguyen and Anh [24]. Furthermore, Nguyen and Sepasian [28] solved the inverse 1-center problem on trees under Chebyshev norm and Hamming distance in $O(n \log n)$ time in the case there exists no topology change. In principle, the problem is solvable in quadratic time. Additionally, the inverse ordered 1-median problem on Chebyshev norm and bottleneck Hamming distance can be solved in $O(n^2 \log n)$ time; see [26].

Although the inverse location problem was intensively studied, there is a limited number of papers related to the inverse obnoxious location problem. Alizadeh and Burkard [3] developed a linear time algorithm to solve the inverse obnoxious 1-center problem. Moreover, Gassner [14] investigated the inverse 1-maxian problem on trees with variable edge lengths and reduced the problem to a minimum cost circulation problem which can be solved in $O(n \log n)$ time.

We focus in this paper the inverse p-maxian problem on trees with $p \geq 2$. This paper is organized as follows. We briefly introduce the optimality criterion of the p-maxian problem on trees as well as formulate the problem under arbitrary cost functions in Section 2. We also state that the problem can be reduced to $O(p^2)$ many inverse 2-maxian problems on trees. Section 3 concentrates on the problem under l_1-norm. We show that the inverse 2-maxian problem can be formulated as a linear program. If the underlying tree is a star, we solve the problem in linear time. We focus on Sections 4 and 5 the inverse 2-maxian problem under Chebyshev norm and bottleneck Hamming distance, respectively. We develop algorithms that solve the corresponding problems in $O(n \log n)$ time.

2. The p-maxian problems on trees

Given a graph $G = (V, E)$, $|V| = n$, each vertex $v_i \in V$ is associated with a nonnegative weight w_i and each edge has a nonnegative length ℓ_e. The length of the shortest path $P(u, v)$ connecting two vertices u and v in G is the distance $d(u, v)$ between these two vertices. A point on G is either a vertex or lies on an edge of the graph. As proposed by Burkard et al. [6], the p-maxian problem on G is to identify a set of p points, say $X = \{x_1, x_2, \ldots, x_p\}$, to maximize the maxian objective function

$$F(X) = \sum_{i=1}^{n} w_i \max_{1 \leq j \leq p} d(v_i, x_j).$$

By the vertex domination property, there exists a p-maxian of G that is the set of p vertices. Now we restrict ourselves to the case where the underlying graph G is a tree, say T. Burkard et al. [6] already showed that $\{a, b\}$ is a 2-maxian of a tree T if $P(a, b)$ is its longest path. We further show that this condition is also the necessary condition as follows.

Theorem 2.1 (2-maxian criterion). *Given two vertices a and b on the tree T, then $\{a, b\}$ is a 2-maxian of T if and only if $P(a, b)$ is the longest path in the tree.*

Proof. For the sufficient condition, see [6]. Assume that $P(a, b)$ is not the longest path of the tree, we will prove that $\{a, b\}$ is not its 2-maxian as well. Let m_{ab} be the midpoint of path $P(a,b)$. By deleting an edge that contains m_{ab}, we can deduce two parts L and R of T which contain a and b, respectively. Then we get

$$F(\{a,b\}) = \sum_{v \in L} w_v d(v,b) + \sum_{v \in R} w_v d(v,a).$$

Moreover, let $P(s,t)$ be the longest path of the tree. We trivially get $d(m_{ab}, a) \leq \max\{d(m_{ab}, s), d(m_{ab}, t)\}$ and $d(m_{ab}, b) \leq \max\{d(m_{ab}, s), d(m_{ab}, t)\}$, and at least one of the two inequalities does not hold with equality. Otherwise, it contradicts the assumption that $P(s,t)$ is the longest path. Therefore, we get

$$d(v,a) \leq \max\{d(v,s), d(v,t)\} \quad \text{for } v \in R$$

and

$$d(v,b) \leq \max\{d(v,s), d(v,t)\} \quad \text{for } v \in L,$$

and at least one of the inequalities will not hold with equality. Finally, we conclude that $F(a,b) < F(s,t)$. □

We now reformulate the optimality criterion in Theorem 2.1. Given two prespecified vertices a and b, and a leaf v, let v_{ab} be the common vertex of three paths $P(a,v)$, $P(b,v)$ and $P(a,b)$. The following result states the necessary and sufficient conditions for $\{a,b\}$ to be the longest path in T.

Corollary 2.2 (Longest path criterion). *$P(a,b)$ is the longest path of T if and only if $d(v, v_{ab}) \leq d(a, v_{ab})$ and $d(v, v_{ab}) \leq d(b, v_{ab})$ for all leaves v in T.*

Next we formulate the inverse version of the p-maxian problem on trees. Given a tree $T = (V,E)$ and a prespecified p-vertex. We can assume without loss of generality that the prespecified p vertices are the leaves of T. The length of each edge e in E can be increased or decreased by an amount p_e or q_e, i.e., the new length of e is $\widetilde{\ell}_e = \ell_e + p_e - q_e$ and is assumed to be nonnegative. We can state the inverse p-maxian on T as follows.

1. The p-vertex becomes a p-maxian of the tree with respect to the new edge lengths $\widetilde{\ell}$.

2. The cost function $\mathcal{C}(p,q)$ is minimized.

3. Modifications are feasible, i.e., $0 \leq p_e \leq \overline{p}_e$ and $0 \leq q_e \leq \overline{q}_e$.

The inverse p-maxian problem can be applied in network design. For instance, a network of computers is presented by a graph. Here, vertices play the role as servers or clients which communicate with each other. Edge lengths represent the transmission time between adjacent vertices. Vertex weights are the communication frequencies of the corresponding vertices. Assume that the designer detects p servers that are harmful to the entire network. He considers them as undesirable servers. On one hand, he aims to modify the transmission time or roughly speaking, the edge lengths, by innovation or prevention so as to make the undesirable servers the p-maxian. On the other hand, the designer wants to minimize the total cost for modifying the network under minisum, minimax, or Hamming distance. This problem can be modeled as an inverse p-maxian problem.

Burkard et al. [6] states that the set S of p vertices is a p-maxian of T if and only if it contains a pair of vertices $\{a, b\}$ such that $P(a, b)$ is the longest path of T. Therefore, the inverse p-maxian problem on a tree can be reduced to $O(p^2)$ many inverse 2-maxian problems on this tree. From here on, we focus on the inverse 2-maxian problem on the tree T.

Consider a monotone cost function $\mathcal{C}$ and $\{a, b\}$ is a pair of leaves which are the prespecified vertices, we get the following property.

Proposition 2.3. *In the optimal solution of the inverse* 2*-maxian problem on* T*, it suffices to increase the lengths of edges in* $P(a, b)$ *and reduce the lengths of edges in* $T \setminus P(a, b)$*.*

By Proposition 2.3, we can set $x_e := p_e$ and $\overline{x}_e := \overline{p}_e$ for $e \in P(a, b)$, and $x_e := q_e$ and $\overline{x}_e = \overline{q}_e$ for $e \in T \setminus P(a, b)$. An edge e is said to be modified by an amount x_e if its modified length is set to $\widetilde{\ell}_e := \ell_e + \text{sign}(e)x_e$, where $\text{sign}(e) = 1$ if $e \in P(a, b)$ and $\text{sign}(e) = -1$ if $e \notin P(a, b)$. Furthermore, denote by $\mathcal{L}$ the set of leaves in the tree T and $\widetilde{d}$ by the distance measure with respect to new edge lengths $\widetilde{\ell}$. In order to make $P(a, b)$ the longest path of T, it must hold that $\widetilde{d}(v, v_{ab}) \leq \widetilde{d}(a, v_{ab})$ or $\widetilde{d}(v, v_{ab}) \leq \widetilde{d}(b, v_{ab})$ for all leaves v in T, see Corollary 2.2. After some elementary computations, we can formulate the inverse 1-maxian problem on T as follows.

$$\text{(2.1)} \qquad \min \mathcal{C}(x) \text{ s.t. } \begin{cases} \displaystyle\sum_{e \in P(a, v_{ab})} x_e + \sum_{e \in P(v, v_{ab})} x_e \geq \mathcal{G}(a, v) & \forall\, v \in \mathcal{L}, \\ \displaystyle\sum_{e \in P(b, v_{ab})} x_e + \sum_{e \in P(v, v_{ab})} x_e \geq \mathcal{G}(b, v) & \forall\, v \in \mathcal{L}, \\ 0 \leq x_e \leq \overline{x}_e & \forall\, e \in E. \end{cases}$$

Here, $\mathcal{G}(\star, v) := d(v, v_{ab}) - d(\star, v_{ab})$, $\star = a, b$, for $v \in \mathcal{L}$ are the gaps between distances.

3. The problem under l_1-norm

Assume that modifying an edge e by a unit amount costs c_e, the objective function in (2.1) under l_1-norm can be written as

$$\mathcal{C}(x) = \sum_{e \in E} c_e x_e.$$

The inverse 2-maxian problem on trees under l_1-norm can be formulated as a linear program. It is therefore solvable in polynomial time. However, an efficient combinatorial algorithm is still unknown.

We now consider the case where the underlying tree is a star graph with center vertex v_0. We can directly deliver the following property from Corollary 2.2.

Corollary 3.1 (Optimality criterion)**.** *Given two leaf nodes a and b in a star graph S. Then, $\{a, b\}$ is a 2-maxian of the star graph if and only if $\ell_{(v_0,a)}$ and $\ell_{(v_0,b)}$ are two largest edges.*

For simplicity, we denote $\ell_{(v_0,a)}$ and $\ell_{(v_0,b)}$ by ℓ_a and ℓ_b, and the cost to modify one unit length of (v_0, a) and (v_0, b) by c_a and c_b. By Corollary 3.1, we consider how to modify the length of edges with minimum cost such that $\widetilde{\ell}_a$ and $\widetilde{\ell}_b$ become the two largest edges in the star. Assume that $\ell_a < \ell_b$, we analyze these two following cases.

1. If $\widetilde{\ell}_a \in [\ell_a, \ell_b]$, we do not increase the length of ℓ_b but decrease the length of $e \neq (v_0, b)$ with $\ell_e > \widetilde{\ell}_a$ to $\widetilde{\ell}_a$. We first presolve the problem as follows.

   - If $\xi = \min_{e\neq(v_0,b):\ell_e>\ell_a} \{\ell_e - \overline{x}_e\} > \ell_a$, then we increase ℓ_a by an amount $\xi - \ell_a$.
   - If $\theta = \ell_a + \overline{x}_{(v_0,a)} < \ell_e$ for $e \neq (v_0, b)$, then we decrease the length of ℓ_e by an amount $\ell_e - \theta$.

   If the presolution is not possible, then the problem is infeasible. Otherwise, we can formulate the problem as a univariate optimization program

   $$\min f(z) = c_a(z - \ell_a) + \sum_{e\neq(v_0,b):\ell_e>z} c_e(\ell_e - z)$$

   where $z = \widetilde{\ell}_a$ and $z \in [\ell_a, \ell_a + \overline{x}_{(v_0,a)}]$.

2. If $\widetilde{\ell}_a > \ell_b$, then the length of e where $\ell_e > \widetilde{\ell}_a$ must be decrease to $\widetilde{\ell}_a$ and the length of ℓ_b must be increase to $\widetilde{\ell}_a$. First, we have to increase ℓ_a an amount $\ell_b - \ell_a$ and presolve the problem as in the first case. If the presolution is not possible then the problem is infeasible. Otherwise, we can formulate the problem as follows.

   $$\min f(z) = (c_a + c_b)(z - \ell_b) + \sum_{e:\ell_e>z} c_e(\ell_e - z)$$

   where $z = \widetilde{\ell}_a = \widetilde{\ell}_b$ and $z \in \left[\ell_b, \min\left\{\ell_a + \overline{x}_{(v_0,a)}, \ell_b + \overline{x}_{(v_0,b)}\right\}\right]$.

In both cases, the function $f(z)$ is a convex function. According to Alizadeh and Burkard [3], the univariate optimization problem can be solved in linear time. To get the optimal solution of the inverse 2-maxian problem on S, we have to solve two problems and get the best one. Therefore, we attain the following result.

Theorem 3.2. *The inverse 2-maxian problem on a star graph can be solved in linear time.*

4. Problem under Chebyshev norm

We recall that modifying an edge e by one unit length costs c_e, the objective function in (2.1) under Chebyshev norm can be written as

$$\mathcal{C}(x) = \max_{e \in E} \{c_e x_e\}.$$

The solution method of the problem is based on the greedy modification. We define a valid modification of edge lengths in T with cost C as follows.

Definition 4.1 (Valid modification)**.** [21] A valid modification with cost C is to modify each edge of T as much as possible such that the cost is limited within C, i.e., we set

$$x_e := \begin{cases} \dfrac{C}{c_e} & \text{if } c_e \overline{x}_e > C, \\ \overline{x}_e & \text{if } c_e \overline{x}_e \le C. \end{cases}$$

We can solve the problem by the following two phases.

Phase 1: Find the interval that contains the optimal cost.

We first sort the costs $\{c_e x_e\}_{e \in E}$ in nondecreasing order and unite the similar values. Then we obtain a sequence of costs

$$c_1 \overline{x}_1 < c_2 \overline{x}_2 < \cdots < c_m \overline{x}_m.$$

Here, $m = O(n)$.

Then we aim to find the smallest index i_0 such that $P(a, b)$ become the longest path by applying the valid modification with cost $c_{i_0} \overline{x}_{i_0}$. The index i_0 can be found by applying a binary search algorithm. If we apply the valid modification with cost $c_k \overline{x}_k$ but the path $P(a, b)$ does not become the longest path then $i_0 > k$. Otherwise, we know $i_0 \le k$.

Let us analyze the complexity to find i_0. In each iteration modifying the edge lengths of the tree costs linear time. Then we find the longest path of the tree in linear time (see Handler [18]), and compare the length of $P(a, b)$ with the longest one in order to decide if Corollary 2.2 holds or not in linear time. As the binary search stops after $O(\log n)$ iterations, this procedure runs in $O(n \log n)$ time.

Assume that we have found the interval $[c_{i_0-1} \overline{x}_{i_0-1}, c_{i_0} \overline{x}_{i_0}]$ that contains the optimal cost. We apply a valid modification with cost $c_{i_0-1} \overline{x}_{i_0-1}$ and get the modified tree $\widetilde{T}$. Also, we update the upper bounds of modifications. The next step is to define a parameter $t \in [0, c_{i_0} \overline{x}_{i_0}]$ and find the smallest value t such that $P(a, b)$ becomes the longest path of $\widetilde{T}$ for a valid modification with cost $c_{i_0} t$.

Phase 2: Find the minimizer t with respect to the optimal objective value in $[0, c_{i_0}\overline{x}_{i_0}]$.

After deleting all edges of the path $P(a,b)$ we get a forest T^{del}. Consider a vertex v in the path $P(a,b)$, $v \neq a, b$. A branch T_v is, by definition, a connected component of T^{del} that contains the vertex v. Denote by $\mathcal{L}(T_v)$ the set of leaves in T_v. Now we try to modify the edge lengths of T so that there is no path $P(v,v')$, for $v \in P(a,b)$, $v \neq a, b$ and $v' \in \mathcal{L}(T_v)$, is longer than $\min\{d(v,a), d(v,b)\}$. To solve this problem, we first apply a valid modification with cost $c_{i_0}t$ for $t \in [0, \overline{x}_{i_0}]$, i.e., one sets $x_e = \frac{c_{i_0}t}{c_e}$ for $e \in E$ and $\overline{x}_e > 0$. Then we consider the modifying lengths of these following paths.

- For a path $P(v,v')$ with $v' \in \mathcal{L}(T_v)$ and $d(v,v') > \min\{d(v,a), d(v,b)\}$, its modifying length is
$$d(v,v') - c_{i_0} \sum_{e \in P(v,v'), \overline{x}_e > 0} \frac{t}{c_e}.$$
- For the paths $P(v,a)$ and $P(v,b)$, we obtain the modifying lengths
$$d(v,a) + c_{i_0} \sum_{e \in P(v,a), \overline{x}_e > 0} \frac{t}{c_e} \quad \text{and} \quad d(v,b) + c_{i_0} \sum_{e \in P(v,b), \overline{x}_e > 0} \frac{t}{c_e}.$$

We first identify the value t_1 and t_2 such that

$$t_1 := \arg\min\max\left\{ d(v,v') - c_{i_0} \sum_{e \in P(v,v'), \overline{x}_e > 0} \frac{t}{c_e}, d(v,a) + c_{i_0} \sum_{e \in P(v,a), \overline{x}_e > 0} \frac{t}{c_e} \right\}, \tag{4.1}$$

$$t_2 := \arg\min\max\left\{ d(v,v') - c_{i_0} \sum_{e \in P(v,v'), \overline{x}_e > 0} \frac{t}{c_e}, d(v,b) + c_{i_0} \sum_{e \in P(v,b), \overline{x}_e > 0} \frac{t}{c_e} \right\}, \tag{4.2}$$

where $v \in P(a,b) \setminus \{a,b\}$, $v' \in \mathcal{L}(T_v)$, $d(v,v') > \min\{d(v,a), d(v,b)\}$ and $t \in [0, \overline{x}_{i_0}]$. As we have to find the minimizer of the upper envelop of linear functions, the algorithm in Section 5 of Gassner [15] will find t_1 and t_2 in linear time provided that all linear functions have been found.

The smallest value t^*, such that the modifying lengths of all paths $P(v,v')$ for $v \in P(a,b) \setminus \{a,b\}$ and $v' \in \mathcal{L}(T_v)$ are not larger than $\min\left\{\widetilde{d}(v,a), \widetilde{d}(v,b)\right\}$, is easily identified as $t^* := \max\{t_1, t_2\}$.

We now analyze the complexity to find all linear functions in the expressions (4.1) and (4.2). We number the vertices in the path $P(a,b)$ from the left to the right side by $a, v_1, v_2, \ldots, v_k, b$, where $k = O(n)$. Denote by $E(S)$ the set of edges in a subtree S. We start with the vertex v_1, it costs $O(|E(T_{v_1})|)$ time (by a breath-first-search procedure) to identify all functions
$$d(v_1,v') - c_{i_0} \sum_{e \in P(v_1,v'), \overline{x}_e > 0} \frac{t}{c_e}$$

for $v' \in \mathcal{L}(T_{v_1})$ and $O(|E(P(a,b))|)$ time to identify

$$d(v_1,\star) + c_{i_0} \sum_{e\in P(v_1,\star),\overline{x}_e>0} \frac{t}{c_e}$$

for $\star = a, b$.

Next we have to find all linear functions with respect to vertex v_2. To identify the functions

$$d(v_2,\star) + c_{i_0} \sum_{e\in P(v_2,\star),\overline{x}_e>0} \frac{t}{c_e}$$

for $\star = a, b$, we have to exchange the modification edge (v_1, v_2) that contribute to the augmentation of path $P(v_2, a)$. We continue the process for vertex $v_3, v_4, \ldots, v_k$ similarly.

In conclusion, it costs $O\left(\sum_{v\in P(a,b)\setminus\{a,b\}} |E(T_v)| + 2\,|E(P(a,b))|\right) = O(n)$ time to complete the procedure to find all linear functions in (4.1) and (4.2). Therefore, we get the following result.

Theorem 4.2. *The inverse* 2*-maxian problem on a tree under Chebyshev norm can be solved in* $O(n \log n)$ *time.*

5. Problem under Hamming distance

First, let us focus on the problem under bottleneck Hamming distance. Assume that modifying one unit length of edge e costs c_e, we aim to minimize the following objective function

$$\max_{e\in E} \{c_e H(x_e)\}.$$

Here, H is a Hamming distance and defined by

$$H(\theta) := \begin{cases} 0 & \text{if } \theta = 0, \\ 1 & \text{otherwise.} \end{cases}$$

By the special structure of the Hamming distance, the objective function receives finitely many values, say $\{c_e : e \in E\}$. Therefore, we can solve the problem by finding the smallest value in $\{c_e : e \in E\}$ such that the optimality criterion in Lemma 2.2 holds.

Number the edges in T by $1, 2, \ldots, m$, and the corresponding costs are $c_1, c_2, \ldots, c_m$ for $m = n - 1$. Let us first sort the costs $\{c_e : e \in E\}$ increasingly and get without loss of generality a sequence

$$c_1 \le c_2 \le \cdots \le c_m.$$

Now we apply a binary search algorithm to find the optimal cost. We start with the cost c_k, $k = \lfloor \frac{m+1}{2} \rfloor$. We modify all the edges $1, 2, \ldots, k-1$. If the optimality criterion holds, we know that the optimal value is less than or equal to c_k. Otherwise, it is larger than c_k. In each iteration we recomputing the length of the longest path in linear time (see Handler [18]) and compare it with the length of $P(a,b)$. Moreover, as the binary search stop after $O(\log m)$ iterations, Phase 1 runs in $O(m \log m) = O(n \log n)$ time.

Theorem 5.1. *The inverse* 2*-maxian problem on trees under bottleneck Hamming distance can be solved in* $O(n \log n)$ *time.*

For the problem under weighted sum Hamming distance, the objective function can be written as $\sum_{e \in E} c_e H(x_e)$. As we can easily reduce the Knapsack problem into an inverse p-maxian problem under weighted sum Hamming distance in polynomial time, we derive a result as follows.

Theorem 5.2. *The inverse* p*-maxian problem on tree under weighted sum Hamming distance is* NP*-hard.*

6. Conclusions

We have addressed the inverse p-maxian problem, $p \geq 2$, under various objective functions. It is shown that the problem can be reduced to $O(p^2)$ many 2-maxian problems. Then we have formulated the inverse 2-maxian problem on trees under l_1-norm as a linear program and solved the problem on star graphs in linear time. Furthermore, the inverse 2-maxian problem on a tree under Chebyshev norm and bottleneck Hamming distance is solvable in $O(n \log n)$ time, where n is the number of vertices in the tree. For future research topics, we will consider the inverse maxian problem on other classes of graphs, e.g., cacti, interval graphs, block graphs, etc.

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