TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 1, pp. 33–47, February 2016
DOI: 10.11650/tjm.20.2016.5851
This paper is available online at http://journal.tms.org.tw

Near-rings of Endo-transition Preserving Functions on Additive Group Semiautomata

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Abstract. An additive group semiautomaton or, in brief, GS-automaton is a generalization of the well known linear state machines. The purpose of this paper is to study the near-ring of endo-transition preserving functions of additive GS-automata. This class of near-rings is a subclass of the celebrated centralizer near-rings, and includes near-rings of infra-endomorphisms. Complete characterizations using both algebraic and graphical properties of the additive GS-automaton such that the near-ring is 0-symmetric or constant are given. Conditions such that this near-ring being simple or being a ring are also provided.

1. Introduction

A triplet (Q, X, δ) with a function $\delta\colon Q \times X \to Q$ is called a *semiautomaton* [12] or *state machine* [9, 11], where Q is the set of states, X the set of inputs and $\delta\colon Q \times X \to Q$ the *state transition function* (or next state function). It is called *faithful* if $\delta(q, x) = \delta(q, y)$ for all $q \in Q$ then $x = y$. Let X^* be the free monoid over X with $\wedge$, the empty string, as identity. The state transition function can be extended to X^* by recursively defining $\delta(q, \wedge) = q$ and $\delta(q, xy) = \delta(\delta(q, x), y)$ for all $q \in Q$ and $x, y \in X$. This system (Q, X, δ) is called a *group semiautomaton* (abbr., GS-automaton) if the set of state $(Q, +)$ is a group [2, 12, 14]. A GS-automaton $\mathcal{S} = (Q, X, \delta)$ is said to be *additive* [5] if there is an input $x_0 \in X$ such that:

(1) $\delta(q, x) = \delta(q, x_0) + \delta(0, x)$ for all $q \in Q$, $x \in X$;

(2) $\delta(q_1 + q_2, x_0) = \delta(q_1, x_0) + \delta(q_2, x_0)$ for all $q_1, q_2 \in Q$.

Condition (1) is called the *decomposition property* and condition (2) is called the *additivity property*. An additive GS-automaton $\mathcal{S} = (Q, X, \delta)$ will also be denoted by $\mathcal{S} = (Q, X, x_0, \delta)$ when the input x_0 is specified. Consequently, the decomposition property or the additivity property implies that $\delta(0, x_0) = 0$. An input $x_0 \in X$ in (1) and (2)

Received January 28, 2015, accepted June 1, 2015.
Communicated by Ching Hung Lam.
2010 *Mathematics Subject Classification*. 16Y30, 68Q70, 05C25.
Key words and phrases. Near-ring, Additive GS-automaton, Endo-transition preserving function, Connected, Fixed point free.

above is therefore called an *endo-input* in a GS-automaton $\mathcal{S} = (Q, X, \delta)$ for it acts like an endomorphism of the underlying group.

The definition of additive GS-automata has its root on linear sequential machines [9, 11]. It can also be viewed as one type of system which abstracts the properties of certain algebraic structures. Any given group $(G, +)$ can be naturally associated with an additive GS-automaton where the state transition function δ is defined by using the binary operation associated with G. Explicitly, consider the GS-automaton $\mathcal{G} = (G, G, \delta)$ where $\delta\colon G \times G \to G$ is defined as $\delta(q, x) = q + x$ for all $q, x \in G$. A quick verification shows that this GS-automaton is additive. The GS-automaton $\mathcal{G}$ had been investigated in [3–5] for their associated syntactic near-rings.

In the following, let $M(Q)$ be the collection of all functions from Q into Q. Recall that $M(Q)$ is a left near-ring using function addition and composition as operations. For details and terminologies about near-rings, please refer to [2, 14] but be aware that in [14] we are using left near-rings instead of right near-rings.

In an additive GS-automaton $\mathcal{S} = (Q, X, \delta)$, we are interested in the *endo-transition preserving functions* of $\mathcal{S}$. A function $f \in M(Q)$ is said to preserve endo-transition if

$$\delta(q, x)f = \delta(qf, x_0)$$

for all $q \in Q$, $x \in X$. Several reasons motivate and support the study of these functions. The collection of all endo-transition preserving functions of $\mathcal{S}$, denoted by $\mathcal{F}(\mathcal{S})$, forms an interesting algebraic object. Indeed it is a subnear-ring of a centralizer near-ring $M_\Omega(Q) = \{f \in M(Q) \mid f\varphi = \varphi f \text{ for all } \varphi \in \Omega\}$ for some semigroup $\Omega \subseteq \mathrm{End}(Q)$ by Proposition 2.1. As a subnear-ring of $M_\Omega(Q)$, the 0-symmetric property of $\mathcal{F}(\mathcal{S})$ extends to $M_\Omega(Q)$ by Theorem 2.6. Also, there is a bijection between $\mathcal{F}(\mathcal{S})$ and the set of transition preserving functions of $\mathcal{S}$. Explicitly, if $f \in \mathcal{F}(\mathcal{S})$ then $f + 1$ satisfies the identity

$$\delta(q, x)(f + 1) = \delta(q(f + 1), x)$$

for all $q \in Q$, $x \in X$ and vice versa. Finally, it includes certain classes on studied near-rings. When $(Q, +)$ is abelian, $X = \{x\}$ and $\eta(q, x) = q+q$ for all $q \in Q$, the GS-automaton $\mathcal{G}_2 = (G, X, \eta)$ is additive and $\mathcal{F}(\mathcal{G}_2)$ is a near-ring of infra-endomorphisms [7].

Given a GS-automaton $\mathcal{S} = (Q, X, \delta)$, let P be a subgroup of Q and Y a subset of X. If the range of the restricted mapping $\delta|_{P\times Y}$ of δ on $P \times Y$ is contained in P, then $(P, Y, \delta|_{P\times Y})$ is said a *group subsemiautomaton* or simply subsemiautomaton of $\mathcal{S}$. We will denote this GS-automaton by (P, Y, δ) if no confusion will arise. If $\mathcal{T} = (P, Y, \delta)$ is a subsemiautomaton of $\mathcal{S}$, it is called a *spanning* (resp., *induced*) subsemiautomaton of $\mathcal{S}$ if $P = Q$ (resp., $Y = X$). These terminologies are consistent with graph theory if we representing $\mathcal{S}$ as a multigraph where Q is the set of vertices and the acting of X on Q via the function δ is the set of direct edges. Explicitly, a directed edge (labeled by x) connects vertices p into q if $\delta(p, x) = q$ for some $x \in X$. The graph of a spanning (resp., induced) subsemiautomaton is just the spanning (resp., induced) subgraph of the multigraph of $\mathcal{S}$.

A state q of a semiautomaton $\mathcal{S} = (Q, X, \delta)$ is called *reachable* from a state p if there exists an input word $w \in X^*$ such that $\delta(p, w) = q$. The state p is called a *predecessor* of q and q is called *successor* of p. In the case that the length of the word $|w| = 1$, the state p is said *adjacent to* q or the state q is *adjacent from* p. Two adjacent states p, q are called (*strongly*) *connected* if p can reach q (and) or q can reach p by some $w \in X$. A subset C of Q is called *connected* if for any two states $p, q \in C$, there exists a finite sequence of states $x_0 = p, x_1, x_2, \ldots, x_n = q$ in C such that x_i and x_{i+1} are connected for all $i = 0, 1, \ldots, n-1$. The set C is called *strongly connected* if every state in C is reachable by any other. Explicitly, for any given states $p, q \in C$, there exists a word $w \in X^*$ such that $\delta(p, w) = q$. The GS-automata $\mathcal{G} = (G, G, \delta)$ defined in previous paragraph is strongly connected. It is proved in Proposition 3.1 that if an additive GS-automaton $\mathcal{S}$ is strongly connected, then $\mathcal{F}(\mathcal{S})$ is a constant near-ring. The converse is shown in Proposition 3.7, it says that if $\mathcal{F}(\mathcal{S})$ is a nonzero constant near-ring, then $\mathcal{S}$ is connected. A maximal connected subset of Q is called a *connected component* of $\mathcal{S}$. The complement of strongly connected is said *totally disconnected*, that is for all $p \neq q \in C$, $\delta(p, x) \neq q$ for all $x \in X$. In an additive GS-automaton $\mathcal{S} = (Q, X, \delta)$, it is shown in Theorem 3.5 that $\mathcal{S}$ is totally disconnected if and only if $\mathcal{F}(\mathcal{S}) = M(Q)$.

If $\mathcal{S}$ is any semiautomaton, then one can always find a connected subsemiautomaton for a given state $q \in Q$ by finding the connected component (denoted $\mathrm{conn}(q)$) in $\mathcal{S}$ which contains q. Be aware that $\mathrm{conn}(q)$ is not necessarily a group subsemiautomaton of a GS-automaton for $\mathrm{conn}(q)$ is in general not a group. Define the *complex* of state 0, denoted $\mathcal{CP}(0) = (\mathrm{cp}(0), X, \delta)$, as the group subsemiautomaton generated by the set

$$\mathrm{reach}(0) = \{q \in Q \mid \text{there exists } w \in X^* \text{ such that } \delta(0, w) = q\}.$$

Observe that $\mathcal{CP}(0)$ is the smallest induced subsemiautomaton of $\mathcal{S}$ which plays an important role in the algebraic theory of GS-automata. It is immediate that $\mathrm{reach}(0)$ is a connected subset of the component $\mathrm{conn}(0)$. In contrast to the linear case [8, 9], $\mathrm{reach}(0)$ need not be strongly connected even when $\mathcal{S}$ is additive [6, Example 2.2].

The state transition function $\delta\colon Q \times X \to Q$ in an additive GS-automaton $\mathcal{S} = (Q, X, \{x_0\}, \delta)$ can be characterized by the input alphabets as an act on the state group Q. Explicitly, Let $\Psi\colon Q \to Q$ and $\Upsilon\colon X \to Q$ be defined as $\delta(q, x_0) = q\Psi$ and $\delta(0, x) = x\Upsilon$ for all $q \in Q$, $x \in X$, respectively. A direct verification shows that Ψ is a group endomorphism of Q, and $\delta(q, x) = q\Psi + x\Upsilon$. The following result quoted from [5, Proposition 1] will be frequently used in the subsequent sections.

Theorem 1.1. *The GS-automaton $\mathcal{S} = (Q, X, \delta)$ is additive if and only if there exist a group endomorphism $\Psi\colon Q \to Q$, a map $\Upsilon\colon X \to Q$ and an input $x_0 \in X$ with $x_0\Upsilon = 0$ such that $\delta(q, x) = q\Psi + x\Upsilon$ for all $q \in Q$, $x \in X$.*

Due to Theorem 1.1, an additive GS-automaton will sometimes be denoted by $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ or $\mathcal{S} = (Q, X, (\Psi, \Upsilon))$ where the state transition function $\delta(q, x) = q\Psi + x\Upsilon$ for all $q \in Q$, $x \in X$. Since the state transition function $\delta\colon Q \times X \to Q$ can be extended to the free monoid X^* via $\delta(q, xy) = \delta(\delta(q, x), y)$ for all $x, y \in X$, the following result extends the domain of Υ to the free monoid X^*. Let $w = x_1x_2 \cdots x_n \in X^*$ be a word of length n where $x_i \in X$ for all $i = 1, 2, \ldots, n$. Then

$$w\Upsilon = \left(\sum_{i=1}^{n-1} x_i\Upsilon\Psi^{n-i}\right) + x_n\Upsilon.$$

Let 1 be the identity mapping on the group Q. The subsemigroup of $\mathrm{End}(Q)$ generated by the set $\{\Psi, 1\}$ is denoted by $\Omega(\mathcal{S})$ or simply Ω whenever no confusion will arise. A subset H of Q is said Ω-*invariant* if $h\varphi \in H$ for all $h \in H$, $\varphi \in \Omega$. It can now be seen that, for $w \in X^*$, $w\Upsilon$ is a finite sum of elements $x_i\Upsilon\varphi_j$ for some $\varphi_j \in \Omega$. Note that $w\Upsilon = \delta(0, w)$ is in fact the state reachable from 0 via the input string w, thus every element in reach(0) is a finite sum of elements in the set $\{x\Upsilon\varphi \mid x \in X, \varphi \in \Omega\}$.

By definition, $\mathcal{CP}(0)$ is the GS-automaton generated by reach(0), the state group cp(0) is thus a subset of the set of finite sums of elements in $\{(\pm)x\Upsilon\varphi \mid x \in X, \varphi \in \Omega\}$. Surprisingly, they are equal [6, Theorem 2.5]. We quote this result as following.

Theorem 1.2. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be an additive GS-automaton. Then the state group of the GS-automaton $\mathcal{CP}(0)$ is generated additively by the set $\{x\Upsilon\varphi \mid x \in X, \varphi \in \Omega\}$ as a subgroup of Q.*

Since elements in Ω are group endomorphisms of G, the set cp(0) is an Ω-invariant subgroup of Q by Theorem 1.2.

Corollary 1.3. *Let $\mathcal{S} = (Q, X, \delta)$ be an additive GS-automaton. Then* cp(0) *is an Ω-invariant subgroup of Q.*

2. Endo-transition preserving functions

If $f \in \mathcal{F}(\mathcal{S})$ for some additive GS-automaton $\mathcal{S} = (Q, X, \{x_0\}, \delta)$, it would be convenient to characterize the mapping f using the identity $\delta(q, x) = q\Psi + x\Upsilon$ shown in Theorem 1.1. Explicitly, $\delta(qf, x_0) = qf\Psi + x_0\Upsilon = qf\Psi$ for $x_0\Upsilon = 0$. Therefore we may rewrite $\mathcal{F}(\mathcal{S})$ as

$$\mathcal{F}(\mathcal{S}) = \{f \in M(Q) \mid \delta(q, x)f = qf\Psi \text{ for all } q \in Q, x \in X\}.$$

If $f \in \mathcal{F}(\mathcal{S})$, $q \in Q$ and $w = x_1x_2\cdots x_n \in X^*$ where $x_i's \in X$ for $i = 1, 2, \ldots, n$, then $\delta(q, w)f = \delta(\delta(q, x_1x_2\cdots x_{n-1}), x_n)f = \delta(q, x_1x_2\cdots x_{n-1})f\Psi$. Inductively, it can be seen that

$$\delta(q, w)f = qf\Psi^n = qf\Psi^{|w|} \quad \text{for all } q \in Q, w \in X^*.$$

The set $\mathcal{F}(\mathcal{S})$ had been studied in [8] for the linear case and in [13] for the homomorphic case, recall that both cases are additive. Note that the zero mapping is contained in $\mathcal{F}(\mathcal{S})$ and thus it is not empty. Let $q \in Q$, $x \in X$ and $f \in \mathcal{F}(\mathcal{S})$. Then $q\Psi f = (q\Psi + x_0\Upsilon)f = \delta(q, x_0)f = qf\Psi$. Consequently, $\Psi f = f\Psi$. Thus $\mathcal{F}(\mathcal{S})$ is contained in the centralizer of Ω in $M(Q)$, denoted by $M_\Omega(Q)$.

Furthermore, let $f, g \in \mathcal{F}(\mathcal{S})$ and $q \in Q$, $x \in X$. Then

$$\begin{aligned}\delta(q, x)(f + g) &= \delta(q, x)f + \delta(q, x)g \\ &= qf\Psi + qg\Psi \\ &= q(f + g)\Psi;\end{aligned}$$

and $\delta(q, x)(fg) = (\delta(q, x)f)g = (qf\Psi)g = q(fg)\Psi$ for $g \in M_\Omega(Q)$ by the previous argument. Thus $f + g, fg \in \mathcal{F}(\mathcal{S})$. Also, by the fact that $0 = \delta(q, x)(f + (-f)) = qf\Psi + \delta(q, x)(-f)$, it follows that $\delta(q, x)(-f) = -qf\Psi = q(-f)\Psi$ or $-f \in \mathcal{F}(\mathcal{S})$. It can easily be seen that $\mathcal{F}(\mathcal{S})$ is indeed a right $M_\Omega(Q)$-subgroup of $M_\Omega(Q)$. As a conclusion, $\mathcal{F}(\mathcal{S})$, as a nonempty subset of $M(Q)$, is indeed a subnear-ring.

Proposition 2.1. *Let $\mathcal{S} = (Q, X, (\Psi, \Upsilon))$ be an additive GS-automaton. Then the set $\mathcal{F}(\mathcal{S})$ is a subnear-ring of $M_\Omega(Q)$. Moreover, it is a right $M_\Omega(Q)$-subgroup of $M_\Omega(Q)$.*

The following example shows that $\mathcal{F}(\mathcal{S}) \subseteq M_\Omega(Q)$ in Proposition 2.1 could be proper.

Example 2.2. Let $(\mathbb{Z}_6, +)$ be the cyclic group of order 6, and $X = \{x, y\}$ where $\delta(q, x) = 2q$ and $\delta(q, y) = 2q + 3$ for all $q \in \mathbb{Z}_6$. Then the GS-automaton $\mathcal{S} = (\mathbb{Z}_6, X, \{x\}, \delta)$ is additive. Let $f \in M(\mathbb{Z}_6)$ where $qf = 5q$. Observe that Ω is generated multiplicatively by $\{\Psi, \mathrm{id}_{\mathbb{Z}_6}\}$ where $\Psi = \widetilde{2}$ and $\mathrm{id}_{\mathbb{Z}_6} = \widetilde{1}$ with $q\widetilde{k} = kq$ for all $k \in \mathbb{Z}$, $q \in \mathbb{Z}_6$. It follows that $\Omega = \left\{\widetilde{2}, \widetilde{4}, \widetilde{1}\right\}$. Evidently, $f \in M_\Omega(\mathbb{Z}_6)$. On the other hand, since $\delta(0, y)f = 3$ and $\delta(0f, x) = \delta(0, x) = 0$, thus $f \notin \mathcal{F}(\mathcal{S})$, and consequently, $\mathcal{F}(\mathcal{S}) \subsetneqq M_\Omega(Q)$.

The subsequent result characterizes the conditions such that $\mathcal{F}(\mathcal{S})$ is a centralizer near-ring $M_\Omega(Q)$.

Theorem 2.3. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be an additive GS-automaton and Ω be the semigroup generated by $\{\Psi, 1\}$. Then the followings are equivalent.*

(1) *$\mathcal{F}(\mathcal{S}) = M_\Omega(Q)$;*

(2) $x\Upsilon = 0$ (*i.e.*, $\delta(0,x) = 0$) *for all* $x \in X$;

(3) $\mathrm{cp}(0) = \{0\}$;

(4) $(\{0\}, X, \delta)$ *is an induced subsemiautomaton of* $\mathcal{S}$;

(5) *The identity mapping* $1 \in \mathcal{F}(\mathcal{S})$.

Proof. We first prove the equivalence of (1) and (2). Assume $\mathcal{F}(\mathcal{S}) = M_\Omega(Q)$. Then, in particular, the identity mapping $1 \in M_\Omega(Q) = \mathcal{F}(\mathcal{S})$. It follows that

$$\delta(q,x) = \delta(q,x)1 = q1\Psi = q\Psi$$

for all $q \in Q$, $x \in X$. Consequently, $x\Upsilon = 0$ for all $x \in X$ by Theorem 1.1. Conversely, assume $x\Upsilon = 0$ for all $x \in X$ and let $f \in M_\Omega(Q)$. Then $\delta(q,x)f = (q\Psi + x\Upsilon)f = q\Psi f = qf\Psi$. Thus $f \in \mathcal{F}(\mathcal{S})$. Hence $\mathcal{F}(\mathcal{S}) = M_\Omega(Q)$ by Proposition 2.1.

The equivalence of (2) and (3) follows from Theorem 1.2. Since $\mathcal{CP}(0) = (\mathrm{cp}(0), X, \delta)$ is the smallest induced subsemiautomaton of $\mathcal{S}$, the equivalence of (3) and (4) is immediate.

Finally, if the identity mapping $1 \in \mathcal{F}(\mathcal{S})$ then, in particular, $x\Upsilon = \delta(0,x) = \delta(0,x)1 = 01\Psi = 0$ for all $x \in X$. On the other hand, if $x\Upsilon = 0$ for all $x \in X$, then $\delta(q,x)1 = q\Psi + x\Upsilon = q1\Psi$, or $1 \in \mathcal{F}(\mathcal{S})$. Hence (5) and (2) are equivalent. □

A common question maybe asked: *When will* $\mathcal{F}(\mathcal{S})$ *be a ring or an abstract affine near-ring?* However, unlike the near-ring of homogeneous functions on a unital R-module, the near-ring $\mathcal{F}(\mathcal{S})$ discussed here is not zero-symmetric and the constant subnear-ring $\mathcal{F}(\mathcal{S})_c$ (if not zero) is not an ideal of $\mathcal{F}(\mathcal{S})$ in general even when $\mathcal{S}$ is additive and Q is a finite abelian group as shown in the following example.

Example 2.4. Let $\mathcal{S} = (\mathbb{Z}_6, \{x\}, \delta)$ with $\delta(q,x) = 4q$ for all $q \in \mathbb{Z}_6$. Then $\mathcal{S}$ is an additive GS-automaton and $\Psi \in \mathrm{End}(\mathbb{Z}_6)$. Let $f \in M(\mathbb{Z}_6)$ defined by $qf = q + 2$ for all $q \in \mathbb{Z}_6$. Then $\delta(q,x)f = 4q + 2 = 4(q+2) = qf\Psi$. It follows that $f \in \mathcal{F}(\mathcal{S})$ but f is not zero-symmetric. Also, the connected components in $\mathcal{S}$ are $C_1 = \{0,3\}$, $C_2 = \{1,4\}$, $C_3 = \{2,5\}$. The function f maps C_1 into C_3, C_2 into C_1 and C_3 into C_2. Thus the connected components are not invariant by the function $f \in \mathcal{F}(\mathcal{S})$.

Moreover, let $h, \theta_2 \in M(\mathbb{Z}_6)$ defined by $qh = 2$ if $q \in \{1,2,4,5\}$, $qh = q$ if $q \in \{0,3\}$ and $q\theta_2 = 2$ for all $q \in \mathbb{Z}_6$. Then $h, \theta_2 \in \mathcal{F}(\mathcal{S})$. Let $g = (h + \theta_2)h - h^2 \in \mathcal{F}(\mathcal{S})$. Observe that $0g = 2$ and $1g = 0$. Thus $g \notin \mathcal{F}(\mathcal{S})_c$ and $\mathcal{F}(\mathcal{S})_c$ is not an ideal of $\mathcal{F}(\mathcal{S})$. Consequently, $\mathcal{F}(\mathcal{S})$ is not an abstract affine near-ring.

For a given function $\varphi \in M(Q)$, let $\mathrm{Fix}(\varphi) = \{q \in Q \mid q\varphi = q\}$ be the set of fixed elements of φ in Q. If θ_a is a constant function in $\mathcal{F}(\mathcal{S})$, then for all $q \in Q$, $x \in X$,

$$a = \delta(q,x)\theta_a = q\theta_a\Psi = a\Psi.$$

This implies that $a \in \mathrm{Fix}(\Psi)$. The converse is also true, for if $a \in \mathrm{Fix}(\Psi)$, then $\delta(q,x)\theta_a = a$ and $q\theta_a\Psi = a\Psi = a$ for all $q \in Q$, $x \in X$. Thus the constant subnear-ring $\mathcal{F}(\mathcal{S})_c$ is characterized.

Proposition 2.5. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Then the constant subnear-ring of $\mathcal{F}(\mathcal{S})$ is $\mathcal{F}(\mathcal{S})_c = \{\theta_a \mid a \in \mathrm{Fix}(\Psi)\}$.*

Therefore $\mathcal{F}(\mathcal{S})$ is 0-symmetric if and only if $\mathrm{Fix}(\Psi) = \{0\}$ or, we say, Ψ is *fixed point free*. This condition is equivalent to $a\Omega = \{a\varphi \mid \varphi \in \Omega\} \neq \{a\}$ for all $a \in Q \setminus \{0\}$. Moreover, if $\theta_a \in \mathcal{F}(\mathcal{S})$ then $\theta_a \in M_\Omega(Q)$ by Proposition 2.1. A quick checking shows that the converse is also true.

Theorem 2.6. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive and Ω be the semigroup generated by $\{\Psi, 1\}$. Then the followings are equivalent.*

(1) *$a\Omega = \{a\varphi \mid \varphi \in \Omega\} \neq \{a\}$ for all $a \in Q \setminus \{0\}$;*

(2) *Ψ is fixed point free;*

(3) *$\mathcal{F}(\mathcal{S})$ is 0-symmetric;*

(4) *$M_\Omega(Q)$ is 0-symmetric.*

As a consequence, $\mathcal{F}(\mathcal{S})$ is 0-symmetric if Ψ is fixed point free. We may ask: *When is $\mathcal{F}(\mathcal{S}) = M_0(Q)$?* If $\delta(q,x) = 0$ for all $q \in Q$, $x \in X$ and $f \in M_0(Q)$, then $\delta(q,x)f = 0f = 0$ and $qf\Psi = \delta(qf, x_0) = 0$ by Theorem 1.1. Thus $\delta(q,x)f = qf\Psi$ and $f \in \mathcal{F}(\mathcal{S})$. Hence $\mathcal{F}(\mathcal{S}) = M_0(Q)$.

In fact, the condition $\delta(q,x) = 0$ for all $q \in Q$, $x \in X$ is also necessary for $\mathcal{F}(\mathcal{S}) = M_0(Q)$ as shown in the next result.

Theorem 2.7. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Then $\mathcal{F}(\mathcal{S}) = M_0(Q)$ if and only if $\delta(q,x) = 0$ for all $q \in Q$, $x \in X$.*

Proof. It remains to show the sufficiency. Assume $\mathcal{F}(\mathcal{S}) = M_0(Q)$. Then the identity map $1 \in \mathcal{F}(\mathcal{S})$ and so $x\Upsilon = 0$ for all $x \in X$ by Theorem 2.3. If $\Psi \neq 0$, then there exist nonzero $a, b \in Q$ such that $a\Psi = b$. Note that $a \neq b$ since Ψ is fixed point free by Theorem 2.6. Define a function $f\colon Q \to Q$ via

$$qf = \begin{cases} b & \text{if } q = b; \\ 0 & \text{otherwise.} \end{cases}$$

Apparently, $f \in M_0(Q) = \mathcal{F}(\mathcal{S})$. As a result,

$$b = bf = a\Psi f = \delta(a, x_0)f = af\Psi = 0\Psi = 0,$$

a contradiction. Hence $\Psi = 0$ and $\delta(q,x) = q\Psi + x\Upsilon = 0$ for all $q \in Q$, $x \in X$ by Theorem 1.1. This completes the proof. $\square$

Proposition 2.8. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Then $\mathcal{F}(\mathcal{S})_0 \lhd \mathcal{F}(\mathcal{S})$ if and only if $cf = 0$ for all $c \in \mathrm{Fix}(\Psi)$, $f \in \mathcal{F}(\mathcal{S})_0$.*

Proof. Let $\mathcal{F}(\mathcal{S})_0 \lhd \mathcal{F}(\mathcal{S})$. In particular, $\mathcal{F}(\mathcal{S})_0$ is left invariant. Let $f \in \mathcal{F}(\mathcal{S})_0$. Recall that the constant mapping $\theta_c \in \mathcal{F}(\mathcal{S})$ for all $c \in \mathrm{Fix}(\Psi)$ by Proposition 2.5, and thus $\theta_c f \in \mathcal{F}(\mathcal{S})_0$ for all $c \in \mathrm{Fix}(\Psi)$. Consequently, $cf = (0\theta_c)f = 0(\theta_c f) = 0$.

Conversely, assume $c\mathcal{F}(\mathcal{S})_0 = 0$ for all $c \in \mathrm{Fix}(\Psi)$. Since, as a near-ring, the 0-symmetric part $\mathcal{F}(\mathcal{S})_0$ is a right ideal of $\mathcal{F}(\mathcal{S})$, it suffices to show that $\mathcal{F}(\mathcal{S})_0$ is left invariant. Given $f \in \mathcal{F}(\mathcal{S})_0$ and $g \in \mathcal{F}(\mathcal{S})$. Note that $0g \in \mathrm{Fix}(\Psi)$, say $0g = d$ for some $d \in \mathrm{Fix}(\Psi)$. Then $0(gf) = (0g)f = df = 0$ by hypothesis. Hence $gf \in \mathcal{F}(\mathcal{S})_0$ and $\mathcal{F}(\mathcal{S})_0$ is left invariant. $\square$

Accordingly, consider the annihilator $(0 : \mathrm{Fix}(\Psi))$ in $\mathcal{F}(\mathcal{S})$ as

$$A(\mathcal{S}) = \{f \in \mathcal{F}(\mathcal{S}) \mid cf = 0 \text{ for all } c \in \mathrm{Fix}(\Psi)\}.$$

Since $0 \in A(\mathcal{S}) \subseteq \mathcal{F}(\mathcal{S})_0$, $A(\mathcal{S})$ is a nonempty subset of $\mathcal{F}(\mathcal{S})_0$. The annihilator $A(\mathcal{S})$ is an ideal of $\mathcal{F}(\mathcal{S})$ by a quick verification or by noting that $\mathrm{Fix}(\Psi)$ is an $\mathcal{F}(\mathcal{S})$-subgroup of Q. Therefore, if $\mathrm{Fix}(\Psi) \neq 0$ then $\mathcal{F}(\mathcal{S})$ is not 0-symmetric by Theorem 2.6 and, consequently, $A(\mathcal{S})$ is a proper ideal of $\mathcal{F}(\mathcal{S})$. The converse of this fact is stated as following result.

Corollary 2.9. *Let $\mathcal{S} = (Q, X, \delta)$ be additive. If $\mathcal{F}(\mathcal{S})$ is simple, then either the annihilator $A(\mathcal{S}) = 0$ or $\mathcal{F}(\mathcal{S})$ is 0-symmetric.*

In the proof of Proposition 2.8, it says that if I is a common ideal of $\mathcal{F}(\mathcal{S})_0$ and $\mathcal{F}(\mathcal{S})$ then $I \subseteq A(\mathcal{S})$. That is, $A(\mathcal{S})$ is the largest common ideal of $\mathcal{F}(\mathcal{S})_0$ and $\mathcal{F}(\mathcal{S})$. Therefore, $\mathcal{F}(\mathcal{S})_0 \lhd \mathcal{F}(\mathcal{S})$ if and only if $\mathcal{F}(\mathcal{S})_0 = A(\mathcal{S})$.

Proposition 2.10. *Let $\mathcal{S} = (Q, X, \delta)$ be additive with $\mathrm{cp}(0) = 0$. Assume Q is a monogenic $\mathcal{F}(\mathcal{S})$-group. Then $\mathcal{F}(\mathcal{S}) \subseteq \mathrm{End}(Q)$ if and only if $\mathcal{F}(\mathcal{S})$ is a ring.*

Proof. If $\mathcal{F}(\mathcal{S}) \subseteq \mathrm{End}(Q)$, then $\mathcal{F}(\mathcal{S})$ is distributive. Since $\mathrm{cp}(0) = 0$, the identity mapping is contained in $\mathcal{F}(\mathcal{S})$ by Theorem 2.3. Thus $\mathcal{F}(\mathcal{S})$ is a distributive near-ring with unity, and therefore $\mathcal{F}(\mathcal{S})$ is a ring.

Conversely, assume $\mathcal{F}(\mathcal{S})$ is a ring. Since Q is monogenic, $d\mathcal{F}(\mathcal{S}) = Q$ for some $d \in Q$. Let $a, b \in Q$, then there exist $g, h \in \mathcal{F}(\mathcal{S})$ such that $dg = a$ and $dh = b$. Let $f \in \mathcal{F}(\mathcal{S})$. Then

$$\begin{aligned}(a+b)f &= (dg + dh)f = d((g+h)f) = d(gf + hf) \\ &= dgf + dhf = af + bf.\end{aligned}$$

Hence f is a group endomorphism of Q, so $\mathcal{F}(\mathcal{S}) \subseteq \mathrm{End}(Q)$. $\square$

3. $\mathcal{F}(\mathcal{S})$ determined by graphical properties

In Proposition 2.5, an element $a \in \mathrm{Fix}(\Psi)$ will determine a constant mapping $\theta_a \in \mathcal{F}(\mathcal{S})$. Moreover, if $f \in \mathcal{F}(\mathcal{S})$, and $q \in \mathrm{reach}(a)$ then $q = \delta(a, w)$ for some $w \in X^*$. It follows that

$$qf = \delta(a, w)f = af\Psi^{|w|} = a\Psi^{|w|}f = af.$$

That is to say, the image of the successors of a by the mapping f in $\mathcal{F}(\mathcal{S})$ is completely determined by a. Also, from the identity $(af)\Psi = (a\Psi)f = af$ shows that the image of a under f is contained in the set $\mathrm{Fix}(\Psi)$. This phenomenon enables us to characterize the near-ring $\mathcal{F}(\mathcal{S})$ when $\mathcal{S}$ is totally disconnected or strongly connected.

Proposition 3.1. *Let $\mathcal{S} = (Q, X, \{x_0\}, \delta)$ be an additive GS-automaton.*

(1) *If $\mathcal{S}$ is totally disconnected, then $\mathcal{F}(\mathcal{S}) = M(Q)$.*

(2) *If $\mathrm{reach}(0) = Q$, then $\mathcal{F}(\mathcal{S})$ is a constant near-ring. In particular, when $\mathcal{S}$ is strongly connected, then $\mathcal{F}(\mathcal{S})$ is a constant near-ring.*

Proof. (1) If $\mathcal{S}$ is totally disconnected, then $\delta(q, x) = q$ for all $q \in Q$, $x \in X$. Thus $\delta(q, x)f = qf = \delta(qf, x_0)$ for all $f \in M(Q)$. Thus $\mathcal{F}(\mathcal{S}) = M(Q)$.

(2) Observe that if $f \in \mathcal{F}(\mathcal{S})$ then, given $w \in X^*$,

$$\delta(0, w)f = 0f\Psi^{|w|} = 0\Psi^{|w|}f = 0f.$$

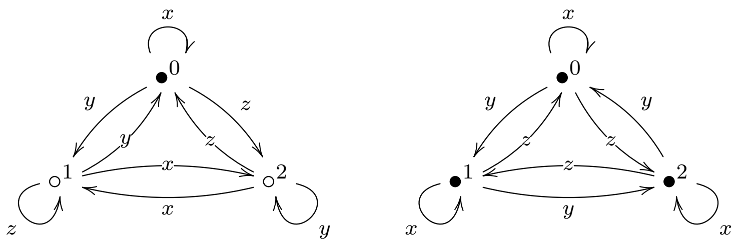
In other words, $qf = 0f$ for any $q \in \mathrm{reach}(0)$. Thus $\mathcal{F}(\mathcal{S})$ is a constant near-ring. Finally, if $\mathcal{S}$ is strongly connected, then $\mathrm{reach}(0) = Q$. □

The Proposition 3.1 shows that $\mathcal{F}(\mathcal{S})$ can not be zero if the additive GS-automaton $\mathcal{S}$ is totally disconnected and $|Q| \geq 2$. The following examples show that when $\mathcal{S}$ is strongly connected, the set $\mathcal{F}(\mathcal{S})$ could be zero or a nontrivial constant near-ring even when Q is a finite cyclic group.

Example 3.2. Let $(Q, +) = \mathbb{Z}_3$ be the cyclic group of order 3, and $X = \{x, y, z\}$. Define $\delta\colon Q \times X \to Q$ via $\delta(q, x) = 2q$, $\delta(q, y) = 2q + 1$ and $\delta(q, z) = 2q + 2$. Then $\mathcal{Z} = (\mathbb{Z}_3, X, \{x\}, \delta)$ is additive. The system $\mathcal{Z}$ is strongly connected and thus $\mathcal{F}(\mathcal{Z})$ is constant by Proposition 3.1. Since $\mathrm{Fix}(\Psi) = \{0\}$, we have $\mathcal{F}(\mathcal{Z})$ is 0-symmetric by Theorem 2.6. Thus $\mathcal{F}(\mathcal{Z}) = \{0\}$.

Moreover, define $\eta\colon Q \times X \to Q$ via $\eta(q, x) = q$, $\eta(q, y) = q+1$ and $\eta(q, z) = q+2$. Then $\mathcal{Y} = (\mathbb{Z}_3, X, \{x\}, \eta)$ is additive. The system $\mathcal{Y}$ is strongly connected and $\mathcal{F}(\mathcal{Y}) = M_c(\mathbb{Z}_3)$

by Propositions 3.1 and 2.5.



Digraph of $\mathcal{Z} = (\mathbb{Z}_3, X, \delta)$ Digraph of $\mathcal{Y} = (\mathbb{Z}_3, X, \eta)$

Example 3.2 motivates the following result.

Corollary 3.3. *Let $\mathcal{S} = (Q, X, (\Psi, \Upsilon))$ be additive and strongly connected. Then $\mathcal{F}(\mathcal{S}) = \{0\}$ if and only if Ψ is fixed point free.*

Proof. Assume that $\mathcal{S} = (Q, X, (\Psi, \Upsilon))$ is additive and strongly connected. Then $\mathcal{F}(\mathcal{S})$ is a constant near-ring by Proposition 3.1(2). Therefore $\mathcal{F}(\mathcal{S}) = \mathcal{F}(\mathcal{S})_c = \{\theta_a \mid a \in \mathrm{Fix}(\Psi)\}$ by Proposition 2.5. Immediately, we have that $\mathcal{F}(\mathcal{S}) = \{0\}$ if and only if Ψ is fixed point free. $\square$

Given a group $(G, +)$, consider its associated GS-automaton $\mathcal{G} = (G, G, \delta)$. For any given $p, q \in G$, $\delta(p, -p+q) = q$ and so $\mathcal{G}$ is strongly connected. Since $\mathcal{G}$ is additive with $\Psi = \Upsilon = \mathrm{id}_G$, $\mathrm{Fix}(\Psi) = G$. It follows that $\mathcal{F}(\mathcal{G}) = M_c(G)$ by Propositions 3.1(2) and 2.5.

Corollary 3.4. *Let $\mathcal{G} = (G, G, \delta)$ be the associated GS-automaton for a given group G with $\delta(q, x) = q + x$ for all $q, x \in G$. Then $\mathcal{F}(\mathcal{G}) = M_c(G)$ the near-ring of constant mappings on G.*

The converse of Proposition 3.1(1) is true, as shown in Theorem 3.5, but the converse of Proposition 3.1(2) is false in general, as expressed in Example 3.6. The following results in this section will demonstrate this assertion.

Theorem 3.5. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Then the followings are equivalent.*

(1) *$\mathcal{S}$ is totally disconnected;*

(2) *$\mathcal{F}(\mathcal{S}) = M(Q)$;*

(3) *$\delta(q, x) = q$ for all $q \in Q$, $x \in X$.*

Proof. The sufficiency of (1) implying (2) follows from Proposition 3.1(1). We may suppose that $|Q| \geq 2$. If there exist $a \neq b \in Q$, $x \in X$ such that $\delta(a, x) = b$, then $bf = \delta(a, x)f = af\Psi$ for all $f \in M(Q)$. Pick a nonzero $c \in Q$ and let $g \in M(Q)$ such that $ag = 0$ and $bg = c$. Consequently, $c = bg = ag\Psi = 0$, a contradiction. Hence $\mathcal{S}$ is totally disconnected and (2) implies (1).

The equivalence of (1) and (3) is immediate. $\square$

The following example provides an additive GS-automaton $\mathcal{S}$, where $\mathcal{F}(\mathcal{S})$ is a nontrivial constant near-ring but in $\mathcal{S}$, $\operatorname{reach}(0) \neq Q$ (and thus $\mathcal{S}$ is not strongly connected). Therefore the converse of Proposition 3.1(2) is not true in general.

Example 3.6. [6, Example 2.2] Let $(Q,+) = \mathbb{Z}$ be the group of integers, and $X = \{x, y\}$. Define $\delta\colon \mathbb{Z} \times X \to \mathbb{Z}$ via $\delta(q,x) = q$ and $\delta(q,y) = q+1$. Then $\mathcal{S} = (Q, X, \{x\}, \delta)$ is additive. Observe that $\operatorname{reach}(0) = \{0\} \cup \mathbb{N} \neq \mathbb{Z}$, $\operatorname{Fix}(\Psi) = \mathbb{Z}$, and $\Omega = \{1\}$.

Let $f \in \mathcal{F}(\mathcal{S})$. Then $\delta(q,y)f = qf\Psi$ implies $(q+1)f = qf$ for all $q \in Q$. Thus $\mathcal{F}(\mathcal{S})$ is a constant near-ring. Indeed $\mathcal{F}(\mathcal{S}) = M_c(\mathbb{Z})$ by Proposition 2.5.

$$\begin{tikzcd}
\cdots \arrow[r, "y"] & \bullet^{-2} \arrow[r, "y"] \arrow[loop below, "x"] & \bullet^{-1} \arrow[r, "y"] \arrow[loop below, "x"] & \bullet^{0} \arrow[r, "y"] \arrow[loop below, "x"] & \bullet^{1} \arrow[r, "y"] \arrow[loop below, "x"] & \bullet^{2} \arrow[r, "y"] \arrow[loop below, "x"] & \cdots
\end{tikzcd}$$

Digraph of $\mathcal{S} = (\mathbb{Z}, X, \delta)$

The example demonstrated in Example 3.6 is connected, which turns out to be true when $\mathcal{F}(\mathcal{S})$ is a nontrivial constant near-ring as presented in the following result.

Proposition 3.7. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive and suppose that $\mathcal{F}(\mathcal{S}) \neq 0$. If $\mathcal{F}(\mathcal{S})$ is a constant near-ring, then $\mathcal{S}$ is connected.*

Proof. Assume that $\mathcal{F}(\mathcal{S})$ is a nontrivial constant near-ring. Then there is a nonzero $a \in \operatorname{Fix}(\Psi)$ by Theorem 2.6. If $\mathcal{S}$ is not connected, then let $C_1 = \operatorname{conn}(0)$ and $C_2 = Q \setminus C_1$. We will construct a nonconstant element in $\mathcal{F}(\mathcal{S})$. Define a function $f \in M(Q)$ such that

$$qf = \begin{cases} a & \text{if } q \in C_1; \\ 0 & \text{if } q \in C_2. \end{cases}$$

Case I. $a \in C_1$. If $q \in C_1$ then $\delta(q,x) \in C_1$ since C_1 is a connected component. Thus $\delta(q,x)f = a = a\Psi = qf\Psi$. If $q \in C_2$ then $\delta(q,x) \in C_2$, otherwise q is adjacent to an element in C_1, contradicting to the choice of q. Thus $\delta(q,x)f = 0 = qf\Psi$.

Case II. $a \in C_2$. Use a similar argument as used in Case I to see that if $q \in C_1$ then $\delta(q,x)f = a = qf\Psi$. If $q \in C_2$ then $\delta(q,x)f = 0 = qf\Psi$. Therefore $f \in \mathcal{F}(\mathcal{S})$ but f is not a constant function. Hence $\mathcal{S}$ is connected. $\square$

The method used in Proposition 3.7 can be generalized to construct elements in $\mathcal{F}(\mathcal{S})$ when $\mathcal{S}$ is not connected and $\operatorname{Fix}(\Psi) \neq \{0\}$. Explicitly, let $\{C_i\}_{i\in I}$ be the collection of all distinct connected components in $\mathcal{S}$ and $\{a_j\}_{j\in J} = \operatorname{Fix}(\Psi)$. Let $\sigma\colon I \to J$ be any function and define $f\colon Q \to Q$ via

$$qf = a_{i\sigma} \quad \text{if } q \in C_i.$$

Since q and $\delta(q,x)$ are in the same connected component, $qf = \delta(q,x)f = a_{i\sigma}$ for all $q \in C_i$, $x \in X$. Thus $qf\Psi = a_{i\sigma} = \delta(q,x)f$, and consequently, $f \in \mathcal{F}(\mathcal{S})$. When I, J are finite, the cardinality $|\mathcal{F}(\mathcal{S})| \geq |J|^{|I|}$. We write this as the following result.

Proposition 3.8. *Let $\mathcal{S} = (Q, X, (\Psi, \Upsilon))$ be additive. If the graph of $\mathcal{S}$ has n distinct connected components and the mapping Ψ has m fixed points, then the near-ring $\mathcal{F}(\mathcal{S})$ contains at least m^n elements.*

It is interesting that Proposition 3.8 can be used to give a combinatorial prove for (1) implying (2) in Theorem 3.5 when Q is finite. Let $|Q| = n$. If $\mathcal{S}$ is totally disconnected, then $\mathcal{S}$ has n connected components and Ψ has n fixed points. It follows that the order $|\mathcal{F}(\mathcal{S})| \geq n^n = |M(Q)|$ by Proposition 3.8 and, consequently, $\mathcal{F}(\mathcal{S}) = M(Q)$.

The following technical lemma is frequently used.

Lemma 3.9. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Suppose $qf = c \in \mathrm{Fix}(\Psi)$ for some $q \in Q$, $f \in \mathcal{F}(\mathcal{S})$. Let a be a predecessor of q and b be a successor of q. Then*

(1) *$af = c$ if Ψ is an automorphism of the group Q.*

(2) *$bf = c$.*

Proof. Let $f \in \mathcal{F}(\mathcal{S})$ and $a, b \in Q$ where a is a predecessor of q and b is a successor of q. Then there are $\mu, \nu \in X^*$ such that $\delta(a, \mu) = q$ and $\delta(q, \nu) = b$. Observe that

$$bf = \delta(q, \nu)f = qf\Psi^{|\nu|} = c\Psi^{|\nu|} = c$$

and

$$af\Psi^{|\mu|} = \delta(a, \mu)f = qf = c = c\Psi^{|\mu|}.$$

Since $\Psi^{|\mu|} \in \mathrm{Aut}(Q)$, thus $af\Psi^{|\mu|} = c\Psi^{|\mu|}$ implies $af = c$. □

Theorem 3.10. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive and Ψ is a group automorphism of Q. Assume that $\mathcal{F}(\mathcal{S}) \neq 0$. Then $\mathcal{S}$ is connected if and only if $\mathcal{F}(\mathcal{S})$ is a constant near-ring.*

Proof. Suppose that $\mathcal{S}$ is connected. Let $f \in \mathcal{F}(\mathcal{S})$. Note that $(0f)\Psi = (0\Psi)f = 0f$ for all $x \in X$ by Proposition 2.1. Hence $0f \in \mathrm{Fix}(\Psi)$, say $0f = c$ for some $c \in \mathrm{Fix}(\Psi)$.

Let $C_1 = \mathrm{reach}(0) \cup \{\text{predecessors of } 0\}$. Then for all $q \in C_1$, $qf = c$ by Lemma 3.9. Inductively, let

$$C_i = C_{i-1} \cup \{q \in Q \mid q \text{ is adjacent to or from some element in } C_{i-1}\}.$$

Then $qf = c$ for all $q \in C_i$ by Lemma 3.9. Since $\mathcal{S}$ is connected, and input words in $\mathcal{S}$ have finite length, thus $\bigcup_{i \in \mathbb{N}} C_i = Q$. Therefore $qf = c$ for all $q \in Q$ and f is constant.

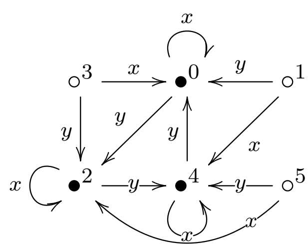
The converse also holds from Proposition 3.7. □

The next example shows that the hypothesis assuming Ψ is a group automorphism in Lemma 3.9(1) and in Theorem 3.10 is not superfluous.

Example 3.11. Let $(Q,+) = \mathbb{Z}_6$ be the cyclic group of order 6, and $X = \{x, y\}$. Define $\delta\colon Q \times X \to Q$ via $\delta(q,x) = 4q$ and $\delta(q,y) = 4q+2$. The system $\mathcal{S} = (\mathbb{Z}_6, X, \{x\}, \delta)$ is additive. The digraph of $\mathcal{S}$ is connected but not strongly connected, and $\mathrm{Fix}(\Psi) = \{0,2,4\}$. Define a function $f \in M(Q)$ via $qf = 3q$. Then $\delta(q,x)f = 12q = 0 = qf\Psi$ and $\delta(q,y)f = 3(4q+2) = 0 = qf\Psi$. Thus $f \in \mathcal{F}(\mathcal{S})$, and $\mathcal{F}(\mathcal{S})$ is not a constant near-ring.

Also, note that 3 is a predecessor of 2 where $2f = 0 \in \mathrm{Fix}(\Psi)$ but $3f = 3 \neq 0$.

In this case, $\mathrm{End}(Q) \cong \mathbb{Z}_6$, $\mathcal{F}(\mathcal{S})_0 = \{0, f\} \subseteq \mathrm{End}(Q)$, $\mathcal{F}(\mathcal{S})_c = \{\theta_0 = 0, \theta_2, \theta_4\}$ and $\mathcal{F}(\mathcal{S})_0 \lhd \mathcal{F}(\mathcal{S})$.



Digraph of $\mathcal{S} = (\mathbb{Z}_6, X, \delta)$

It is now appropriate to answer the question: *When will $\mathcal{F}(\mathcal{S}) = M_c(Q)$ for an additive GS-automaton $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$?* Recall that $\mathcal{F}(\mathcal{S})_c = \{\theta_a \mid a \in \mathrm{Fix}(\Psi)\}$ by Proposition 2.5. It follows that $\mathrm{Fix}(\Psi) = Q$ or $\Psi = 1$ be the identity map of Q is necessary for $\mathcal{F}(\mathcal{S}) = M_c(Q)$.

By letting $\Psi = 1$, then the hypothesis of Theorem 3.10 is fulfilled and thus $\mathcal{F}(\mathcal{S})$ is a constant near-ring equals to $M_c(Q)$ if $\mathcal{S}$ is connected and vice versa.

Recall that there is a directed edge from q to p if there is an alphabet $x \in X$ such that $\delta(q,x) = p$. In this case, $\delta(q,x) = q + x\Upsilon$ and so $q + x\Upsilon = p$ or $q = p - x\Upsilon$ is equivalent to q is adjacent to p or p is adjacent from q. If $\mathcal{S}$ is connected then, for any given $q \in Q$, there is a route (undirected) connects 0 and q. Explicitly, say $q_0 = 0$, $q_1, \ldots, q_n = q$ is this route where each pairs q_i, q_{i+1} are adjacent for $i = 0, 1, \ldots, n-1$. If q_i is adjacent to (from) q_{i+1}, then $q_i + x_i\Upsilon = q_{i+1}$ ($q_i - x_i\Upsilon = q_{i+1}$) for $i = 0, 1, \ldots, n-1$, respectively. By abusing the notation, we write this as $q_i + (\pm)x_i\Upsilon = q_{i+1}$ for $i = 0, 1, \ldots, n-1$, where $(\pm)$ means *plus* or *minus* depending on q_i is adjacent to or from q_{i+1}. Inductively, by adding these n identities $q_i + (\pm)x_i\Upsilon = q_{i+1}$ for $i = 0, 1, \ldots, n-1$, we get

$$\sum_{i=0}^{n-1}(\pm)x_i\Upsilon = q_0 + \sum_{i=0}^{n-1}(\pm)x_i\Upsilon = q_n = q.$$

Thus $q \in \mathrm{cp}(0)$, or equivalently, $Q = \mathrm{cp}(0)$.

Conversely, if $Q = \mathrm{cp}(0)$ then, for any given $q \in Q$, $q = \sum_{j=1}^{n}(\pm)x_j\Upsilon$ where $x_j \in X$ for all $j = 1, 2, \ldots, n$ by Theorem 1.2. That means there is a route (undirected) from 0 to q and thus $\mathcal{S}$ is connected. We write this as the following.

Theorem 3.12. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Then the followings are equivalent.*

(1) $\mathcal{F}(\mathcal{S}) = M_c(Q)$;

(2) $\delta(q, x) = q + x\Upsilon$ *for all* $q \in Q$, $x \in X$ *and* $\mathcal{S}$ *is connected;*

(3) $\delta(q, x) = q + x\Upsilon$ *for all* $q \in Q$, $x \in X$ *and* $Q = \mathrm{cp}(0)$.

Motivated by Example 3.11 in which the 0-symmetric subnear-ring is an ideal of $\mathcal{F}(\mathcal{S})$, we have the following characterization.

Proposition 3.13. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Assume $\mathcal{F}(\mathcal{S})$ is not 0-symmetric. If $\mathcal{F}(\mathcal{S})$ is simple, then each connected component of $\mathcal{S}$ contains at least one element in* $\mathrm{Fix}(\Psi)$.

Proof. Since $\mathcal{F}(\mathcal{S})$ is not 0-symmetric, $\mathrm{Fix}(\Psi) \neq 0$ by Theorem 2.6. Pick a nonzero $d \in \mathrm{Fix}(\Psi)$. To the contrary, assume there is a connected component C contains no element in $\mathrm{Fix}(\Psi)$. Note that $Q \setminus C$ is not empty for $\mathrm{conn}(0) \subseteq Q \setminus C$. Define a function $f \colon Q \to Q$ via

$$qf = \begin{cases} d & \text{if } q \in C; \\ 0 & \text{otherwise.} \end{cases}$$

Note that both q and $\delta(q, x)$ are in the same component. If $q \in C$, then $\delta(q, x)f = d = qf\Psi$ for all $x \in X$. If $q \notin C$, then $\delta(q, x)f = 0 = qf\Psi$. Hence $f \in \mathcal{F}(\mathcal{S})$. Note that $f \in A(\mathcal{S}) = \{f \in \mathcal{F}(\mathcal{S}) \mid cf = 0 \text{ for all } c \in \mathrm{Fix}(\Psi)\}$ and thus $A(\mathcal{S})$ is a nonzero ideal of $\mathcal{F}(\mathcal{S})$. This contradicts the simplicity of $\mathcal{F}(\mathcal{S})$. □

The converse of Proposition 3.13 is false in general. In Example 3.11, $\mathcal{S}$ is connected and $\mathcal{F}(\mathcal{S})$ is not 0-symmetric but $\mathcal{F}(\mathcal{S})$ is not simple. Seeing that $\mathcal{F}(\mathcal{S})_0$ is simple in this example and recall that $A(\mathcal{S})$ is also an ideal of $\mathcal{F}(\mathcal{S})_0$, we modify Proposition 3.13 as follows.

Proposition 3.14. *Let $\mathcal{S} = (Q, X, \delta = (\Psi, \Upsilon))$ be additive. Assume $\mathcal{F}(\mathcal{S})$ is not 0-symmetric. If $\mathcal{F}(\mathcal{S})_0$ is simple, then each connected component of $\mathcal{S}$ contains at least one element in* $\mathrm{Fix}(\Psi)$.

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