TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 1, pp. 129–141, February 2016
DOI: 10.11650/tjm.20.2016.6079
This paper is available online at http://journal.tms.org.tw

Arbitrary Decay of Energy for a Viscoelastic Problem with Balakrishnan-Taylor Damping

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Abstract. In this paper we consider a viscoelastic problem with Balakrishnan-Taylor damping

$$u_{tt} - \Big(a + b \left\|\nabla u\right\|^2 + \sigma(\nabla u, \nabla u_t)\Big) \Delta u + \int_0^t g(t-s)\Delta u(s)\, ds = 0$$

with Dirichlet boundary condition. We establish a decay result of the energy of solutions for the problem without imposing the usual relation between the relaxation function g and its derivative. This result generalizes earlier ones to an arbitrary rate of decay, which is not necessarily of exponential or polynomial decay.

1. Introduction

In this paper we consider the following viscoelastic problem with Balakrishnan-Taylor damping

$$u_{tt} - \Big(a + b \left\|\nabla u\right\|^2 + \sigma(\nabla u, \nabla u_t)\Big) \Delta u + \int_0^t g(t-s)\Delta u(s)\, ds = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for } x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, a, b, σ are positive constants, g is a relaxation function which will be specified later. From the physical point of view, problem (1.1)-(1.3) is related to the panel flutter equation and spillover problem with memory.

In the absence of the Balakrishnan-Taylor damping ($\sigma = 0$), problem (1.1)-(1.3) has been extensively studied and several results concerning existence, nonexistence and asymptotic behavior have been established (see e.g. [6, 9, 10] for the case $g = 0$ and [3, 7] for the

Received March 26, 2015, accepted June 1, 2015.
Communicated by Yingfei Yi.
2010 *Mathematics Subject Classification*. 35L70, 35B41.
Key words and phrases. Arbitrary decay rate, Balakrishnan-Taylor damping, Memory.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (Grant No. 2014R1A1A3A04049561).

case $b = 0$). When $\sigma = g = 0$, problem (1.1)-(1.3) reduces to the well-known Kirchhoff equation which has been introduced by Kirchhoff [5] in order to describe the nonlinear vibrations of an elastic string, and when $b = \sigma = 0$, problem (1.1)-(1.3) forms a linear viscoelastic equation used to investigate the motion of viscoelastic materials.

Balakrishnan-Taylor damping was proposed by Balakrishnan and Taylor [1] and Bass and Zes [2]. Since then, some authors have discussed results on existence and asymptotic behavior of a class of equations with Balakrishnan-Taylor damping (see [8, 13, 15] and references therein). Tatar and Zaraï [13, 15] investigated exponential and polynomial decay results under the classical condition $g'(t) \le -\zeta g(t)$ and $g'(t) \le -\zeta (g(t))^{1+\frac{1}{p}}$, $p > 2$, for some $\zeta > 0$, respectively. Later, Mu and Ma [8] extended these results by proving a general decay rate of energy under the condition $g'(t) \le -\zeta(t)g(t)$, where $\zeta(t)$ is a nonincreasing and positive function.

On the other hand, Fabrizio and Polidoro [4] obtained an exponential decay rate of solutions to a linear viscoelastic wave equation under the condition $g'(t) \le 0$ and $e^{\alpha t} g(t) \in L^1(0,\infty)$ for some $\alpha > 0$. Tatar [12] weakened this assumption as

$$g'(t) \le 0 \quad \text{and} \quad \zeta(t)g(t) \in L^1(0,\infty), \tag{1.4}$$

where $\zeta(t)$ is a nonnegative function, and established an arbitrary decay rate for a linear viscoelastic wave equation by introducing an appropriate new functional in the modified energy.

Inspired by these results, we improve earlier ones concerning exponential decay for problem (1.1)-(1.3) by imposing the condition (1.4) on the relaxation function g. The remainder of the paper is organized as follows. In Section 2, we give some preliminaries related to problem (1.1)-(1.3). In Section 3, we prove an arbitrary decay result.

2. Preliminaries

We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^1(\Omega)$. For a Hilbert space X, we denote $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ the inner product and norm of X, respectively. For simplicity, we denote $(\cdot,\cdot)_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let λ be the smallest positive constant such that

$$\lambda \|u\|^2 \le \|\nabla u\|^2 \quad \text{for } u \in H_0^1(\Omega). \tag{2.1}$$

As in [12], we impose the following conditions on the relaxation function g:

(G1) $g\colon [0,\infty) \to (0,\infty)$ is a continuous, nonincreasing and almost everywhere differentiable function satisfying

$$\int_0^\infty g(s)\, ds := l < a. \tag{2.2}$$

(G2) There exists a nondecreasing function $\zeta(t) > 0$ such that

$$\frac{\zeta'(t)}{\zeta(t)} := \eta(t) \text{ is a decreasing function and } \int_0^\infty g(s)\zeta(s)\,ds < \infty. \tag{2.3}$$

By standard Galerkin method, we get the existence result (see e.g. [3, 14]):

Theorem 2.1. *Assume that* (G1) *holds. Then, for every* $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ *there exists a unique solution* u *to problem* (1.1)-(1.3) *such that*

$$u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \quad u_t \in L^\infty(0, T; H_0^1(\Omega)), \quad u_{tt} \in L^\infty(0, T; L^2(\Omega)).$$

3. Arbitrary decay of energy

In this section we prove an arbitrary decay rate of the energy of solutions to problem (1.1)-(1.3). We define the energy of problem (1.1)-(1.3) by

$$E(t = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(a - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{b}{4}\|\nabla u(t)\|^4 + \frac{1}{2}(g\Box\nabla u)(t), \tag{3.1}$$

where $(g\Box\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,ds$.

Lemma 3.1. *The energy* $E(t)$ *satisfies*

$$E'(t) \le -\frac{\sigma}{4}\left(\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 + \frac{1}{2}(g'\Box\nabla u)(t) \quad \text{for } t > 0. \tag{3.2}$$

Proof. Multiplying (1.1) by $u_t(t)$, which makes sense because $u_t \in L^\infty(0, T; H_0^1(\Omega))$, we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t(t)\|^2 + \frac{a}{2}\|\nabla u(t)\|^2 + \frac{b}{4}\|\nabla u(t)\|^4\right) \\ &= -\frac{\sigma}{4}\left(\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 + \int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))\,ds. \end{aligned} \tag{3.3}$$

A direct calculation ensures

$$\begin{aligned} &\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))\,ds \\ &= -\frac{1}{2}\frac{d}{dt}(g\Box\nabla u)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)\,ds\,\|\nabla u(t)\|^2\right) - \frac{g(t)}{2}\|\nabla u(t)\|^2 + \frac{1}{2}(g'\Box\nabla u)(t). \end{aligned}$$

Applying this to the right-hand side of (3.3), we complete the proof. □

To demonstrate the stability of problem (1.1)-(1.3), we introduce the following notations as in [11, 12]. For every measurable set $\mathcal{A} \subset \mathbb{R}^+$, we define the probability measure $\widehat{g}$ by

$$\widehat{g}(\mathcal{A}) = \frac{1}{l} \int_{\mathcal{A}} g(s)\, ds. \tag{3.4}$$

The flatness set of g is defined by

$$\mathcal{F}_g = \left\{ s \in \mathbb{R}^+ : g(s) > 0 \text{ and } g'(s) = 0 \right\}. \tag{3.5}$$

Now let us define the perturbed functional by

$$L(t) = ME(t) + \gamma_1 \Phi(t) + \gamma_2 \Psi(t) + \gamma_3 \Xi(t),$$

where M and γ_i $(i = 1, 2, 3)$ are positive constants to be specified later,

$$\Phi(t) = (u_t(t), u(t)) + \frac{\sigma}{4} \left\| \nabla u(t) \right\|^4,$$

$$\Psi(t) = - \int_0^t g(t-s)(u(t) - u(s), u_t(t))\, ds,$$

and

$$\Xi(t) = \int_0^t G_\zeta(t-s) \left\| \nabla u(s) \right\|^2 ds, \tag{3.6}$$

here

$$G_\zeta(t) = \zeta^{-1}(t) \int_t^\infty g(s) \zeta(s)\, ds. \tag{3.7}$$

Remark 3.2. The function Ξ given in (3.6) was first introduced by Tatar [12] to get an arbitrary decay rate for a linear viscoelastic equation.

Lemma 3.3. *Assume that* (G1) *holds. Then, for $M > 0$ large there exist positive constants α_1 and α_2 such that*

$$\alpha_1 (E(t) + \Xi(t)) \le L(t) \le \alpha_2 (E(t) + \Xi(t)).$$

Proof. Young's inequality, Holder's inequality and (2.1) imply

$$\begin{aligned}
|\Phi(t)| &= \left| (u_t(t), u(t)) + \frac{\sigma}{4} \left\| \nabla u(t) \right\|^4 \right| \\
&\le \frac{1}{2} \left\| u_t(t) \right\|^2 + \frac{1}{2\lambda} \left\| \nabla u(t) \right\|^2 + \frac{\sigma}{4} \left\| \nabla u(t) \right\|^4 \\
&\le \frac{1}{2} \left\| u_t(t) \right\|^2 + \frac{1}{2\lambda(1-l)} \left(1 - \int_0^t g(s)\, ds \right) \left\| \nabla u(t) \right\|^2 + \frac{\sigma}{4} \left\| \nabla u(t) \right\|^4 \\
&\le C_1 E(t)
\end{aligned}$$

and

$$\begin{aligned}|\Psi(t)| &\le \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(\int_0^t g(t-s)\|u(t)-u(s)\|\,ds\right)^2\\ &\le \frac{1}{2}\|u_t(t)\|^2 + \frac{l}{2\lambda}(g\Box\nabla u)(t)\\ &\le C_2E(t),\end{aligned}$$

where $C_1 = \max\left\{1, \frac{1}{\lambda(1-l)}, \frac{\sigma}{b}\right\}$ and $C_2 = \max\left\{1, \frac{l}{\lambda}\right\}$. Thus we obtain

$$|L(t) - ME(t) - \gamma_3\Xi(t)| \le (\gamma_1C_1 + \gamma_2C_2)E(t).$$

Choosing $M > 0$ large and putting $\alpha_1 = \min\{M - \gamma_1C_1 - \gamma_2C_2, \gamma_3\}$, $\alpha_2 = \min\{M+\gamma_1C_1 + \gamma_2C_2, \gamma_3\}$, we complete the proof. □

Lemma 3.4. *Assume that* (G1) *holds. Then* Φ *satisfies*

$$\begin{aligned}\Phi'(t) &\le \|u_t(t)\|^2 - \left(a - \frac{l}{2}\right)\|\nabla u(t)\|^2 - b\|\nabla u(t)\|^4 - \frac{1}{2}(g\Box\nabla u)(t)\\ &\quad + \frac{1}{2}\int_0^t g(t-s)\|\nabla u(s)\|^2\,ds,\end{aligned} \tag{3.8}$$

Proof. From (1.1)-(1.2), we have

$$\begin{aligned}\Phi'(t) &= \|u_t(t)\|^2 + (u(t), u_{tt}(t)) + \sigma\|\nabla u(t)\|^2(\nabla u(t), \nabla u_t(t))\\ &= \|u_t(t)\|^2 - a\|\nabla u(t)\|^2 - b\|\nabla u(t)\|^4 + \int_0^t g(t-s)(\nabla u(s), \nabla u(t))\,ds.\end{aligned} \tag{3.9}$$

Substituting the following relation (see [12, Lemma 2])

$$\begin{aligned}\int_0^t g(t-s)(\nabla u(s), \nabla u(t))\,ds &= -\frac{1}{2}(g\Box\nabla u)(t) + \frac{1}{2}\int_0^t g(t-s)\|\nabla u(t)\|\,ds\\ &\quad + \frac{1}{2}\int_0^t g(t-s)\|\nabla u(s)\|^2\,ds\end{aligned} \tag{3.10}$$

into the last term of (3.9) and using (2.2), we complete the proof. □

Lemma 3.5. *Assume that* (G1) *holds. Then, for any positive constant* δ_i $(i = 1, 2, 3)$ *and* *all measurable sets $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A}=\mathbb{R}^+\setminus\mathcal{B}$ it holds that*

(3.11)
$$\begin{aligned}\Psi'(t) \le &-\left(\int_0^t g(s)\,ds-\delta_2\right)\|u_t(t)\|^2+\delta_3 b\,\|\nabla u(t)\|^4\\ &+\left\{\left(a-\int_0^t g(s)\,ds\right)\left(\delta_1+\frac{3l\widehat{g}(\mathcal{B})}{2}\right)\right\}\|\nabla u(t)\|^2\\ &+l\left\{\left(a-\int_0^t g(s)\,ds\right)\frac{1}{4\delta_1}+1+\frac{1}{\delta_1}\right\}\int_{\mathcal{A}\cap[0,t]} g(t-s)\,\|\nabla u(t)-\nabla u(s)\|^2\,ds\\ &+l(1+\delta_1)\widehat{g}(\mathcal{B})\int_{\mathcal{B}\cap[0,t]} g(t-s)\,\|\nabla u(t)-\nabla u(s)\|^2\,ds\\ &+\frac{1}{2}\left(a-\int_0^t g(s)\,ds\right)\int_{\mathcal{B}\cap[0,t]} g(t-s)\,\|\nabla u(s)\|^2\,ds-\frac{g(0)}{4\delta_2\lambda}(g'\square\nabla u)(t)\\ &+\frac{2\sigma^2E(0)}{a-l}\left(\frac{d}{dt}\|\nabla u(t)\|^2\right)^2+\left(\frac{l}{16}+\frac{lbE(0)}{2\delta_3(a-l)}\right)(g\square\nabla u)(t).\end{aligned}$$

Proof. From (1.1) and (1.2), we have

$$\begin{aligned}\Psi'(t)=&-\left(\int_0^t g(s)\,ds\right)\|u_t(t)\|^2-\int_0^t g'(t-s)(u(t)-u(s),u_t(t))\,ds\\ &+\left(a+b\,\|\nabla u(t)\|^2+\sigma(\nabla u(t),\nabla u_t(t))\right)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s),\nabla u(t))\,ds\\ &-\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s),\int_0^t g(t-s)\nabla u(s)\,ds\right)ds.\end{aligned}$$

Since

$$\begin{aligned}&-\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s),\int_0^t g(t-s)\nabla u(s)\,ds\right)ds\\ &=\int_0^t g(t-s)\int_0^t g(t-\tau)(\nabla u(\tau)-\nabla u(t),\nabla u(t)-\nabla u(s))\,d\tau\,ds\\ &\quad-\int_0^t g(t-s)\int_0^t g(t-\tau)(\nabla u(t),\nabla u(t)-\nabla u(s))\,d\tau\,ds\\ &=\left\|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2\\ &\quad-\left(\int_0^t g(s)\,ds\right)\int_0^t g(t-s)(\nabla u(t),\nabla u(t)-\nabla u(s))\,ds,\end{aligned}$$

it follows that

$$
\begin{aligned}
\Psi'(t) &= -\left(\int_0^t g(s)\,ds\right)\|u_t(t)\|^2 - \int_0^t g'(t-s)(u(t)-u(s),u_t(t))\,ds \\
&\quad + \left(a - \int_0^t g(s)\,ds\right)\int_0^t g(t-s)(\nabla u(t),\nabla u(t)-\nabla u(s))\,ds \\
&\quad + b\,\|\nabla u(t)\|^2 \int_0^t g(t-s)(\nabla u(t),\nabla u(t)-\nabla u(s))\,ds \\
&\quad + \sigma(\nabla u(t),\nabla u_t(t))\int_0^t g(t-s)(\nabla u(t),\nabla u(t)-\nabla u(s))\,ds \\
&\quad + \left\|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2 \\
&:= -\left(\int_0^t g(s)\,ds\right)\|u_t(t)\|^2 + I_1 + \left(a - \int_0^t g(s)\,ds\right) I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{3.12}
$$

Now we will estimate the terms on right hand side of (3.12). For all measurable sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} = \mathbb{R}^+ \setminus \mathcal{B}$ and any $\delta_1 > 0$, we have from (G1) that

$$
\begin{aligned}
|I_2| &= \Bigg|\int_{\mathcal{A}\cap[0,t]} g(t-s)(\nabla u(t),\nabla u(t)-\nabla u(s))\,ds \\
&\quad + \left(\int_{\mathcal{B}\cap[0,t]} g(s)\,ds\right)\|\nabla u(t)\|^2 - \int_{\mathcal{B}\cap[0,t]} g(t-s)(\nabla u(t),\nabla u(s))\,ds\Bigg| \\
&\le \delta_1\|\nabla u(t)\|^2 + \frac{l}{4\delta_1}\int_{\mathcal{A}\cap[0,t]} g(t-s)\,\|\nabla u(t)-\nabla u(s)\|^2\,ds \\
&\quad + \frac{3}{2}\left(\int_{\mathcal{B}\cap[0,t]} g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}\int_{\mathcal{B}\cap[0,t]} g(t-s)\,\|\nabla u(s)\|^2\,ds \\
&\le \left(\delta_1 + \frac{3l}{2}\widehat{g}(\mathcal{B})\right)\|\nabla u(t)\|^2 + \frac{l}{4\delta_1}\int_{\mathcal{A}\cap[0,t]} g(t-s)\,\|\nabla u(t)-\nabla u(s)\|^2\,ds \\
&\quad + \frac{1}{2}\int_{\mathcal{B}\cap[0,t]} g(t-s)\,\|\nabla u(s)\|^2\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
|I_5| &= \left\|\int_{\mathcal{A}\cap[0,t]} g(t-s)(\nabla u(t)-\nabla u(s))\,ds + \int_{\mathcal{B}\cap[0,t]} g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2 \\
&= \left\|\int_{\mathcal{A}\cap[0,t]} g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2 + \left\|\int_{\mathcal{B}\cap[0,t]} g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2 \\
&\quad + 2\left(\int_{\mathcal{A}\cap[0,t]} g(t-s)(\nabla u(t)-\nabla u(s))\,ds, \int_{\mathcal{B}\cap[0,t]} g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right) \\
&\le \left(1+\frac{1}{\delta_1}\right)\left(\int_{\mathcal{A}\cap[0,t]} g(t-s)\,ds\right)\int_{\mathcal{A}\cap[0,t]} g(t-s)\,\|\nabla u(t)-\nabla u(s)\|^2\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+(1+\delta_1)\left(\int_{\mathcal{B}\cap[0,t]} g(t-s)\,ds\right)\int_{\mathcal{B}\cap[0,t]} g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^2\,ds\\
\le&\left(1+\frac{1}{\delta_1}\right)l\int_{\mathcal{A}\cap[0,t]} g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^2\,ds\\
&+(1+\delta_1)l\widehat{g}(\mathcal{B})\int_{\mathcal{B}\cap[0,t]} g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^2\,ds.
\end{aligned}
$$

For any $\delta_2>0$, we get

$$
|I_1|\le\delta_2\left\|u_t(t)\right\|^2-\frac{g(0)}{4\delta_2\lambda}(g'\Box\nabla u)(t).
$$

For any $\delta_3>0$, it follows that

$$
\begin{aligned}
|I_3|&\le b\left\|\nabla u(t)\right\|^2\left(\delta_3\left\|\nabla u(t)\right\|^2+\frac{l}{4\delta_3}(g\Box\nabla u)(t)\right)\\
&=\delta_3 b\left\|\nabla u(t)\right\|^4+\frac{lb}{4\delta_3}\left\|\nabla u(t)\right\|^2(g\Box\nabla u)(t)\\
&\le\delta_3 b\left\|\nabla u(t)\right\|^4+\frac{lbE(0)}{2\delta_3(a-l)}(g\Box\nabla u)(t),
\end{aligned}
$$

in the last inequality it is used (3.1) and the fact $E(t)\le E(0)$. It holds that

$$
\begin{aligned}
|I_4|&\le\sigma^2\left\|\nabla u(t)\right\|^2\left(\frac{d}{dt}\left\|\nabla u(t)\right\|^2\right)^2+\frac{l}{16}(g\Box\nabla u)(t)\\
&\le\frac{2\sigma^2E(0)}{a-l}\left(\frac{d}{dt}\left\|\nabla u(t)\right\|^2\right)^2+\frac{l}{16}(g\Box\nabla u)(t).
\end{aligned}
\tag{3.13}
$$

Substituting these estimates into (3.12), we get the desired result. □

Lemma 3.6. *Assume that* (G1) *and* (G2) *hold. Then, for any positive constant* δ_i $(i=1,2,3)$ *and all measurable sets* $\mathcal{A}$ *and* $\mathcal{B}$ *with* $\mathcal{A}=\mathbb{R}^+\setminus\mathcal{B}$ *it holds that*

(3.14)

$$
\begin{aligned}
L'(t)\le&-\left\{\gamma_2\left(\int_0^t g(s)\,ds-\delta_2\right)-\gamma_1\right\}\left\|u_t(t)\right\|^2-(\gamma_1-\gamma_2\delta_3)b\left\|\nabla u(t)\right\|^4\\
&+\left[\gamma_2\left\{\left(a-\int_0^t g(s)\,ds\right)\left(\delta_1+\frac{3l\widehat{g}(\mathcal{B})}{2}\right)\right\}-\gamma_1\left(a-\frac{l}{2}\right)+\gamma_3G_\zeta(0)\right]\left\|\nabla u(t)\right\|^2\\
&+\gamma_2 l\left\{\left(a-\int_0^t g(s)\,ds\right)\frac{1}{4\delta_1}+1+\frac{1}{\delta_1}\right\}\int_{\mathcal{A}\cap[0,t]} g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^2\,ds\\
&+\gamma_2 l(1+\delta_1)\widehat{g}(\mathcal{B})\int_{\mathcal{B}\cap[0,t]} g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^2\,ds\\
&+\frac{\gamma_2}{2}\left(a-\int_0^t g(s)\,ds\right)\int_{\mathcal{B}\cap[0,t]} g(t-s)\left\|\nabla u(s)\right\|^2\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\frac{M}{2}-\frac{\gamma_2 g(0)}{4\delta_2\lambda}\right\}(g'\square\nabla u)(t)+\left(\frac{\gamma_2 l}{16}+\frac{\gamma_2 lbE(0)}{2\delta_3(a-l)}-\frac{\gamma_1}{2}\right)(g\square\nabla u)(t)\\
&+\left(\frac{\gamma_1}{2}-\gamma_3\right)\int_0^t g(t-s)\left\|\nabla u(s)\right\|^2\,ds-\gamma_3\eta(t)\Xi(t)\\
&-\left(\frac{M\sigma}{4}-\frac{2\gamma_2\sigma^2E(0)}{a-l}\right)\left(\frac{d}{dt}\left\|\nabla u(t)\right\|^2\right)^2.
\end{aligned}
$$

Proof. From (2.3), we get

$$
\begin{aligned}
\Xi'(t)&=G_\zeta(0)\left\|\nabla u(t)\right\|^2+\int_0^t G'_\zeta(t-s)\left\|\nabla u(s)\right\|^2\,ds\\
&=G_\zeta(0)\left\|\nabla u(t)\right\|^2-\int_0^t\left(\frac{\zeta'(t-s)}{\zeta(t-s)}G_\zeta(t-s)+g(t-s)\right)\left\|\nabla u(s)\right\|^2\,ds\\
&\le G_\zeta(0)\left\|\nabla u(t)\right\|^2-\eta(t)\Xi(t)-\int_0^t g(t-s)\left\|\nabla u(s)\right\|^2\,ds.
\end{aligned}
\tag{3.15}
$$

Combining (3.2), (3.8), (3.11) and (3.15), we get the desired result. □

Let $\int_0^{t^*} g(s)\,ds := g^*$ for $t^*>0$, then our main theorem reads as:

Theorem 3.7. *Assume that* (G1), (G2), $\widehat{g}(\mathcal{F}_g)<\frac{1}{16}$ *and* $E(0)<\frac{l(a-l)}{8b}$ *hold. Then there exist positive constants* t^*, C_0, *and* ω *such that if* $G_\zeta(0)<\frac{(32a-13l)g^*-3al}{32a}$ *then*

$$
E(t)\le C_0\zeta(t)^{-\omega}\quad \text{for } t\ge t^*. \tag{3.16}
$$

Proof. For $n\in\mathbb{N}$, as in [11, 12], we introduce the sets

$$
\mathcal{A}_n=\left\{s\in\mathbb{R}^+ : ng'(s)+g(s)\le 0\right\}\quad \text{and}\quad \mathcal{B}_n=\mathbb{R}^+\setminus\mathcal{A}_n.
$$

It is easy to show that

$$
\bigcup_{n=1}^{\infty}\mathcal{A}_n=\mathbb{R}^+\setminus\left\{\mathcal{F}_g\cup\mathcal{N}_g\right\},
$$

where $\mathcal{F}_g$ is given in (3.5) and $\mathcal{N}_g$ is the null set where g' is not defined. Since $\mathcal{B}_{n+1}\subset\mathcal{B}_n$ for all n and $\bigcap_{n=1}^{\infty}\mathcal{B}_n=\mathcal{F}_g\cup\mathcal{N}_g$, we get

$$
\lim_{n\to\infty}\widehat{g}(\mathcal{B}_n)=\widehat{g}(\mathcal{F}_g). \tag{3.17}
$$

Since g is positive, we have $\int_0^t g(s)\,ds\ge\int_0^{t^*} g(s)\,ds := g^*$ for all $t\ge t^*$. Thus, taking

$\mathcal{A} = \mathcal{A}_n$ and $\mathcal{B} = \mathcal{B}_n$ in (3.14), we see that

$$\begin{aligned} L'(t) \leq & - \{\gamma_2(g^* - \delta_2) - \gamma_1\} \|u_t(t)\|^2 - (\gamma_1 - \gamma_2\delta_3) b \|\nabla u(t)\|^4 \\ & + \left\{ \gamma_2(a - g^*) \left(\delta_1 + \frac{3l\widehat{g}(\mathcal{B}_n)}{2} \right) - \gamma_1 \left(a - \frac{l}{2} \right) + \gamma_3 G_\zeta(0) \right\} \|\nabla u(t)\|^2 \\ & + \left\{ \gamma_2 l \left(\frac{a - g^*}{4\delta_1} + 1 + \frac{1}{\delta_1} \right) - \frac{1}{n} \left(\frac{M}{2} - \frac{\gamma_2 g(0)}{4\delta_2 \lambda} \right) \right\} \\ & \times \int_{\mathcal{A}_n \cap [0,t]} g(t-s) \|\nabla u(t) - \nabla u(s)\|^2 \, ds \\ & + \left[\gamma_2 l \left\{ (1 + \delta_1) \widehat{g}(\mathcal{B}_n) + \frac{1}{16} + \frac{bE(0)}{2\delta_3(a - l)} \right\} - \frac{\gamma_1}{2} \right] (g \Box \nabla u)(t) \\ & + \left\{ \frac{\gamma_2}{2}(a - g^*) + \frac{\gamma_1}{2} - \gamma_3 \right\} \int_0^t g(t-s) \|\nabla u(s)\|^2 \, ds - \gamma_3 \eta(t) \Xi(t) \\ & - \left(\frac{M\sigma}{4} - \frac{2\gamma_2 \sigma^2 E(0)}{a - l} \right) \left(\frac{d}{dt} \|\nabla u(t)\|^2 \right)^2 \end{aligned} \tag{3.18}$$

for $t \geq t^*$. By choosing $\gamma_1 = (g^* - \epsilon)\gamma_2$ for small $0 < \epsilon < g^*$ and $\delta_3 = \frac{g^* - \epsilon}{2}$, (3.18) becomes

$$\begin{aligned} L'(t) \leq & -\gamma_2(\epsilon - \delta_2) \|u_t(t)\|^2 - \frac{\gamma_2 b(g^* - \epsilon)}{2} \|\nabla u(t)\|^4 \\ & + \left[\gamma_2(a - g^*) \left(\delta_1 + \frac{3l\widehat{g}(\mathcal{B}_n)}{2} \right) - \{\kappa + (1 - \kappa)\} (g^* - \epsilon)\gamma_2 \left(a - \frac{l}{2} \right) + \gamma_3 G_\zeta(0) \right] \\ & \times \|\nabla u(t)\|^2 \\ & + \left\{ \gamma_2 l \left(\frac{a - g^*}{4\delta_1} + 1 + \frac{1}{\delta_1} \right) - \frac{1}{n} \left(\frac{M}{2} - \frac{\gamma_2 g(0)}{4\delta_2 \lambda} \right) \right\} \\ & \times \int_{\mathcal{A}_n \cap [0,t]} g(t-s) \|\nabla u(t) - \nabla u(s)\|^2 \, ds \\ & + \left[\gamma_2 l \left\{ (1 + \delta_1)\widehat{g}(\mathcal{B}_n) + \frac{1}{16} + \frac{bE(0)}{(g^* - \epsilon)(a - l)} \right\} - \frac{(g^* - \epsilon)\gamma_2}{2} \right] (g \Box \nabla u)(t) \\ & + \left\{ \frac{\gamma_2(a - \epsilon)}{2} - \gamma_3 \right\} \int_0^t g(t-s) \|\nabla u(s)\|^2 \, ds - \gamma_3 \eta(t) \Xi(t) \\ & - \left(\frac{M\sigma}{4} - \frac{2\gamma_2 \sigma^2 E(0)}{a - l} \right) \left(\frac{d}{dt} \|\nabla u(t)\|^2 \right)^2 \end{aligned} \tag{3.19}$$

for $t \geq t^*$, where $\kappa = \frac{3l(a-g^*)}{16g^*(2a-l)}$. Owing to $l = \int_0^\infty g(s)\, ds$ and $E(0) < \frac{l(a-l)}{8b}$, there exists $t_1 > 0$ large such that

$$\frac{l}{2} < g^* \quad \text{and} \quad \frac{8bE(0)}{a - l} < g^* < l \quad \text{for } t^* \geq t_1,$$

and then there exists a constant $\epsilon_1 > 0$ small such that

$$\frac{l}{2} < g^* - \epsilon \quad \text{and} \quad \frac{8bE(0)}{a - l} < g^* - \epsilon < l \quad \text{for } t^* \geq t_1 \text{ and } 0 < \epsilon \leq \epsilon_1. \tag{3.20}$$

Since $\widehat{g}(\mathcal{F}_g) < \frac{1}{16}$, from (3.17) there exists $n_1 \in \mathbb{N}$ large such that $\widehat{g}(\mathcal{B}_n) < \frac{1}{16}$ for $n \geq n_1$. Thus we get that for $n \geq n_1$, $t^* \geq t_1$, and $0 < \epsilon \leq \epsilon_1$,

$$l\left(\widehat{g}(\mathcal{B}_n) + \frac{1}{16} + \frac{bE(0)}{(g^*-\epsilon)(a-l)}\right) - \frac{g^*-\epsilon}{2} < \frac{l}{4} - \frac{g^*-\epsilon}{2} < 0. \tag{3.21}$$

It is also noted that

$$(a-g^*)\frac{3l\widehat{g}(\mathcal{B}_n)}{2} - \kappa g^*\left(a-\frac{l}{2}\right) < (a-g^*)\frac{3l}{32} - \kappa g^*\left(a-\frac{l}{2}\right) = 0 \quad \text{for } n \geq n_1.$$

So, we can choose a positive constant ϵ_2 with $\epsilon_2 \leq \epsilon_1$ such that

$$(a-g^*)\frac{3l\widehat{g}(\mathcal{B}_n)}{2} - \kappa(g^*-\epsilon)\left(a-\frac{l}{2}\right) < 0 \quad \text{for } n \geq n_1,\ 0 < \epsilon \leq \epsilon_2. \tag{3.22}$$

Thus, from (3.21) and (3.22), we can take $\delta_1 > 0$ small enough such that, for $n \geq n_1$, $t \geq t_1$ and $0 < \epsilon \leq \epsilon_2$,

$$l\left\{(1+\delta_1)\widehat{g}(\mathcal{B}_n) + \frac{1}{16} + \frac{bE(0)}{(g^*-\epsilon)(a-l)}\right\} - \frac{g^*-\epsilon}{2} < 0 \tag{3.23}$$

and

$$(a-g^*)\left(\delta_1 + \frac{3l\widehat{g}(\mathcal{B}_n)}{2}\right) - \kappa(g^*-\epsilon)\left(a-\frac{l}{2}\right) < 0. \tag{3.24}$$

On the other hand, we can choose a constant t_2 with $t_2 \geq t_1$ so that $\frac{3al}{32a-13l} < g^* < l$ for $t^* \geq t_2$, and hence we find $1-\kappa > 0$ for $t^* \geq t_2$.

Once n_1, t_2 and ϵ_2 are fixed, we take $n = n_1$, $t^* = t_2$, $\epsilon = \epsilon_2$. Next we choose γ_2 and γ_3 satisfying

$$\frac{a\gamma_2}{2} < \gamma_3 < \frac{\gamma_2\{(32a-13l)g^* - 3al\}}{64G_\zeta(0)},$$

which is valid under the condition $G_\zeta(0) < \frac{(32a-13l)g^*-3al}{32a}$. Then we get

$$\frac{\gamma_2(a-\epsilon)}{2} - \gamma_3 < 0 \tag{3.25}$$

and

$$\begin{aligned} &\gamma_3 G_\zeta(0) - \gamma_2(1-\kappa)(g^*-\epsilon)\left(a-\frac{l}{2}\right) \\ &< \frac{\gamma_2\{(32a-13l)g^*-3al\}}{64} - \gamma_2\left(1-\frac{3l(a-g^*)}{16g^*(2a-l)}\right)(g^*-\epsilon)\left(a-\frac{l}{2}\right) \\ &= \gamma_2\{(32a-13l)g^*-3al\}\left(\frac{1}{64}-\frac{g^*-\epsilon}{32g^*}\right) \\ &< 0, \end{aligned} \tag{3.26}$$

we used the fact that $\frac{g^*-\epsilon}{g^*} > \frac{1}{2}$ in the last inequality.

Finally, we take $\delta_2 > 0$ small enough and $M > 0$ large enough so that

$$\epsilon - \delta_2 > 0, \tag{3.27}$$

$$\frac{M\sigma}{2} - \frac{2\gamma_2\sigma^2 E(0)}{a-l} > 0 \tag{3.28}$$

and

$$\gamma_2 l\left(\frac{a-g^*}{4\delta_1} + 1 + \frac{1}{\delta_1}\right) - \frac{1}{n}\left(\frac{M}{2} - \frac{\gamma_2 g(0)}{4\delta_2\lambda}\right) < 0. \tag{3.29}$$

Adapting (3.23)-(3.29) to (3.19), using the fact that $\eta(t)$ is decreasing and Lemma 3.4, we arrive at

$$\begin{aligned} L'(t) &\le -C_3 E(t) - \gamma_3\eta(t)\Xi(t) \le -C_3\frac{\eta(t)}{\eta(t^*)}E(t) - \gamma_3\eta(t)\Xi(t) \\ &\le -C_4\eta(t)(E(t)+\Xi(t)) \le -\omega\eta(t)L(t) \quad \text{for } t \ge t^*, \end{aligned} \tag{3.30}$$

where $C_3 > 0$, $C_4 = \min\left\{\frac{C_3}{\eta(t^*)}, \gamma_3\right\}$ and $\omega = \frac{C_4}{\alpha_2}$. This and Lemma 3.4 give that

$$\begin{aligned} \alpha_1(E(t)+\Xi(t)) \le L(t) &\le L(t^*)e^{-\omega\int_{t^*}^t \eta(s)\,ds} \\ &= L(t^*)e^{-\omega\int_{t^*}^t \frac{\zeta'(s)}{\zeta(s)}\,ds} = L(t^*)e^{-\omega\ln\frac{\zeta(t)}{\zeta(t^*)}} \\ &= L(t^*)(\zeta(t^*))^{\omega}(\zeta(t))^{-\omega} \quad \text{for } t \ge t^*. \end{aligned} \tag{3.31}$$

Making use of the fact $\Xi(t) \ge 0$, we get the desired result. □

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