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Algebraic Properties of Cauchy Singular Integral Operators on the Unit Circle

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Abstract. In this paper we study algebraic properties of singular integral operators with Cauchy kernel on the L^2 space of the unit circle. We give an operator equation characterization for this class of Cauchy singular integral operators. This characterization provides a direct connection between the singular integral operators and multiplication operators. We then use this characterization to study when two Cauchy singular integral operators commute. Our approach also leads to generalizations of several results on normal Cauchy singular integral operators obtained recently by Nakazi and Yamamoto.

1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane. Let $L^2 = L^2(\mathbb{T})$ be the set of all square-integrable functions on $\mathbb{T}$. Each function $f \in L^2$ has a Fourier series expansion

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \quad \text{for } \theta \in [0, 2\pi]$$

and

$$\|f(z)\|^2 = \int_{\mathbb{T}} |f(z)|^2 \, dm(z) = \sum_{n=-\infty}^{\infty} |f_n|^2$$

where $m(z)$ is the normalized Lebesgue measure on $\mathbb{T}$. The function $f(e^{i\theta})$ has a unique harmonic extension into the open unit disk D as follows

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n r^{|n|} e^{in\theta}, \quad 0 \le r \le 1.$$

Let L^∞ be the set of essentially bounded functions on $\mathbb{T}$. Given $\varphi \in L^\infty$, the multiplication operator M_φ is defined by

$$M_\varphi f = \varphi f, \quad f \in L^2.$$

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Note that $\|M_\varphi\| = \|\varphi\|_\infty$. If $\varphi_1, \varphi_2 \in L^\infty$, then

$$M_{\varphi_1} M_{\varphi_2} = M_{\varphi_1 \varphi_2} = M_{\varphi_2} M_{\varphi_1}.$$

The Hardy space H^2 is the closed subspace of L^2 spanned by analytic polynomials. In other words, each $f \in H^2$ has a Fourier series expansion

$$f(e^{i\theta}) = \sum_{n=0}^{\infty} f_n e^{in\theta} \quad \text{for } \theta \in [0, 2\pi],$$

and we can view f as an analytic function inside the unit disk D with power series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad |z| < 1.$$

Let H^∞ be the set of all bounded analytic functions on D.

Given $\varphi \in L^\infty$, the Toeplitz operator $T_\varphi \colon H^2 \to H^2$ is defined by

$$T_\varphi(f) = P(\varphi f), \quad f \in H^2,$$

and the Hankel operator $H_\varphi \colon H^2 \to L^2 \ominus H^2 = \overline{zH^2}$ is defined by

$$H_\varphi(f) = Q(\varphi f), \quad g \in H^2$$

where P and $Q = I - P$ denote the orthogonal projections that map L^2 onto H^2 and $L^2 \ominus H^2 = \overline{zH^2}$ respectively. Some basic algebraic properties of Toeplitz operators were developed in Brown and Halmos [2]. See also [1] for literatures on Toeplitz operators and Hankel operators.

Let $\alpha, \beta \in L^\infty$, the Cauchy singular integral operator $S_{\alpha,\beta} \colon L^2 \to L^2$ is defined by

$$S_{\alpha,\beta}(f) = \alpha P f + \beta Q f, \quad f \in L^2.$$

It is clear that

$$\begin{aligned} \|S_{\alpha,\beta}(f)\| &\le \|\alpha P f\| + \|\beta Q f\| \le \|\alpha\|_\infty \|Pf\| + \|\beta\|_\infty \|Qf\| \\ &\le (\|\alpha\|_\infty + \|\beta\|_\infty) \|f\|, \end{aligned}$$

so $S_{\alpha,\beta}$ is a bounded operator. See a recent survey on norms of some classical singular integral operators in [9]. The operator $S_{\alpha,\beta}$ has an integral representation with Cauchy kernel,

$$S_{\alpha,\beta}(f) = \frac{\alpha(z) + \beta(z)}{2} f(z) + \frac{\alpha(z) - \beta(z)}{2} \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Singular integral operators and singular integral equations have been studied extensively in literature. The two volumes [4] and [5] by Gohberg and Krupnik are classical.

Singular integral equations also have wide and important applications as demonstrated in a recent book [10] by Mandal and Chakrabarti.

Recently Nakazi and Yamamoto in [12] characterized when $S_{\alpha,\beta}$ is normal. The operator $S_{\alpha,\beta}$ has a close connection with Toeplitz and Hankel operators via a 2 by 2 block operator form (Lemma 3.1 in [12]). This connection plays a key role in deriving most results in [12].

In this paper we use a more direct approach by characterizing this class of singular integral operators as the solutions to an operator equation, see Proposition 2.1. Our approach provides the insight into how this class of singular integral operators is intimately related to the multiplication operators.

Research on singular integral operators has focused on boundedness, invertibility and Fredholm theory. In this paper several basic algebraic properties of the singular integral operator $S_{\alpha,\beta}$ are obtained. A couple of results from [12] are generalized and proved in simpler and more direct ways. Because we consider more general questions, the algebra involved is somewhat lengthy and more challenging. It is also possible to use 2 by 2 Toeplitz and Hankel block operator forms as in [12] and the algebra will be even more demanding and may be impossible in some instances. However our insights do come from working with related problems on Toeplitz and Hankel operators in [6] and [7]. Furthermore, this study shows that this class of operators has many interesting properties and these operators are natural extensions of multiplication operators, Toeplitz operators and Hankel operators.

We outline our plan. In Section 2, we show that singular integral operators or products of singular operators satisfy an operator equation. In Section 3, we characterize when the product of two singular integral operators is also a singular integral operator. This characterization enables us to identify some subalgebras of singular integral operators. We then study when the product of a singular operator and the adjoint of another singular operator is also a singular integral operator. As an application, we recover the results about isometric and unitary singular integral operators in [12]. We prove that $S_{\alpha,\beta}$ is a coisometry if and only if it is a unitary operator.

In Section 4, we prove essentially two singular integral operators commute if and only if one is a multiple of the other. In Section 5, we discuss when a singular operator and the adjoint of another singular integral operator commute. As a corollary, we obtain normal singular integral operators discovered in [12]. In Section 6, we show that most results from previous sections are also valid for singular integral operators defined on the L^p space of the unit circle $\mathbb{T}$.

2. A characterization of singular integral operators

Let $B(L^2)$ be the algebra of all bounded linear operators on L^2. Let S denote the set of all singular integral operators

$$S = \left\{S_{\alpha,\beta} \in B(L^2) \mid \alpha, \beta \in L^\infty\right\}.$$

Note that $S_{\alpha,\alpha} = M_\alpha$ and $S_{\alpha,\beta} = M_\beta + S_{\alpha-\beta,0}$. By the Spectral Theorem for normal operators, the commutant of M_z is the set of all multiplication operators on L^2. Set

$$\begin{aligned} M &= \left\{M_\alpha \in B(L^2) \mid \alpha \in L^\infty\right\} \\ &= \left\{A \in B(L^2) \mid M_z A = A M_z\right\}. \end{aligned}$$

Let G be a subset of $B(L^2)$, we define

$$G^* = \{A^* \mid A \in G\}.$$

The set G is said to be self-adjoint if $G = G^*$. The set M is a commutative C^*-algebra. The set S is neither an algebra nor a self-adjoint set.

Let $e_n = z^n$ and $e_{-n} = z^{-n} = \overline{z}^n$ for $n \geq 0$ where $z = e^{i\theta}$. For $f \in L^2$, the Fourier series of f is

$$f = \sum_{n=-\infty}^{\infty} f_n e_n = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}.$$

Thus f_{-1} denotes the Fourier coefficient corresponding to the term e_{-1}.

For two operators $C, D \in B(L^2)$, let $[C, D] = CD - DC$ denote the commutator of C and D. For $x, y \in L^2$, let $x \otimes y$ denote the rank one operator defined by $[x \otimes y]h = \langle h, y\rangle\, x$ for $h \in L^2$. The following proposition characterizes S as the set of all operators whose commutators with M_z are special rank one operators.

Proposition 2.1. *Let $A \in B(L^2)$. Then $A \in S$ if and only if there exists a $\psi \in L^\infty$ such that*

$$[A, M_z] = \psi \otimes e_{-1}. \tag{2.1}$$

In this case $A = S_{\psi+\beta,\beta}$ for some $\beta \in L^\infty$.

Proof. Let $A = S_{\alpha,\beta} \in S$ for some $\alpha, \beta \in L^\infty$. Let $f = \sum_{n=-\infty}^{\infty} f_n z^n \in L^2$, then

$$\begin{aligned} S_{\alpha,\beta} M_z(f) &= \alpha P[zf] + \beta Q[zf] \\ &= \alpha[zPf + f_{-1}] + \beta[zQf - f_{-1}] \\ &= z\alpha Pf + z\beta Qf + (\alpha - \beta) f_{-1} \\ &= M_z S_{\alpha,\beta}(f) + [(\alpha - \beta) \otimes e_{-1}](f). \end{aligned}$$

This proves (2.1) with $\psi = (\alpha - \beta) \in L^\infty$.

Now assume $A \in B(L^2)$ and (2.1) holds. By the above argument

$$S_{\psi,0}M_z - M_z S_{\psi,0} = \psi \otimes e_{-1}.$$

Therefore

$$\begin{aligned}(A - S_{\psi,0})M_z - M_z(A - S_{\psi,0}) &= (AM_z - AM_z) - (S_{\psi,0}M_z - S_{\psi,0}M_z)\\ &= \psi \otimes e_{-1} - \psi \otimes e_{-1} = 0.\end{aligned}$$

and

$$(A - S_{\psi,0})M_z = M_z(A - S_{\psi,0}).$$

By the Spectral Theorem, $A - S_{\psi,0} = M_\beta = S_{\beta,\beta}$ for some $\beta \in L^\infty$. Thus $A = S_{\psi,0} + S_{\beta,\beta} = S_{\psi+\beta,\beta}$. □

The adjoint $S^*_{\alpha,\beta}$ in general is not in S. The following result tells us when $S^*_{\alpha,\beta}$ belongs to S.

Proposition 2.2. *The adjoint $S^*_{\alpha,\beta} \in S$ if and only if $(\alpha - \beta) = \lambda$ for some constant λ. In this case $S^*_{\alpha,\beta} = S_{\overline{\alpha},\overline{\beta}}$.*

Proof. By Proposition 2.1, $S^*_{\alpha,\beta} \in S$ if and only if

$$S^*_{\alpha,\beta}M_z - M_z S^*_{\alpha,\beta} = \psi \otimes e_{-1}$$

for some $\psi \in L^\infty$. But $M_z^* M_z = M_z M_z^* = I$ and

$$\begin{aligned}S^*_{\alpha,\beta}M_z - M_z S^*_{\alpha,\beta} &= M_z(M_z^* S^*_{\alpha,\beta} - S^*_{\alpha,\beta}M_z^*)M_z\\ &= M_z[S_{\alpha,\beta}, M_z]^* M_z\\ &= M_z[e_{-1} \otimes (\alpha - \beta)]M_z\\ &= e_0 \otimes \overline{z}(\alpha - \beta).\end{aligned}$$

Thus

$$\psi \otimes e_{-1} = e_0 \otimes \overline{z}(\alpha - \beta)$$

and $(\alpha - \beta)\overline{z} = \lambda e_{-1}$ and $\psi = \overline{\lambda} e_0$ for some complex number λ. In this case

$$S^*_{\alpha,\beta} = (M_\beta + S_{\alpha-\beta,0})^* = M_{\overline{\beta}} + S_{\overline{\lambda},0} = S_{\overline{\alpha},\overline{\beta}}.$$

The proof is complete. □

The following set M_+ is a self-adjoint subset of S and it is slightly larger than M.

$$M_+ = \{S_{\beta+\lambda,\beta} \mid \beta \in L^\infty, \lambda \in \mathbb{C}\}.$$

The following corollary is Theorem 2.1 in [12]. See also related work on self-adjoint singular integral operators [8].

Corollary 2.3. [12] *$S_{\alpha,\beta}$ is self-adjoint if and only if α and β are real valued functions and $(\alpha - \beta)$ is a real constant.*

Proof. If $S^*_{\alpha,\beta} = S_{\alpha,\beta} \in S$, by Proposition 2.2, $(\alpha - \beta)$ is a constant. Furthermore $S^*_{\alpha,\beta} = S_{\overline{\alpha},\overline{\beta}} = S_{\alpha,\beta}$ implies that $\alpha = \overline{\alpha}$ and $\beta = \overline{\beta}$. □

The set M is a subalgebra of S. Are there other subalgebras of S? To answer this question, we need to determine when the product of two operators from S belongs to S. The following lemma derives operator equations for the products of operators from S or S^*.

Lemma 2.4. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Then*

$$[S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2}, M_z] = (\alpha_1 - \beta_1) \otimes S^*_{\alpha_2,\beta_2} e_{-1} + S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2) \otimes e_{-1}.$$

Similarly,

$$[S_{\alpha_1,\beta_1} S^*_{\alpha_2,\beta_2}, M_z] = \alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2$$

and

$$[S^*_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}, M_z] = e_0 \otimes S^*_{\alpha_1,\beta_1} \overline{z}(\alpha_2 - \beta_2) + S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) \otimes e_{-1}.$$

Proof. By a direct computation,

$$\begin{aligned}
[S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2}, M_z] &= [S_{\alpha_1,\beta_1}, M_z] S_{\alpha_2,\beta_2} + S_{\alpha_1,\beta_1}[S_{\alpha_2,\beta_2}, M_z] \\
&= [(\alpha_1 - \beta_1) \otimes e_{-1}] S_{\alpha_2,\beta_2} + S_{\alpha_1,\beta_1}[(\alpha_2 - \beta_2) \otimes e_{-1}] \\
&= (\alpha_1 - \beta_1) \otimes S^*_{\alpha_2,\beta_2} e_{-1} + S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2) \otimes e_{-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[S_{\alpha_1,\beta_1} S^*_{\alpha_2,\beta_2}, M_z] &= [S_{\alpha_1,\beta_1}, M_z] S^*_{\alpha_2,\beta_2} + S_{\alpha_1,\beta_1}[S^*_{\alpha_2,\beta_2}, M_z] \\
&= [S_{\alpha_1,\beta_1}, M_z] S^*_{\alpha_2,\beta_2} + S_{\alpha_1,\beta_1} M_z [S_{\alpha_2,\beta_2}, M_z]^* M_z \\
&= (\alpha_1 - \beta_1) \otimes S_{\alpha_2,\beta_2} e_{-1} + S_{\alpha_1,\beta_1} e_0 \otimes \overline{z}(\alpha_2 - \beta_2) \\
&= (\alpha_1 - \beta_1) \otimes \overline{z}\beta_2 + \alpha_1 \otimes \overline{z}(\alpha_2 - \beta_2) \\
&= \alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2
\end{aligned}$$

and

$$\begin{aligned}[S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}, M_z] &= [S^*_{\alpha_2,\beta_2}, M_z]S_{\alpha_1,\beta_1} + S^*_{\alpha_2,\beta_2}[S_{\alpha_1,\beta_1}, M_z] \\ &= M_z[S_{\alpha_2,\beta_2}, M_z]^* M_z S_{\alpha_1,\beta_1} + S^*_{\alpha_2,\beta_2}[S_{\alpha_1,\beta_1}, M_z] \\ &= e_0 \otimes S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2) + S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) \otimes e_{-1}.\end{aligned}$$

The proof is complete. □

We now establish that finite sums of products of singular integral operators are models for operators whose commutators with M_z are of finite rank.

Theorem 2.5. *Let $A \in B(L^2)$. Then*

$$[A, M_z] = \sum_{k=1}^{n} \varphi_k \otimes \psi_k \tag{2.2}$$

where $\varphi_k, \psi_k \in L^\infty$ if and only if

$$A = M_\alpha + \sum_{k=1}^{m} S_{\alpha_k,\beta_k} S^*_{\gamma_k,\delta_k} \tag{2.3}$$

where $\alpha, \alpha_k, \beta_k, \gamma_k, \delta_k \in L^\infty$ and $m = \frac{n}{2}$ if n is even and $m = \frac{n+1}{2}$ if n is odd. In particular, if

$$[A, M_z] = \varphi \otimes \psi$$

where $\varphi, \psi \in L^\infty$, then

$$A = M_\alpha + M_\varphi S^*_{z\psi,0} \quad \text{or} \quad A = M_\beta + S_{\varphi,0} M^*_{z\psi}$$

for some $\alpha, \beta \in L^\infty$.

Proof. One direction is clear from Lemma 2.4. Assume now (2.2) holds, we will prove (2.3). We will demonstrate the result for $n = 2$ and $n = 1$. The general case will be clear. Assume

$$[A, M_z] = \varphi_1 \otimes \psi_1 + \varphi_2 \otimes \psi_2.$$

We work backward by using Lemma 2.4 and setting

$$\varphi_1 \otimes \psi_1 + \varphi_2 \otimes \psi_2 = \alpha_1 \otimes \overline{z}\gamma_1 - \beta_1 \otimes \overline{z}\delta_1. \tag{2.4}$$

One set of solutions is

$$\alpha_1 = \varphi_1, \quad \beta_1 = -\varphi_2 \quad \gamma_1 = z\psi_1, \quad \delta_1 = z\psi_2.$$

Therefore by Lemma 2.4 and (2.4),

$$[A - S_{\alpha_1,\beta_1}S^*_{\gamma_1,\delta_1}, M_z] = [A, M_z] - [S_{\alpha_1,\beta_1}S^*_{\gamma_1,\delta_1}, M] = 0.$$

By the Spectral Theorem,

$$A - S_{\alpha_1,\beta_1}S^*_{\gamma_1,\delta_1} = M_\alpha$$

for some $\alpha \in L^\infty$. The proves the case $n = 2$. For $n = 1$, if

$$[A, M_z] = \varphi \otimes \psi,$$

then we work backward by using Lemma 2.4 and setting

$$\varphi \otimes \psi = \alpha_1 \otimes \overline{z}\gamma_1 - \beta_1 \otimes \overline{z}\delta_1.$$

We set

$$\alpha_1 = \beta_1 = \varphi, \quad \gamma_1 = z\psi, \quad \delta_1 = 0.$$

Then $A = M_\alpha + M_\varphi S^*_{z\psi,0}$ for some $\alpha \in L^\infty$. The proof of $A = M_\beta + S_{\varphi,0}M^*_{z\psi}$ is similar. □

3. Products of singular integral operators

We first study when the product of two operators from S belongs to S. This result will help us identify some subalgebras of S.

Theorem 3.1. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Assume S_{α_1,β_1} is not in M. Then $S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2} \in S$ if and only if $\alpha_2 \in H^\infty$, $\beta_2 \in \overline{H^\infty}$. In this case $S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2} = S_{\alpha_1\alpha_2,\beta_1\beta_2}$.*

Proof. By Proposition 2.1, $S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2} \in S$ if and only if

$$[S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2}, M_z] = \psi \otimes e_{-1} \tag{3.1}$$

for some $\psi \in L^\infty$. By Lemma 2.4,

$$\begin{aligned}[S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2}, M_z] &= (\alpha_1 - \beta_1) \otimes S^*_{\alpha_2,\beta_2}e_{-1} + S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2) \otimes e_{-1} \\ &= \psi \otimes e_{-1}.\end{aligned}$$

By assumption $\alpha_1 - \beta_1 \neq 0$, therefore, there exists a complex number λ such that

$$S^*_{\alpha_2,\beta_2}e_{-1} = \overline{\lambda}e_{-1}, \tag{3.2}$$

$$S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2) = -\lambda(\alpha_1 - \beta_1) + \psi. \tag{3.3}$$

Note that $S^*_{\alpha_2,\beta_2}(f) = P[\overline{\alpha_2}f] + Q[\overline{\beta_2}f]$. We have from (3.2)

$$\left(P[\overline{\alpha_2}e_{-1}] + Q[\overline{\beta_2}e_{-1}]\right) = \overline{\lambda}\overline{z},$$

which implies that $\alpha_2 \in H^\infty$ and $\beta_2 = \lambda + \sum_{n=-\infty}^{-1} \beta_{2n} z^n \in \overline{H^\infty}$. Now (3.3) becomes

$$\alpha_1 P[\alpha_2 - \beta_2] + \beta_1 Q[\alpha_2 - \beta_2] = -\lambda(\alpha_1 - \beta_1) + \psi,$$
$$\alpha_1(\alpha_2 - \lambda) + \beta_1(-\beta_2 + \lambda) = -\lambda(\alpha_1 - \beta_1) + \psi.$$

Therefore

$$\alpha_1\alpha_2 = \beta_1\beta_2 + \psi. \tag{3.4}$$

Since $\alpha_2 \in H^\infty$ and $\beta_2 \in \overline{H^\infty}$,

$$\begin{aligned} S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} f &= S_{\alpha_1,\beta_1}(\alpha_2 P f + \beta_2 Q f) \\ &= \alpha_1 P[\alpha_2 P f + \beta_2 Q f] + \beta_1 Q[\alpha_2 P f + \beta_2 Q f] \\ &= \alpha_1\alpha_2 P f + \beta_1\beta_2 Q f \\ &= S_{\alpha_1\alpha_2,\beta_1\beta_2} f = S_{\alpha_1\alpha_2,\alpha_1\alpha_2-\psi} f = S_{\beta_1\beta_2+\psi,\beta_1\beta_2} f. \end{aligned}$$

The proof is complete. □

Remark 3.2. (a) If $S_{\alpha_1,\beta_1} = M_{\alpha_1}$, then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} \in S$ for any S_{α_2,β_2} and $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = M_{\alpha_1} S_{\alpha_2,\beta_2} = S_{\alpha_1\alpha_2,\alpha_1\beta_2}$.

(b) Assume $S_{\alpha_1,\beta_1} \notin M$ and $S_{\alpha_2,\beta_2} = M_{\alpha_2}$. If $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} \in S$, then by the above theorem, α_2 is a constant, $S_{\alpha_2,\beta_2} = \alpha_2 I$ and $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_1\alpha_2,\beta_1\alpha_2}$.

(c) If $\alpha_2 \in H^\infty$, $\beta_2 \in \overline{H^\infty}$, then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_1\alpha_2,\beta_1\beta_2}$. This formula is valid for more general singular integral operators, see equation (6.3) in [4].

Recall M_α for some $\alpha \in L^\infty$ is invertible if and only if α is invertible in L^∞, and in this case $M_\alpha^{-1} = M_{\alpha^{-1}}$. Let erange$(\alpha)$ denote the essential range of α. Thus $\sigma(M_\alpha) =$ erange(α) where $\sigma(M_\alpha)$ denote the spectrum of M_α. We now characterize when $S_{\alpha,\beta}$ is invertible and whose inverse is also in S.

Corollary 3.3. *Let $S_{\alpha,\beta} \in S$ for some $\alpha, \beta \in L^\infty$. Assume $S_{\alpha,\beta} \notin M$. Then $S_{\alpha,\beta}$ is invertible and $S_{\alpha,\beta}^{-1} \in S$ if and only if $\alpha, \overline{\beta} \in H^\infty$ and α, $\overline{\beta}$ are invertible in H^∞. In this case $S_{\alpha,\beta}^{-1} = S_{\alpha^{-1},\beta^{-1}}$. Thus, if $\alpha, \overline{\beta} \in H^\infty$, then $\sigma(S_{\alpha,\beta}) \subseteq \mathrm{erange}(\alpha) \cup \mathrm{erange}(\beta)$.*

Proof. If S_{α_1,β_1} is the inverse of $S_{\alpha,\beta}$, then $S_{\alpha_1,\beta_1} S_{\alpha,\beta} = S_{\alpha_1\alpha,\beta_1\beta} = I \in S$. Since $S_{\alpha_1,\beta_1} \notin M$ (otherwise $S_{\alpha,\beta} \in M$), by Theorem 3.1, $\alpha, \overline{\beta} \in H^\infty$ and $\alpha_1\alpha = \beta_1\beta = 1$. The result follows. □

The proof of Theorem 3.1 also tells us when $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} \in M$. This corresponds to the case $\psi = 0$ in (3.4).

Corollary 3.4. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Assume S_{α_1,β_1} is not in M. Then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} \in M$ if and only if $\alpha_2 \in H^\infty$, $\beta_2 \in \overline{H^\infty}$ and $\alpha_1\alpha_2 = \beta_1\beta_2$. In this case $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = M_{\beta_1\beta_2}$.*

We will need the following important result (a part of F. and M. Riesz theorem), see Corollary 4.2 on page 62 of [3].

Lemma 3.5. *If $\alpha \in H^2$ and α is not identically zero, then $\alpha \neq 0$ almost everywhere on the unit circle $\mathbb{T}$.*

The next result characterizes when $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = 0$.

Corollary 3.6. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Assume S_{α_1,β_1} is not in M and $S_{\alpha_2,\beta_2} \neq 0$. Then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = 0$ if and only if one of the following two statements holds.*

(i) $\alpha_1 \neq 0$, $\beta_1 = 0$, $\alpha_2 = 0$ *and* $\beta_2 \in \overline{H^\infty}$.

(ii) $\beta_1 \neq 0$, $\alpha_1 = 0$, $\beta_2 = 0$ *and* $\alpha_2 \in H^\infty$.

Proof. By Corollary 3.4, $\alpha_1 \alpha_2 = \beta_1 \beta_2 = 0$. The result follows from this equation and Lemma 3.5 since $\alpha_2 \in H^\infty$ and $\beta_2 \in \overline{H^\infty}$. □

We now identify some subalgebras of S which are closely related to the subalgebras of H^∞.

Corollary 3.7. *The largest subalgebra K of S not contained in M is*

$$K = \left\{ S_{\alpha,\beta} \mid \alpha \in H^\infty, \beta \in \overline{H^\infty} \right\}.$$

Proof. Let K be a subalgebra of S not contained in M. Let $S_{\alpha,\beta} \in S$ but $S_{\alpha,\beta} \notin M$. By Theorem 3.1, $S^2_{\alpha,\beta} \in K \subset S$ implies that $\alpha \in H^\infty$, $\beta \in \overline{H^\infty}$. Let S_{α_1,β_1} be an arbitrary element of K, then $S_{\alpha,\beta} S_{\alpha_1,\beta_1} \in K \subset S$ implies that $\alpha_1 \in H^\infty$, $\beta_1 \in \overline{H^\infty}$. The proof is complete. □

If $S_{\alpha,\beta} \in K$, then $S^*_{\alpha,\beta} \notin S$ unless both α and β are constants (by Proposition 2.2). For any two fixed inner functions θ_1, θ_2, the following set K_{θ_1,θ_2} is a subalgebra of S.

$$K_{\theta_1,\theta_2} = \left\{ S_{\theta_1 \alpha, \overline{\theta_2} \beta} \mid \alpha \in H^\infty, \beta \in \overline{H^\infty} \right\} = S_{\theta_1, \overline{\theta_2}} K.$$

Next we consider when $S^*_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}$ belongs to S. This characterization is slightly more complicated.

Proposition 3.8. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Then $S^*_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1} \in S$ if and only if $\alpha_1(\overline{\alpha_2} - \overline{\beta_2}), \overline{\beta_1}(\alpha_2 - \beta_2) \in H^\infty$. In this case $S^*_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1} = S_{\overline{\alpha_2}\alpha_1, \overline{\beta_2}\beta_1}$.*

Proof. By Proposition 2.1, $S^*_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1} \in S$ if and only if

$$[S^*_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}, M_z] = \psi \otimes e_{-1}$$

for some $\psi \in L^\infty$. By Lemma 2.4,

$$\begin{aligned}[S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}, M_z] &= e_0 \otimes S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2) + S^*_{\alpha_2,\beta_2}(\alpha_1-\beta_1)\otimes e_{-1}\\ &= \psi \otimes e_{-1}.\end{aligned}$$

Therefore there exists a complex number λ such that

$$\begin{aligned}S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2) &= \overline{\lambda}e_{-1}\\ \lambda e_0 + S^*_{\alpha_2,\beta_2}(\alpha_1-\beta_1) &= \psi.\end{aligned}$$

It follows that

$$P\left[\overline{\alpha_1}\overline{z}(\alpha_2-\beta_2)\right] + Q\left[\overline{\beta_1}\overline{z}(\alpha_2-\beta_2)\right] = \overline{\lambda}\overline{z} \tag{3.5}$$

$$\lambda + P\left[\overline{\alpha_2}(\alpha_1-\beta_1)\right] + Q\left[\overline{\beta_2}(\alpha_1-\beta_1)\right] = \psi.$$

Therefore $\overline{\alpha_1}(\alpha_2-\beta_2) = \overline{h_1}$ and $\overline{\beta_1}(\alpha_2-\beta_2) = \overline{\lambda} + zh_2$ for some $h_1, h_2 \in H^\infty$. Now

$$\begin{aligned}S^*_{\alpha_2,\beta}S_{\alpha_1,\beta_1}f &= P\left[\overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf\right] + Q\left[\overline{\beta_2}\alpha_1 Pf + \overline{\beta_2}\beta_1 Qf\right]\\ &= P\left[\alpha_1\left(\overline{\alpha_2}-\overline{\beta_2}\right)Pf + \beta_1\left(\overline{\alpha_2}-\overline{\beta_2}\right)Qf\right] + \overline{\beta_2}\alpha_1 Pf + \overline{\beta_2}\beta_1 Qf\\ &= \alpha_1\left(\overline{\alpha_2}-\overline{\beta_2}\right)Pf + \overline{\beta_2}\alpha_1 Pf + \overline{\beta_2}\beta_1 Qf\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\beta_2}\beta_1 Qf = S_{\overline{\alpha_2}\alpha_1,\overline{\beta_2}\beta_1}f.\end{aligned}$$

The proof is complete. □

Corollary 3.9. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Then*

(a) $S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} \in M$ *if and only if* $\alpha_1(\overline{\alpha_2}-\overline{\beta_2}), \overline{\beta_1}(\alpha_2-\beta_2) \in H^\infty$ *and* $\overline{\alpha_2}\alpha_1 = \overline{\beta_2}\beta_1$. *In this case* $S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = M_{\overline{\alpha_2}\alpha_1}$.

(b) *If* $S_{\alpha_1,\beta_1} \neq 0$ *and* $S_{\alpha_2,\beta_2} \neq 0$, *then* $S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = 0$ *if and only if and one of the following statement holds.*

   (i) $\alpha_1 \neq 0$, $\beta_2 \neq 0$, $\beta_1 = 0$, $\alpha_2 = 0$ *and* $\alpha_1\overline{\beta_2} \in H^\infty$.

   (ii) $\alpha_2 \neq 0$, $\beta_1 \neq 0$, $\alpha_1 = 0$, $\beta_2 = 0$ *and* $\overline{\beta_1}\alpha_2 \in H^\infty$.

Proof. Part (a) follows from the proof of Proposition 3.8. We now prove (b). If $S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = 0$, then $\overline{\alpha_2}\alpha_1 = \overline{\beta_2}\beta_1 = 0$. Since $\alpha_1\overline{\beta_2} \in H^\infty$, by Lemma 3.5, either $\alpha_1\overline{\beta_2} = 0$ or $\alpha_1\overline{\beta_2} \neq 0$ almost everywhere on $\mathbb{T}$. Thus if both α_1 and $\beta_2$are not zero functions, then $\overline{\alpha_2}\alpha_1 = \overline{\beta_2}\beta_1 = 0$ imply that both β_1 and α_2 are zero functions. This proves (i). The proof of (ii) is similar. □

Corollary 3.10. *Let $\alpha \in L^\infty$ and assume α is invertible in L^∞. Also let $h \in H^\infty$ and assume h is invertible in H^∞. Then $S_{\alpha,h\alpha}$ is left invertible with left inverse $S^*_{1/\overline{\alpha},1/(\overline{h\alpha})}$.*

By Proposition 3.12, $S^*_{1/\overline{\alpha},1/(\overline{h\alpha})}$ is in general not a right inverse of $S_{\alpha,h\alpha}$ unless h is a constant.

Corollary 3.11. [12] *The operator $S_{\alpha,\beta}$ is an isometry if and only if $|\alpha| = |\beta| = 1$ and $\alpha = \theta\beta$ for some inner function θ. The operator $S_{\alpha,\beta}$ is a unitary operator if and only if $|\alpha| = |\beta| = 1$ and $\alpha = \lambda\beta$ for some unimodular constant λ.*

Proof. The operator $S_{\alpha,\beta}$ is an isometry if and only if $S^*_{\alpha,\beta}S_{\alpha,\beta} = I$. By Corollary 3.9, $\overline{\alpha}\alpha = \overline{\beta}\beta = 1$. The function $\alpha(\overline{\alpha} - \overline{\beta}) = 1 - \alpha\overline{\beta} \in H^\infty$ implies that $\alpha\overline{\beta} \in H^\infty$. But $\left|\alpha\overline{\beta}\right| = 1$, so $\alpha\overline{\beta} = \theta \in H^\infty$ is an inner function. Thus $\alpha = \alpha\overline{\beta}\beta = \theta\beta$.

If $S_{\alpha,\beta} = S_{\theta f,\beta}$ is a unitary operator, then

$$\begin{aligned}\beta &= S_{\theta f,\beta}S^*_{\theta\beta,\beta}\beta = S_{\theta\beta,\beta}\left[P(\overline{\theta\beta}\beta) + Q(\overline{\beta}\beta)\right] \\ &= S_{\theta\beta,\beta}P(\overline{\theta}) = \theta\beta P(\overline{\theta}).\end{aligned}$$

Thus $\theta P(\overline{\theta}) = 1$ since $|\beta| = 1$. Therefore θ is a unimodular constant. □

It is natural to ask when $S_{\alpha,\beta}$ is a coisometry. We can answer this question by studying product $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}$.

Proposition 3.12. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S$. Assume $S_{\alpha_1,\beta_1} \neq 0$. Then $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} \in S$ if and only if one of the following four statements holds.*

(i) *$\alpha_1 = 0$ and β_2 is a constant.*

(ii) *$\beta_1 = 0$ and α_2 is a constant.*

(iii) *Both α_2 and β_2 are constants.*

(iv) *$\beta_1 = \overline{\lambda}\alpha_1, \alpha_2 = \lambda\beta_2 + \mu$ for two constants λ and μ.*

In all cases, $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}$.

Proof. By Proposition 2.1, $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} \in S$ if and only if

$$[S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}, M_z] = \psi \otimes e_{-1}$$

for some $\psi \in L^\infty$. By Lemma 2.4,

$$[S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}, M_z] = \alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2 = \psi \otimes e_{-1}. \tag{3.6}$$

There are two cases. Either α_2 and β_2 are linearly dependent or α_1 and β_1 are linearly dependent.

Assume α_2 and β_2 are linearly dependent. If $\beta_2 = 0$ and $\psi = 0$, then $\alpha_1 = 0$ (assume $\alpha_2 \neq 0$). This leads to statement (i) with $\beta_2 = 0$. If $\beta_2 = 0$ and $\psi \neq 0$, then by (3.6), α_2 is a constant. This leads to statement (iii) with $\beta_2 = 0$. If $\beta_2 \neq 0$, then

$$\alpha_2 = \lambda\beta_2 \tag{3.7}$$

for some constant λ. Now equation (3.6) becomes

$$\left(\overline{\lambda}\alpha_1 - \beta_1\right) \otimes \overline{z}\beta_2 = \psi \otimes e_{-1}.$$

If $\psi = 0$, then $\overline{\lambda}\alpha_1 - \beta_1 = 0$. This leads to statement (iv) with $\mu = 0$. If $\psi \neq 0$, $\overline{z}\beta_2 = \mu e_{-1}$ for some constant μ, therefore β_2 is a constant. This leads to statement (iii).

Assume now α_1 and β_1 are linearly dependent but α_2 and β_2 are linearly independent. If $\beta_1 = 0$, then (3.6) implies that α_2 is a constant (assume $\alpha_1 \neq 0$). This leads to statement (ii). If $\beta_1 \neq 0$, then

$$\alpha_1 = \lambda\beta_1. \tag{3.8}$$

Now equation (3.6) becomes

$$\beta_1 \otimes \overline{z}(\overline{\lambda}\alpha_2 - \beta_2) = \psi \otimes e_{-1}.$$

If $\psi = 0$, then $\overline{\lambda}\alpha_2 - \beta_2 = 0$, which is impossible. If $\psi \neq 0$, there exists a constant $\mu \neq 0$ such that

$$(\overline{\lambda}\alpha_2 - \beta_2) = -\mu, \quad \overline{\mu}\beta_1 = \psi. \tag{3.9}$$

If $\lambda = 0$, then by (3.8) and (3.9), $\alpha_1 = 0$ and β_2 is a constant. This leads to statement (i). If $\lambda \neq 0$, this leads to statement (iv). In this case, by (3.8) and (3.9),

$$\begin{aligned}
S_{\alpha_1,\beta_1} S^*_{\alpha_2,\beta_2} f &= S_{\alpha_1,\beta_1}\left[P\left[\overline{\alpha_2} f\right] + Q\left[\overline{\beta_2} f\right]\right] \\
&= \alpha_1 P\left[\overline{\alpha_2} f\right] + \beta_1 Q\left[\overline{\beta_2} f\right] \\
&= \lambda\beta_1 P\left[\overline{\alpha_2} f\right] + \beta_1 Q\left[\left(\lambda\overline{\alpha_2} + \overline{\mu}\right) f\right] \\
&= \lambda\beta_1\overline{\alpha_2} f + \beta_1\overline{\mu} Q\left[f\right] \\
&= S_{\lambda\beta_1\overline{\alpha_2},\lambda\beta_1\overline{\alpha_2}+\overline{\mu}\beta_1} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}.
\end{aligned}$$

The proof is complete. □

Corollary 3.13. *Let $\beta \in L^\infty$. Assume β is invertible in L^∞ and λ is a nonzero constant. Then $S_{\lambda\beta,\beta}$ is right invertible with right inverse $S^*_{1/\overline{\beta\lambda},1/\overline{\beta}}$.*

In fact $S^*_{1/\overline{\beta\lambda},1/\overline{\beta}}$ is an inverse of $S_{\lambda\beta,\beta}$, see Corollary 3.10.

Corollary 3.11 shows that $S_{\alpha,\beta}$ can be an isometry but not a unitary operator. Surprisingly we show that $S_{\alpha,\beta}$ is a coisometry implies that $S_{\alpha,\beta}$ is in fact a unitary operator.

Corollary 3.14. *The operator $S_{\alpha,\beta}$ is a coisometry if and only if $S_{\alpha,\beta}$ is a unitary operator.*

Proof. The operator $S_{\alpha,\beta}$ is a coisometry if and only if $S_{\alpha,\beta}S^*_{\alpha,\beta} = I$. By Proposition 3.12, $S_{\alpha,\beta}S^*_{\alpha,\beta} = S_{\alpha\overline{\alpha},\beta\overline{\beta}} = I$, thus $|\alpha| = |\beta| = 1$. If statement (iii) in Proposition 3.12 holds, then $S_{\alpha,\beta}$ is a multiple of the identity, so $S_{\alpha,\beta}$ is a unitary operator. If statement (iv) in Proposition 3.12 holds, then $\alpha = \lambda\beta$ for some unimodular constant σ and $S_{\alpha,\beta}$ is a unitary operator by Corollary 3.11. □

4. Commuting singular integral operators

In this section we discuss when two operators from S commute. In contrast with the set M, where any two operators from M always commute, we prove that two operators from S commute if and only if one is essentially a scalar multiple of the other operator. Recall that the algebra K is defined by

$$K = \left\{ S_{\alpha,\beta} \mid \alpha \in H^\infty, \beta \in \overline{H^\infty} \right\}.$$

If either S_{α_1,β_1} or S_{α_2,β_2} is a scalar multiple of identity operator, then $S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$. Thus in the next theorem, we assume both S_{α_1,β_1} and S_{α_2,β_2} are not scalar multiples of the identity operator.

Theorem 4.1. *If $S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$, then one of the following statements holds.*

(i) *Both S_{α_1,β_1} and S_{α_2,β_2} are in M.*

(ii) *Both S_{α_1,β_1} and S_{α_2,β_2} are in K.*

(iii) *$S_{\alpha_1,\beta_1} = \lambda S_{\alpha_2,\beta_2} + \mu I$ for some constants λ and μ.*

Proof. Assume (i) does not hold. We will prove either (ii) or (iii) holds. By Lemma 2.4, $S_{\alpha_1,\beta_1}S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$ implies that

$$\begin{aligned} &(\alpha_1 - \beta_1) \otimes S^*_{\alpha_2,\beta_2}e_{-1} + S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2) \otimes e_{-1} \\ &= (\alpha_2 - \beta_2) \otimes S^*_{\alpha_1,\beta_1}e_{-1} + S_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) \otimes e_{-1}. \end{aligned} \tag{4.1}$$

There are three cases.

Case A. Either $\alpha_1 - \beta_1 = 0$ or $\alpha_2 - \beta_2 = 0$. Without loss of generality, assume $\alpha_1 - \beta_1 = 0$ and $\alpha_2 - \beta_2 \neq 0$. Then by (4.1),

$$S^*_{\alpha_1,\beta_1}e_{-1} = \sigma e_{-1}$$

for some constant σ. This implies that $\alpha_1 = \beta_1 = \overline{\sigma}$, so $S_{\alpha_1,\beta_1} = \overline{\sigma}I$.

Case B. $\alpha_1 - \beta_1 \neq 0$, $\alpha_2 - \beta_2 \neq 0$ and $(\alpha_1 - \beta_1) \neq \lambda(\alpha_2 - \beta_2)$ for any constant λ. Then by (4.1),

$$S^*_{\alpha_2,\beta_2} e_{-1} = \mu e_{-1}, \quad S^*_{\alpha_1,\beta_1} e_{-1} = \sigma e_{-1} \tag{4.2}$$

for some constants μ and σ. Thus

$$\overline{\mu}(\alpha_1 - \beta_1) + S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2) = \overline{\sigma}(\alpha_2 - \beta_2) + S_{\alpha_2,\beta_2}(\alpha_1 - \beta_1).$$

Equation (4.2) is the same as (3.2). Thus equation (4.2) implies that $\alpha_2 \in H^\infty$, $\beta_2 \in \overline{H^\infty}$ and $\alpha_1 \in H^\infty$, $\beta_1 \in \overline{H^\infty}$. Therefore both S_{α_1,β_1} and S_{α_2,β_2} are in K and $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1} = S_{\alpha_1\alpha_2,\beta_1\beta_2}$ This leads to (ii).

Case C. $\alpha_1 - \beta_1 \neq 0$, $\alpha_2 - \beta_2 \neq 0$ and $(\alpha_1 - \beta_1) = \lambda(\alpha_2 - \beta_2)$ for some constant λ. Note that $\lambda\alpha_2 - \alpha_1 = \lambda\beta_2 - \beta_1$. Then (4.1) becomes

$$\begin{aligned}
&(\alpha_2 - \beta_2) \otimes \left(\overline{\lambda} S^*_{\alpha_2,\beta_2} e_{-1} - S^*_{\alpha_1,\beta_1} e_{-1}\right) \\
&= (S_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) - S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2)) \otimes e_{-1} \\
&= (S_{\alpha_2,\beta_2}\lambda(\alpha_2 - \beta_2) - S_{\alpha_1,\beta_1}(\alpha_2 - \beta_2)) \otimes e_{-1} \\
&= S_{\lambda\alpha_2-\alpha_1,\lambda\beta_2-\beta_1}(\alpha_2 - \beta_2) \otimes e_{-1} \\
&= [(\lambda\alpha_2 - \alpha_1)(\alpha_2 - \beta_2)] \otimes e_{-1}.
\end{aligned}$$

Therefore, there exists a constant μ such that

$$\overline{\lambda} S^*_{\alpha_2,\beta_2} e_{-1} - S^*_{\alpha_1,\beta_1} e_{-1} = \overline{\mu} e_{-1} \tag{4.3}$$

$$(\lambda\alpha_2 - \alpha_1)(\alpha_2 - \beta_2) = \mu(\alpha_2 - \beta_2). \tag{4.4}$$

It follows from (4.3) that

$$\left(P\left[\overline{\lambda}\overline{\alpha_2}z - \overline{\alpha_1}z\right] + Q\left[\lambda\overline{\beta_2}\overline{z} - \overline{\beta_1}\overline{z}\right]\right) = \overline{\mu z}.$$

Hence $\overline{\lambda}\overline{\alpha_2} - \overline{\alpha_1} = \overline{h_1}$, $\overline{\lambda\beta_2} - \beta_1 = \overline{\mu} + zh_2$ for some $h_1, h_2 \in H^\infty$. Equation (4.4) becomes

$$(\lambda\alpha_2 - \alpha_1 - \mu)(\alpha_2 - \beta_2) = 0.$$

But $\lambda\alpha_2 - \alpha_1 - \mu = h_1 - \mu \in H^\infty$. By Lemma 3.5, either $\alpha_2 - \beta_2 = 0$ or $\lambda\alpha_2 - \alpha_1 - \mu = 0$. In the case $\alpha_2 - \beta_2 = 0$, we have $(\alpha_1 - \beta_1) = \lambda(\alpha_2 - \beta_2) = 0$, so both S_{α_1,β_1} and S_{α_2,β_2} are in M which is excluded at the beginning of the proof. Therefore $\lambda\alpha_2 - \alpha_1 - \mu = 0$. Furthermore $\lambda\beta_2 - \beta_1 = \lambda\alpha_2 - \alpha_1 = \mu$. That is, $\alpha_1 = \lambda\alpha_2 - \mu$ and $\beta_1 = \lambda\beta_2 - \mu$. In conclusion $S_{\alpha_1,\beta_1} = \lambda S_{\alpha_2,\beta_2} - \mu I$. □

Remark 4.2. If S_{α_1,β_1} (or S_{α_2,β_2}) is in K but not in M, then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}$ implies that S_{α_2,β_2} is also in K. If S_{α_1,β_1} is in M, then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}$ implies that either S_{α_2,β_2} is also in M or $S_{\alpha_1,\beta_1} = cI$ for some constant c.

5. Commuting singular integral operators and their adjoints

In this section we study when $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$. As a corollary, we recover the result in [12] about when S_{α_1,β_1} is a normal operator. The proof of Theorem 5.6 is elementary by using Lemma 2.4, but it is quite long and somewhat complicated. We divide the proof into several lemmas. If either S_{α_1,β_1} or S_{α_2,β_2} is a scalar multiple of identity operator, then $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$. Thus in this section, we assume both S_{α_1,β_1} and S_{α_2,β_2} are not scalar multiples of the identity operator.

Lemma 5.1. *If $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$, then*

$$\alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2 = e_0 \otimes S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2) + S^*_{\alpha_2,\beta_2}(\alpha_1-\beta_1)\otimes e_{-1}. \tag{5.1}$$

Proof. Note that $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$ implies that

$$\left[S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}, M_z\right] = \left[S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}, M_z\right].$$

By Lemma 2.4,

$$\alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2 = e_0 \otimes S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2) + S^*_{\alpha_2,\beta_2}(\alpha_1-\beta_1)\otimes e_{-1}.$$

The proof is complete. □

Lemma 5.2. *If both sides of* (5.1) *are zero operators, then one of the following statements holds.*

(i) *Both S_{α_1,β_1} and S_{α_2,β_2} are in M.*

(ii) *$\alpha_2 = \beta_1 = 0$ and $\alpha_1\overline{\beta_2} \in H^\infty$. In this case $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = 0$.*

(iii) *$\alpha_1 = \beta_2 = 0$ and $\overline{\beta_1}\alpha_2 \in H^\infty$. In this case $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = 0$.*

(iv) *There exist some constants λ ($\lambda \neq 0,1$) and μ such that*

$$\alpha_1 = \lambda\beta_1, \quad \beta_2 = \overline{\lambda}\alpha_2, \quad \overline{\alpha_1}\alpha_2 = \mu.$$

In this case $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = \overline{\mu}I$.

Proof. If both sides of (5.1) are zero operators and $\beta_1 = 0$, then $\alpha_1 \otimes \overline{z}\alpha_2 = 0$ implies that $\alpha_2 = 0$. If the right side of (5.1) is a zero operator, then

$$S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2) = \eta e_{-1}, \quad S^*_{\alpha_2,\beta_2}(\alpha_1-\beta_1) = -\overline{\eta}e_0$$

for some constant η. Therefore

$$S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2) = P\left[\overline{\alpha_1}\overline{z}(0-\beta_2)\right] = \eta\overline{z},$$

which implies that $\overline{\alpha_1}\beta_2 \in \overline{H^\infty}$. This leads to (ii). If both sides of (5.1) are zero operators and $\beta_1 \neq 0$, then

$$\alpha_1 = \lambda\beta_1, \quad \beta_2 = \overline{\lambda}\alpha_2,$$
$$S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2) = \eta e_{-1}, \quad S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) = -\overline{\eta}e_0$$

for some constants λ and η.

If $\lambda = 0$, then $\alpha_1 = \beta_2 = 0$ and $\overline{\beta_1}\alpha_2 \in H^\infty$. This leads to statement (iii).

If $\lambda = 1$, then both S_{α_1,β_1} and S_{α_2,β_2} are in M. This leads to statement (i).

If $\lambda \neq 0$ and $\lambda \neq 1$, then

$$P\left[\overline{\alpha_1 z}(1-\overline{\lambda})\alpha_2\right] + Q\left[\overline{\beta_1}\overline{z}(1-\overline{\lambda})\alpha_2\right] = \eta\overline{z},$$

which implies that $\overline{\alpha_1}\alpha_2 = \overline{\lambda\beta_1}\alpha_2 \in \overline{H^\infty}$, $\overline{\beta_1}\alpha_2 \in H^\infty$. Hence $\overline{\alpha_1}\alpha_2$ is a constant. In this case it is easy to verify that $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = \overline{\mu}I$. This leads to statement (iv). □

Lemma 5.3. *If both sides of* (5.1) *are rank one operators, then one of the following statements holds.*

(1) α_2 *and* β_2 *are constants. In this case* $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}$.

(2) α_1 *and* β_1 *are constants. In this case* $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}$.

(3a) $\alpha_1 = 0$, β_2 *is a constant, and* $\overline{\beta_1}(\alpha_2 - \beta_2) \in H^\infty$.

(3b) $\alpha_1 = \lambda\beta_1$, $\overline{\lambda}\alpha_2 - \beta_2 = \delta$ *for some constants* $\lambda \neq 0$ *and* δ, *and* $\overline{\beta_1}(\alpha_2 - \beta_2)$ *is a constant. In cases* (3a) *and* (3b), $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}$.

(4a) $\beta_2 = 0$, α_1 *is a constant, and* $\alpha_2\left(\overline{\beta_1} - \overline{\alpha_1}\right) \in H^\infty$.

(4b) $\beta_2 = \eta\alpha_2$, $\alpha_1 - \overline{\eta}\beta_1 = \delta$ *for some constants* $\eta \neq 0$ *and* δ, *and* $\overline{\beta_2}(\alpha_1 - \beta_2)$ *is a constant. In cases* (4a) *and* (4b), $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S^*_{\overline{\alpha_1}\alpha_2,\overline{\beta_1}\beta_2}$.

(5a) $\alpha_2 = 0$, β_1 *is a constant, and* $\overline{\beta_2}(\alpha_1 - \beta_1) \in H^\infty$. *In this case* $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S^*_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}$.

(5b) $\beta_1 = 0$, α_2 *is a constant, and* $\alpha_1(\overline{\alpha_2} - \overline{\beta_2}) \in H^\infty$. *In this case* $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}$.

Proof. We first assume $\alpha_2 \neq 0$ and $\beta_1 \neq 0$. If the left side of (5.1) is of rank one, then there exist constants η and λ such that either

$$\beta_2 = \eta\alpha_2 \tag{5.2}$$

or

$$\alpha_1 = \lambda\beta_1. \tag{5.3}$$

If the right side of (5.1) is of rank one, then there exist constants σ and μ such that either

$$S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2) = \sigma e_{-1} \tag{5.4}$$

or

$$S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) = \mu e_0. \tag{5.5}$$

There are four cases. We consider them one by one.

Case A. (5.2) and (5.4) hold. Plugging (5.2) and (5.4) into (5.1), we have

$$(\alpha_1 - \overline{\eta}\beta_1) \otimes \overline{z}\alpha_2 = \left[\overline{\sigma}e_0 + S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1)\right] \otimes e_{-1}.$$

Therefore α_2 and β_2 are constants. This leads to statement (1).

Case B. (5.3) and (5.5) hold. Plugging (5.3) and (5.5) into (5.1), we have

$$\beta_1 \otimes \overline{z}\left(\overline{\lambda}\alpha_2 - \beta_2\right) = e_0 \otimes \left[S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2) + \overline{\mu}e_{-1}\right].$$

Therefore α_1 and β_1 are constants. This leads to statement (2).

Case C. (5.3) and (5.4) hold. Plugging (5.3) and (5.4) into (5.1), we have

$$\beta_1 \otimes \overline{z}\left(\overline{\lambda}\alpha_2 - \beta_2\right) = \left[\overline{\sigma}e_0 + S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1)\right] \otimes e_{-1}.$$

Therefore $\overline{\lambda}\alpha_2 - \beta_2$ is a constant. In this case, by Part (iv) of Proposition 3.12,

$$S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S_{\alpha_1\overline{\alpha_2},\beta_1\overline{\beta_2}}.$$

Therefore $S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} \in S$. By Proposition 3.8, $\alpha_1(\overline{\alpha_2} - \overline{\beta_2}), \overline{\beta_1}(\alpha_2 - \beta_2) \in H^\infty$. If $\lambda = 0$, then we have (3a). If $\lambda \neq 0$, then $\lambda\beta_1(\overline{\alpha_2} - \overline{\beta_2}), \overline{\beta_1}(\alpha_2 - \beta_2) \in H^\infty$ implies that $\overline{\beta_1}(\alpha_2 - \beta_2)$ is a constant. This is (3b).

Case D. (5.2) and (5.5) hold. Plugging (5.2) and (5.5) into (5.1), we have

$$(\alpha_1 - \overline{\eta}\beta_1) \otimes \overline{z}\alpha_2 = e_0 \otimes \left[S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2) + \overline{\mu}e_{-1}\right]. \tag{5.6}$$

Therefore $\alpha_1 - \overline{\eta}\beta_1 = \delta$ for some constant δ. Case D in some sense is dual to Case C by considering $\left(S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}\right)^* = \left(S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}\right)^*$. Here we give a direct proof which demonstrates that Case C can also be proved directly without using Propositions 3.12 and 3.8. By $\alpha_1 - \overline{\eta}\beta_1 = \delta$ and (5.6),

$$S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2) + \overline{\mu}e_{-1} = \overline{\delta}\overline{z}\alpha_2.$$

Therefore

$$P\left[\overline{\alpha_1 z}(\alpha_2-\beta_2)-\overline{\delta}\overline{z}\alpha_2\right]+Q\left[\overline{\beta_1}\overline{z}(\alpha_2-\beta_2)-\overline{\delta}\overline{z}\alpha_2\right]=-\overline{\mu}e_{-1}.$$

Hence

$$\overline{\alpha_1}(\alpha_2-\beta_2)-\overline{\delta}\alpha_2\in\overline{H^\infty},\quad \overline{\beta_1}(\alpha_2-\beta_2)-\overline{\delta}\alpha_2\in H^\infty.$$

Using $\beta_2=\eta\alpha_2$, $\alpha_1-\overline{\eta}\beta_1=\delta$, we have

$$\begin{aligned}\overline{\alpha_1}(\alpha_2-\beta_2)-\overline{\delta}\alpha_2&=\eta\alpha_2\left(\overline{\beta_1}-\overline{\alpha_1}\right)\in\overline{H^\infty},\\ \overline{\beta_1}(\alpha_2-\beta_2)-\overline{\delta}\alpha_2&=\alpha_2\left(\overline{\beta_1}-\overline{\alpha_1}\right)\in H^\infty.\end{aligned}$$

If $\eta=0$, we have (4a). If $\eta\neq 0$, we have (4b). In this case we can verify that

$$\left(S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}\right)^*=\left(S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}\right)^*=S_{\overline{\alpha_1}\alpha_2,\overline{\beta_1}\beta_2}.$$

We now deal with the situation $\alpha_2=0$ or $\beta_1=0$. The proof is similar, so we will be brief. Since the right side of (5.1) is of rank one, there exist constants σ and μ such that either

$$S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2-\beta_2)=\sigma e_{-1} \tag{5.7}$$

or

$$S^*_{\alpha_2,\beta_2}(\alpha_1-\beta_1)=\mu e_0. \tag{5.8}$$

There are four cases. When $\alpha_2=0$ and (5.7) holds, equation (5.1) implies that β_2 is a constant. This leads to statement (1). When $\alpha_2=0$ and (5.8) holds, equation (5.1) implies that β_1 is a constant and $\alpha_1(\overline{\alpha_2}-\overline{\beta_2})\in H^\infty$. This leads to statement (5a). When $\beta_1=0$ and (5.7) holds, equation (5.1) implies that α_2 is a constant and $\alpha_1(\overline{\alpha_2}-\overline{\beta_2})\in H^\infty$. This leads to statement (5b). When $\beta_1=0$ and (5.8) holds, equation (5.1) implies that α_1 is a constant. This leads to statement (2). The proof is complete. □

Lemma 5.4. *If both sides of* (5.1) *are rank two operators, then one of the following statements holds.*

(i) β_1 *and* α_2 *are constants, and* $\overline{\beta_2}(\alpha_1-\beta_1)-\overline{\alpha_2}\alpha_1\in H^\infty$ *(equivalently* $\overline{\alpha_1}(\alpha_2-\beta_2)+\overline{\beta}_1\beta_2\in H^\infty$*).*

(ii) α_1 *and* β_2 *are constants, and* $\overline{\beta_1}(\alpha_2-\beta_2)-\overline{\alpha_1}\alpha_2\in H^\infty$ *(equivalently* $\alpha_2(\overline{\alpha_1}-\overline{\beta_1})+\overline{\beta}_2\beta_1\in H^\infty$*).*

(iii) *There exist some constants* $\lambda\neq 0$, δ_1 *and* δ_2 *such that*

$$\alpha_1=\lambda\beta_1+\delta_1,\quad \beta_2=\overline{\lambda}\alpha_2+\delta_2$$

and $\overline{\beta_1}(\alpha_2-\beta_2)-\overline{\delta_1}\alpha_2$ *is a constant (equivalently* $\alpha_2(\overline{\alpha_1}-\overline{\beta_1})+\delta_2\overline{\beta_1}$ *is a constant).*

In all cases $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$.

Proof. Recall (5.1),

$$\alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2 = e_0 \otimes f + g \otimes e_{-1} \tag{5.9}$$

where

$$f = S^*_{\alpha_1,\beta_1}\overline{z}(\alpha_2 - \beta_2), \quad g = S^*_{\alpha_2,\beta_2}(\alpha_1 - \beta_1).$$

If both sides of (5.9) are rank two operators, then there exist constants λ_1, λ_2, η_1, η_2 such that

$$\begin{bmatrix} e_0 \\ g \end{bmatrix} = \begin{bmatrix} \eta_1 & \eta_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}. \tag{5.10}$$

Plugging the above equation into left side of (5.9), we have

$$\alpha_1 \otimes \overline{z}\alpha_2 - \beta_1 \otimes \overline{z}\beta_2 = \alpha_1 \otimes (\overline{\eta_1}f + \overline{\lambda_1}e_{-1}) + \beta_2 \otimes (\overline{\eta_2}f + \overline{\lambda_2}e_{-1}).$$

Thus

$$\begin{bmatrix} \overline{\eta_1} & \overline{\lambda_1} \\ \overline{\eta_2} & \overline{\lambda_2} \end{bmatrix} \begin{bmatrix} f \\ e_{-1} \end{bmatrix} = \begin{bmatrix} \overline{z}\alpha_2 \\ -\overline{z}\beta_2 \end{bmatrix},$$

$$\Omega \begin{bmatrix} \overline{\lambda_2} & -\overline{\lambda_1} \\ -\overline{\eta_2} & \overline{\eta_1} \end{bmatrix} \begin{bmatrix} \overline{z}\alpha_2 \\ -\overline{z}\beta_2 \end{bmatrix} = \begin{bmatrix} f \\ e_{-1} \end{bmatrix} \tag{5.11}$$

where $\Omega = 1/\left(\overline{\eta_1}\overline{\lambda_2} - \overline{\eta_2}\overline{\lambda_1}\right)$. Equations (5.10) and (5.11) imply that

$$e_0 = \eta_1\alpha_1 + \eta_2\beta_1 \tag{5.12}$$

$$e_{-1} = -\Omega\overline{z}\left(\overline{\eta_2}\alpha_2 + \overline{\eta_1}\beta_2\right). \tag{5.13}$$

If $\eta_1 = 0$, then β_1 and α_2 are constants. Equation (5.9) becomes

$$(\alpha_1\overline{\alpha_2} - g) \otimes e_{-1} = e_0 \otimes (f + \overline{z}\beta_2\overline{\beta_1}).$$

Therefore

$$\alpha_1\overline{\alpha_2} - g = \mu e_0, \quad f + \overline{z}\beta_2\overline{\beta_1} = \overline{\mu}e_{-1}$$

for some constant μ. It follows that

$$\alpha_1\overline{\alpha_2} - P\left[\overline{\alpha_2}(\alpha_1 - \beta_1)\right] + Q\left[\overline{\beta_2}(\alpha_1 - \beta_1)\right] = \mu e_0,$$
$$P\left[\overline{\alpha_1}\overline{z}(\alpha_2 - \beta_2)\right] + Q\left[\overline{\beta_1}\overline{z}(\alpha_2 - \beta_2)\right] + \overline{z}\beta_2\overline{\beta_1} = \overline{\mu}e_{-1}.$$

Therefore

$$Q\left[\overline{\beta_2}(\alpha_1-\beta_1)-\alpha_1\overline{\alpha_2}\right]=\mu e_0-\overline{\alpha_2}\beta_1 e_0,$$
$$P\left[\overline{\alpha_1 z}(\alpha_2-\beta_2)+\overline{z}\beta_2\overline{\beta_1}\right]=\overline{\mu}e_{-1}-\overline{\beta_1}\alpha_2 e_{-1}.$$

So, $\overline{\beta_2}(\alpha_1-\beta_1)-\alpha_1\overline{\alpha_2}\in H^\infty$ (equivalently $\overline{\alpha_1}(\alpha_2-\beta_2)+\beta_2\overline{\beta_1}\in\overline{H^\infty}$). This leads to statement (i).

If $\eta_1\neq 0$, we rewrite the (5.12) and (5.13) as

$$\alpha_1=\lambda\beta_1+\delta_1,\quad \beta_2=\overline{\lambda}\alpha_2+\delta_2 \tag{5.14}$$

for some constants $\lambda\neq 0$, δ_1, δ_2. Now plugging these two relations into (5.10), we have

$$\delta_1\otimes\overline{z}\alpha_2-\beta_1\otimes\overline{z}\delta_2=e_0\otimes f+g\otimes e_{-1},$$
$$e_0\otimes\left(f-\overline{\delta_1}\overline{z}\alpha_2\right)+\left(g+\overline{\delta_2}\beta_1\right)\otimes e_{-1}=0.$$

Therefore there exists a constant μ such that

$$\left(f-\overline{\delta_1}\overline{z}\alpha_2\right)=\mu e_{-1}$$
$$g+\overline{\delta_2}\beta_1=-\overline{\mu}.$$

It follows that

$$P\left[\overline{\alpha_1 z}(\alpha_2-\beta_2)\right]+Q\left[\overline{\beta_1}\overline{z}(\alpha_2-\beta_2)\right]=\overline{\mu}e_{-1}+\overline{\delta}_1\overline{z}\alpha_2$$
$$P\left[\overline{\alpha_2}(\alpha_1-\beta_1)\right]+Q\left[\overline{\beta_2}(\alpha_1-\beta_1)\right]=-\overline{\mu}-\overline{\delta}_2\beta_1.$$

Equivalently

$$P\left[\overline{\alpha_1 z}(\alpha_2-\beta_2)-\overline{\delta}_1\overline{z}\alpha_2\right]+Q\left[\overline{\beta_1}\overline{z}(\alpha_2-\beta_2)-\overline{\delta}_1\overline{z}\alpha_2\right]=\overline{\mu}e_{-1}$$
$$P\left[\overline{\alpha_2}(\alpha_1-\beta_1)+\overline{\delta}_2\beta_1\right]+Q\left[\overline{\beta_2}(\alpha_1-\beta_1)+\overline{\delta}_2\beta_1\right]=-\overline{\mu}.$$

Therefore

$$\overline{h_1}:=\overline{\alpha_1}(\alpha_2-\beta_2)-\overline{\delta}_1\alpha_2\in\overline{H^\infty}$$
$$h_2:=\overline{\beta_1}(\alpha_2-\beta_2)-\overline{\delta}_1\alpha_2\in H^\infty$$
$$\overline{h_3}:=\overline{\alpha_2}(\alpha_1-\beta_1)+\overline{\delta}_2\beta_1\in\overline{H^\infty}$$
$$h_4:=\overline{\beta_2}(\alpha_1-\beta_1)+\overline{\delta}_2\beta_1\in H^\infty.$$

By using (5.14) and a direct computation

$$h_1=-h_4,\quad h_2=-h_3,\quad \overline{h_1}=\overline{\lambda}h_2 \tag{5.15}$$

$$h_2 = (1-\overline{\lambda})\overline{\beta_1}\alpha_2 - \delta_2\overline{\beta_1} - \overline{\delta}_1\alpha_2.$$

Therefore, if $\lambda \neq 0$, then h_1, h_2, h_3, h_4 are all constants (equivalently one of them is a constant). This leads to statement (iii). If $\lambda = 0$, then α_1 and β_2 are constants and $h_1 = -h_4 = 0$ and $h_2, h_3 \in H^\infty$. This leads to statement (ii). In both cases, note that

$$\begin{aligned}\left(\overline{\beta_2} - \overline{\alpha_2}\right)\alpha_1 &= -\left(h_1 + \delta_1\overline{\alpha_2}\right)\\ \left(\overline{\beta_2} - \overline{\alpha_2}\right)\beta_1 &= \left(\overline{h_2} + \delta_1\overline{\alpha_2}\right)\end{aligned}$$

where $h_1, \overline{h_2} \in H^\infty$. We have, for any $f \in L^2$,

$$\begin{aligned}S^*_{\alpha_2,\beta}S_{\alpha_1,\beta_1}f &= P\left[\overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf\right] + Q\left[\overline{\beta_2}\alpha_1 Pf + \overline{\beta_2}\beta_1 Qf\right]\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf + Q\left[\left(\overline{\beta_2} - \overline{\alpha_2}\right)\alpha_1 Pf + \left(\overline{\beta_2} - \overline{\alpha_2}\right)\beta_1 Qf\right]\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf + Q\left[-\left(h_1 + \delta_1\overline{\alpha_2}\right)Pf - \left(\overline{h_2} + \delta_1\overline{\alpha_2}\right)Qf\right]\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf - Q\left[\delta_1\overline{\alpha_2}Pf\right] - \overline{h_2}Qf - Q\left[\delta_1\overline{\alpha_2}Qf\right]\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf - Q\left[\delta_1\overline{\alpha_2}f\right] - \overline{h_2}Qf\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\alpha_2}\beta_1 Qf - Q\left[\delta_1\overline{\alpha_2}f\right] + \left[\left(\overline{\beta_2} - \overline{\alpha_2}\right)\beta_1 + \delta_1\overline{\alpha_2}\right]Qf\\ &= \overline{\alpha_2}\alpha_1 Pf + \overline{\beta_2}\beta_1 Qf - \delta_1 Q\left[\overline{\alpha_2}f\right] + \delta_1\overline{\alpha_2}Qf\\ &= \overline{\alpha_2}(\lambda\beta_1 + \delta_1)Pf + (\lambda\overline{\alpha_2} + \overline{\delta_2})\beta_1 Qf - \delta_1 Q\left[\overline{\alpha_2}f\right] + \delta_1\overline{\alpha_2}Qf\\ &= \overline{\alpha_2}\lambda\beta_1 f + \delta_1\overline{\alpha_2}f + \overline{\delta_2}\beta_1 Qf - \delta_1 Q\left[\overline{\alpha_2}f\right]\\ &= \alpha_1\overline{\alpha_2}f + \overline{\delta_2}\beta_1 Q[f] - \delta_1 Q\left[\overline{\alpha_2}f\right].\end{aligned} \tag{5.16}$$

Note also

$$\begin{aligned}S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}f &= \alpha_1 P\left[\overline{\alpha_2}f\right] + \beta_1 Q\left[\overline{\beta_2}f\right]\\ &= \alpha_1\overline{\alpha_2}f + \beta_1 Q\left[\overline{\beta_2}f\right] - \alpha_1 Q\left[\overline{\alpha_2}f\right]\\ &= \alpha_1\overline{\alpha_2}f + \beta_1 Q\left[\left(\lambda\overline{\alpha_2} + \overline{\delta_2}\right)f\right] - \left(\lambda\beta_1 + \delta_1\right)Q\left[\overline{\alpha_2}f\right]\\ &= \alpha_1\overline{\alpha_2}f + \overline{\delta_2}\beta_1 Q[f] - \delta_1 Q\left[\overline{\alpha_2}f\right].\end{aligned}$$

Therefore $S^*_{\alpha_2,\beta}S_{\alpha_1,\beta_1} = S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}$. □

Remark 5.5. The verification of $S^*_{\alpha_2,\beta}S_{\alpha_1,\beta_1} = S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}$ as in (5.16) is lengthy. Since

$$\left[S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}, M_z\right] = \left[S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}, M_z\right],$$

we have $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} - S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1} = M_\alpha$ for some $\alpha \in L^\infty$. We need only to verify $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2}e_0 - S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}e_0 = \alpha = 0$, which will be slightly shorter than the verification as in (5.16). However we choose the above more direct method.

Summarizing the above three lemmas, we have the following theorem.

Theorem 5.6. $S_{\alpha_1,\beta_1}S^*_{\alpha_2,\beta_2} = S^*_{\alpha_2,\beta_2}S_{\alpha_1,\beta_1}$ *if and only if one of the following six statements holds.*

(i) *Both* S_{α_1,β_1} *and* S_{α_2,β_2} *are in* M.

(ii) α_1 *and* β_1 *are constants.*

(iii) α_2 *and* β_2 *are constants.*

(iv) β_1 *and* α_2 *are constants, and* $\overline{\beta_2}(\alpha_1 - \beta_1) - \overline{\alpha_2}\alpha_1 \in H^\infty$.

(v) α_1 *and* β_2 *are constants, and* $\overline{\beta_1}(\alpha_2 - \beta_2) - \overline{\alpha_1}\alpha_2 \in H^\infty$.

(vi) $\alpha_1 = \lambda\beta_1 + \delta_1$, $\beta_2 = \overline{\lambda}\alpha_2 + \delta_2$ *for some constants* $\lambda \neq 0$, δ_1, δ_2, *and* $\overline{\beta_1}(\alpha_2 - \beta_2) - \overline{\delta_1}\alpha_2$ *is a constant.*

Proof. We need to explain the results from Lemmas 5.2, 5.3 and 5.4. Item (i) in Lemma 5.2 is statement (i) here. Items (ii), (iii) and (iv) in Lemma 5.2 are included in above statements (iv), (v) and (vi) respectively. Items (1) and (2) from Lemma 5.3 correspond to statements (i) and (ii). Item (3a) or (4a) from Lemma 5.3 corresponds to statement (v) with $\alpha_1 = 0$ or $\beta_2 = 0$. Item (3b) or (4b) from Lemma 5.3 corresponds to statement (vi) with $\delta_1 = 0$ or $\delta_2 = 0$. Item (5a) or (5b) from Lemma 5.3 corresponds to statement (iv) with $\alpha_2 = 0$ or $\beta_1 = 0$. Items (i), (ii) and (iii) from Lemma 5.4 correspond to statements (iv), (v) and (vi) respectively. The proof is complete. □

The statement (vi) with $\lambda = 0$ is included in the statement (v). The statement (vi) with $\lambda = 1$ and $\delta_1 = \delta_2 = 0$ reduces to statement (i).

Corollary 5.7. [12] *The operator* $S_{\alpha,\beta}$ *is normal if and only if one of the following two statement holds.*

(i) α, β *are constants.*

(ii) $\alpha = \lambda\beta + \delta$ *for some constants* δ *and* λ *with* $|\lambda| = 1$, *and* $(\lambda - 1)|\beta|^2 + \delta\overline{\beta} - \overline{\delta}\lambda\beta$ *is a constant.*

We can represent Condition (ii) slightly more explicitly by looking at the cases $\lambda = 1$ and $\lambda \neq 1$ separately. We refer to Theorems 3.1 and 3.2 in [12] for more details.

We state the above result as a corollary of Theorem 5.6. However we can prove this corollary directly by following the proof of Theorem 5.6. The direct proof of this corollary is considerably simpler than the proof of Theorem 5.6 (see Appendix). This direct proof also reveals the following insight.

Corollary 5.8. *The operator $S_{\alpha,\beta}$ is normal if and only if one of the following two statement holds.*

(i) $S_{\alpha,\beta} = M_\beta + S_{\delta,0}$ *where δ is a constant such that $\delta\overline{\beta} - \overline{\delta}\beta$ is a constant.*

(ii) $S_{\alpha,\beta} = \lambda S_{\alpha_1,\beta_1} + \mu I$ *for some constants λ and μ and unitary operator S_{α_1,β_1}.*

The operator $S_{\alpha,\beta}$ in (i) above is a normal operator in the set M_+.

6. Singular integral operators on L^p spaces

As seen in [4], the singular integral operator $S_{\alpha,\beta}$ is also defined on any L^p space. In this section we extend some results for $S_{\alpha,\beta}$ on L^2 to $S_{\alpha,\beta}$ on L^p. Since several of our problems are purely algebraic, appropriate interpretations of them on L^p spaces are not difficult. Nevertheless we include these interpretations to demonstrate the power of our techniques in this more general context and to provide a motivation of further studying singular integral operators on L^p space.

For $1 < p < \infty$, let L^p denote the usual Lebesgue space of the unit circle $\mathbb{T}$ and H^p denote the Hardy space of the unit disk. We can identify H^p as a closed subspace of L^p. Let P denote the projection of L^p onto H^p. By M. Riesz's Theorem (Theorem 2.3 on page 108 of [3]), P is a bounded operator (but P is not bounded for $p = 1$ and $p = \infty$). Let $\alpha, \beta \in L^\infty$ and $Q = I - P$. The operator $S_{\alpha,\beta}$ on L^p defined by

$$S_{\alpha,\beta} f = \alpha P f + \beta Q f, \quad f \in L^p$$

is bounded. The operator $S_{\alpha,\beta}$ in terms of Cauchy integral formula is

$$(S_{\alpha,\beta} f)(z) = \frac{\alpha(z) + \beta(z)}{2} f(z) + \frac{\alpha(z) - \beta(z)}{2} \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Let

$$S_p = \left\{ S_{\alpha,\beta} \in B(L^p) \mid \alpha, \beta \in L^\infty \right\}.$$

Let q be such that $1/p + 1/q = 1$ and L^q be the dual space of L^p. For $x \in L^p$ and $y \in L^q$, let $x \otimes y$ denote the rank one operator defined on L^p by $[x \otimes y]\, h = \langle h, y \rangle\, x$ for $h \in L^p$ where $\langle h, y \rangle$ is the duality pairing,

$$\langle h, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \overline{y(e^{i\theta})} \, d\theta.$$

Again $S^*_{\alpha,\beta} \in B(L^q)$ is defined by

$$S^*_{\alpha,\beta} g = P\left[\overline{\alpha} g\right] + Q\left[\overline{\beta} g\right], \quad g \in L^q.$$

This can be verified by a direct computation, for $f \in L^p$ and $g \in L^q$,

$$\begin{aligned}\langle S_{\alpha,\beta} f, g\rangle &= \langle \alpha P f + \beta Q f, g\rangle = \langle P f, \overline{\alpha} g\rangle + \langle Q f, \overline{\beta} g\rangle \\ &= \langle f, P[\overline{\alpha} g]\rangle + \langle f, Q[\overline{\beta} g]\rangle = \langle f, S^*_{\alpha,\beta} g\rangle .\end{aligned}$$

We can also view $S^*_{\alpha,\beta}$ as a bounded operator on L^p. To distinguish, we use a different notation. Let $T_{\alpha,\beta} \in B(L^p)$ be the operator defined by

$$T_{\alpha,\beta} f = P[\overline{\alpha} f] + Q[\overline{\beta} f], \quad g \in L^p.$$

Here is the analogue of Proposition 2.1.

Proposition 6.1. *Let $A \in B(L^p)$. Then $A \in S_p$ if and only if there exists a $\psi \in L^\infty$ such that*

$$[A, M_z] = \psi \otimes e_{-1}.$$

In this case $A = S_{\psi+\beta,\beta}$ for some $\beta \in L^\infty$.

Even though proofs of the following two results on L^p are similar to the corresponding proofs on L^2, we demonstrate these proofs carefully. The proofs are simple but they illustrate the needed adaptation. We will skip the more complicated proofs.

Lemma 6.2. *$M_{\overline{z}} T_{\alpha,\beta} - T_{\alpha,\beta} M_{\overline{z}} = e_{-1} \otimes (\alpha - \beta)$.*

Proof. For $f \in L^p$,

$$\begin{aligned}T_{\alpha,\beta} M_{\overline{z}} f &= P[\overline{z}\overline{\alpha} f] + Q[\overline{z}\overline{\beta} f] \\ &= \overline{z} P[\overline{\alpha} f] - \overline{z}(\overline{\alpha} f)_0 + \overline{z} Q[\overline{\beta} f] + \overline{z}(\overline{\beta} f)_0 \\ &= M_{\overline{z}} T_{\alpha,\beta} f + \langle (\overline{\beta} - \overline{\alpha}) f, 1\rangle e_{-1} \\ &= M_{\overline{z}} T_{\alpha,\beta} f - [e_{-1} \otimes (\alpha - \beta)] f\end{aligned}$$

where $(\overline{\alpha} f)_0$ denotes the constant term in the Fourier series of $\overline{\alpha} f$. □

Proposition 6.3. *The operator $T_{\alpha,\beta} \in S_p$ if and only if $(\alpha - \beta) = \lambda$ for some constant λ. In this case $T_{\alpha,\beta} = S_{\overline{\alpha},\overline{\beta}}$.*

Proof. By Proposition (6.1), $T_{\alpha,\beta} \in S$ if and only if

$$T_{\alpha,\beta} M_z - M_z T_{\alpha,\beta} = \psi \otimes e_{-1}$$

for some $\psi \in L^\infty$. But

$$\begin{aligned}T_{\alpha,\beta} M_z - M_z T_{\alpha,\beta} &= M_z (M_{\overline{z}} T_{\alpha,\beta} - T_{\alpha,\beta} M_{\overline{z}}) M_z \\ &= M_z [e_{-1} \otimes (\alpha - \beta)] M_z \\ &= e_0 \otimes \overline{z}(\alpha - \beta).\end{aligned}$$

Thus

$$\psi \otimes e_{-1} = e_0 \otimes \overline{z}(\alpha - \beta)$$

and $(\alpha - \beta)\overline{z} = \lambda e_{-1}$ and $\psi = \overline{\lambda} e_0$ for some complex number λ. In this case

$$T_{\alpha,\beta} = M_{\overline{\beta}} + S_{\overline{\lambda},0} = S_{\overline{\alpha},\overline{\beta}}.$$

The proof is complete. □

Corollary 6.4. *$T_{\alpha,\beta} = S_{\alpha,\beta}$ on L^p if and only if α and β are real valued functions and $(\alpha - \beta)$ is a real constant.*

Recall by definition the operator $S_{\alpha,\beta}$ on the Banach space L^p is self-adjoint if $\langle S_{\alpha,\beta} f, g\rangle$ is real for all $f \in L^p$ and $g \in L^q$. The condition $T_{\alpha,\beta} = S_{\alpha,\beta}$ is equivalent to $S_{\alpha,\beta}$ being self-adjoint.

Lemma 2.4 also holds on L^p. We state part of it.

Lemma 6.5. *Let $S_{\alpha_1,\beta_1}, T_{\alpha_2,\beta_2} \in B(L^p)$. Then*

$$[T_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}, M_z] = e_0 \otimes T_{\alpha_2,\beta_2}\overline{z}(\alpha_2 - \beta_2) + T_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) \otimes e_{-1}. \tag{6.1}$$

Proof. Since $\alpha_2, \beta_2 \in L^\infty$, by M. Riesz's Theorem, $T_{\alpha_2,\beta_2}\overline{z}(\alpha_2 - \beta_2) \in L^q$, $T_{\alpha_2,\beta_2}(\alpha_1 - \beta_1) \in L^p$. Thus the operator on the right side of (6.1) is a bounded operator on L^p. □

Next we select a few results and state them without proofs.

Theorem 6.6. *Let $S_{\alpha_1,\beta_1}, S_{\alpha_2,\beta_2} \in S_p$. Assume $\alpha_1 \neq \beta_1$. Then $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} \in S_p$ if and only if $\alpha_2 \in H^\infty$, $\beta_2 \in \overline{H^\infty}$. In this case $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_1\alpha_2,\beta_1\beta_2}$.*

Theorem 6.7. *The operator $S_{\alpha,\beta}$ on L^p is an isometry if and only if $|\alpha| = |\beta| = 1$ and $\alpha = \theta\beta$ for some inner function $\theta \in H^\infty$. The operator $S_{\alpha,\beta}$ is an invertible isometry if and only if $|\alpha| = |\beta| = 1$ and $\alpha = \lambda\beta$ for some unimodular constant λ. The operator $T_{\alpha,\beta}$ on L^p is an isometry if and only if $|\alpha| = |\beta| = 1$ and $\alpha = \lambda\beta$ for some unimodular constant λ.*

Theorem 6.8. *Assume $S_{\alpha_1,\beta_1} S_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2} S_{\alpha_1,\beta_1}$ on L^p. Then one of the following statements holds.*

(i) *$\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.*

(ii) *$\alpha_1, \overline{\beta_1} \in H^\infty$ and $\alpha_2, \overline{\beta_2} \in H^\infty$.*

(iii) *$S_{\alpha_1,\beta_1} = \lambda S_{\alpha_2,\beta_2} + \mu I$ for some constants λ and μ.*

Theorem 6.9. *If $S_{\alpha,\beta} T_{\alpha,\beta} = T_{\alpha,\beta} S_{\alpha,\beta}$ on L^p, then one of the following two statement holds.*

(i) $S_{\alpha,\beta} = M_\beta + S_{\delta,0}$ *where* δ *is a constant such that* $\delta\overline{\beta} - \overline{\delta}\beta$ *is a constant.*

(ii) $S_{\alpha,\beta} = \lambda S_{\alpha_1,\beta_1} + \mu I$ *for some constants* λ *and* μ*. Furthermore* S_{α_1,β_1} *is a unitary operator.*

7. Appendix

In this appendix we provide a direct derivation of normal singular integral operators obtained in [12].

Theorem 7.1. *Assume one of* α *and* β *is not a constant and* $\alpha - \beta \neq 0$*. Then* $S_{\alpha,\beta}$ *is normal if and only if* $\alpha = \lambda\beta + \delta$ *for some constants* δ *and* λ *with* $|\lambda| = 1$*, and* $(\lambda - 1)|\beta|^2 + \delta\overline{\beta} - \overline{\delta}\lambda\beta$ *is a constant.*

Proof. If $S_{\alpha,\beta}$ is normal, then $[S_{\alpha,\beta}S^*_{\alpha,\beta}, M_z] = [S^*_{\alpha,\beta}S_{\alpha,\beta}, M_z]$. By Lemma 2.4,

$$\alpha \otimes \overline{z}\alpha - \beta \otimes \overline{z}\beta = e_0 \otimes S^*_{\alpha,\beta}\overline{z}(\alpha - \beta) + S^*_{\alpha,\beta}(\alpha - \beta) \otimes e_{-1}. \tag{7.1}$$

We divide the proof into three cases.

Case 1. Both sides of (7.1) are rank zero operators. Then

$$\alpha = \lambda\beta$$

for some constant $\lambda \neq 1$. Plugging this into the left side of (7.1), we have

$$\left(|\lambda|^2 - 1\right)\beta \otimes \overline{z}\beta = 0,$$

so $|\lambda|^2 = 1$. Furthermore

$$e_0 \otimes S^*_{\alpha,\beta}\overline{z}(\alpha - \beta) + S^*_{\alpha,\beta}(\alpha - \beta) \otimes e_{-1} = 0$$

implies that

$$S^*_{\alpha,\beta}(\alpha - \beta) = \mu, \quad S^*_{\alpha,\beta}\overline{z}(\alpha - \beta) = -\overline{\mu}e_{-1}.$$

Thus

$$P\left[\overline{\alpha}(\lambda - 1)\beta\right] + Q\left[\overline{\beta}(\lambda - 1)\beta\right] = u$$
$$P\left[\overline{\alpha z}(\lambda - 1)\beta\right] + Q\left[\overline{\beta}\overline{z}(\lambda - 1)\beta\right] = -\overline{\mu z}.$$

Therefore $\overline{\lambda}(\lambda - 1)|\beta|^2 \in \overline{H^\infty}$, $(\lambda - 1)|\beta|^2 \in H^\infty$ and $|\beta|^2 = \sigma$ for some constant σ. In this case it is easy to verify that $S_{\alpha,\beta}S^*_{\alpha,\beta} = S^*_{\alpha,\beta}S_{\alpha,\beta} = S_{\alpha\overline{\alpha},\beta\overline{\beta}} = \sigma I$. In fact $S_{\alpha,\beta}$ is a multiple of a unitary operator.

Case 2. Both sides of (7.1) are rank one operators. Then

$$\alpha = \lambda\beta, \quad S^*_{\alpha,\beta}(\alpha - \beta) = \mu$$

for some constants λ and μ. Plugging these relations into (7.1), we have

$$\left(|\lambda|^2 - 1\right) \beta \otimes \overline{z}\beta = e_0 \otimes \left(S^*_{\alpha,\beta}\overline{z}(\alpha - \beta) + \overline{\mu} e_{-1}\right).$$

Therefore β is a constant and $\alpha = \lambda\beta$ is also a constant. This is impossible by the assumption.

Case 3. Both sides of (7.1) are rank two operators. By (7.1), e_0 is a linear combination of α and β. Therefore $\alpha = \lambda\beta + \delta$ for some constants $\delta \neq 0$ and λ. We consider two cases.

Case 3a. $\lambda = 1$. Plugging $\alpha = \beta + \delta$ into (7.1), we have

$$(\beta + \delta) \otimes \overline{z}\delta + \delta \otimes \overline{z}\beta = e_0 \otimes S^*_{\alpha,\beta}\overline{z}\delta + S^*_{\alpha,\beta}\delta \otimes e_{-1}.$$

Thus

$$\begin{aligned} e_0 \otimes \left[S^*_{\alpha,\beta}\overline{z}\delta - \overline{z}\overline{\delta}\beta\right] + \left[S^*_{\alpha,\beta}\delta - \overline{\delta}(\beta + \delta)\right] \otimes e_{-1} &= 0, \\ P\left[\overline{\alpha z}\delta - \overline{z}\overline{\delta}\beta\right] + P\left[\overline{\beta}\overline{z}\delta - \overline{z}\overline{\delta}\beta\right] &= \mu, \\ P\left[\overline{\alpha}\delta - \overline{\delta}(\beta + \delta)\right] + P\left[\overline{\beta}\delta - \overline{\delta}(\beta + \delta)\right] &= -\mu. \end{aligned}$$

Hence $\overline{\beta}\delta - \overline{\delta}\beta \in \overline{H^\infty}$, $\overline{\beta}\delta - \overline{\delta}\beta \in H^\infty$ and $\overline{\beta}\delta - \overline{\delta}\beta$ is a constant. In this case we see that $S_{\alpha,\beta} = M_\beta + S_{\delta,0}$ is normal.

Case 3b. $\lambda \neq 1$. Let

$$\sigma = \delta/(\lambda - 1), \quad \alpha_1 = \alpha + \sigma, \quad \beta_1 = \beta + \sigma.$$

Then

$$\lambda\beta_1 = \lambda\beta + \lambda\sigma = \alpha - \delta + \lambda\sigma = \alpha_1 - \sigma - \delta + \lambda\sigma = \alpha_1.$$

Since $S_{\alpha_1,\beta_1} = S_{\alpha,\beta} + \sigma I$, $S_{\alpha,\beta}$ is normal if and only if S_{α_1,β_1} is normal, this reduces to case (1) and S_{α_1,β_1} is a multiple of a unitary operator. Thus $|\lambda| = 1$ and

$$|\beta_1|^2 = |\beta + \sigma|^2 = \frac{1}{(\lambda - 1)}\left[(\lambda - 1)|\beta|^2 + \delta\overline{\beta} - \overline{\delta}\lambda\beta\right] + |\sigma|^2$$

is a constant. The proof is complete. □

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