TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 2, pp. 311–332, April 2016
DOI: 10.11650/tjm.20.2016.5722
This paper is available online at http://journal.tms.org.tw

Meridian Surfaces of Elliptic or Hyperbolic Type with Pointwise 1-type Gauss Map in Minkowski 4-space

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Abstract. In the present paper we consider a special class of spacelike surfaces in the Minkowski 4-space which are one-parameter systems of meridians of the rotational hypersurface with timelike or spacelike axis. They are called meridian surfaces of elliptic or hyperbolic type, respectively. We study these surfaces with respect to their Gauss map. We find all meridian surfaces of elliptic or hyperbolic type with harmonic Gauss map and give the complete classification of meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map.

1. Introduction

The study of submanifolds of Euclidean space or pseudo-Euclidean space via the notion of finite type immersions began in the late 1970's with the papers [6,8] of B.-Y. Chen and has been extensively carried out since then. An isometric immersion $x\colon M \to \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ (or pseudo-Euclidean space $\mathbb{E}^m_s$) is said to be of *finite type* [6], if x identified with the position vector field of M in $\mathbb{E}^m$ (or $\mathbb{E}^m_s$) can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

where x_0 is a constant map, $x_1, x_2, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, then M is said to be of *k-type*. Many results on finite type immersions have been collected in the survey paper [9].

The notion of finite type immersion is naturally extended to the Gauss map G on M by B.-Y. Chen and P. Piccinni [11]. Thus, a submanifold M of an Euclidean (or pseudo-Euclidean space) is said to have *1-type Gauss map* G, if G satisfies $\Delta G = a(G + C)$ for some $a \in \mathbb{R}$ and some constant vector C (see, for example, [3–5, 17]).

Received December 30, 2014, accepted August 3, 2015.
Communicated by Sai-Kee Yeung.
2010 *Mathematics Subject Classification*. Primary: 53A35; Secondary: 53B20, 53B25.
Key words and phrases. Meridian surfaces in Minkowski space, Finite type immersions, Harmonic Gauss map, Pointwise 1-type Gauss map.
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However, the Laplacian of the Gauss map of some well-known surfaces such as the helicoid, the catenoid, and the right cone in the Euclidean 3-space $\mathbb{E}^3$, the helicoids of 1st, 2nd, and 3rd kind, conjugate of Enneper's surface of 2nd kind and B-scrolls in the Minkowski 3-space $\mathbb{E}^3_1$, the generalized catenoids, and Enneper's hypersurfaces in $\mathbb{E}^{n+1}_1$ takes a somewhat different form, namely, $\Delta G = \lambda(G + C)$ for some non-constant smooth function λ and some constant vector C. Therefore, it is worth studying the class of surfaces satisfying such an equation.

We use the following definition: a submanifold M of the Euclidean space $\mathbb{E}^m$ (or pseudo-Euclidean space $\mathbb{E}^m_s$) is said to have *pointwise* 1*-type Gauss map* if its Gauss map G satisfies

$$\Delta G = \lambda(G + C)$$

for some non-zero smooth function λ on M and some constant vector C. A pointwise 1-type Gauss map is called *proper* if the function λ is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of *first kind* if the vector C is zero. Otherwise, it is said to be of *second kind* [10].

Classification results on surfaces with pointwise 1-type Gauss map in Minkowski space have been obtained in the last few years. For example, in [20] Y. Kim and D. Yoon studied ruled surfaces with 1-type Gauss map in Minkowski space $\mathbb{E}^m_1$ and gave a complete classification of null scrolls with 1-type Gauss map. The classification of ruled surfaces with pointwise 1-type Gauss map of first kind in Minkowski space $\mathbb{E}^3_1$ is given in [18]. Ruled surfaces with pointwise 1-type Gauss map of second kind in Minkowski 3-space were classified in [12]. The complete classification of flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ is given in [19].

Basic source of examples of surfaces in the four-dimensional Euclidean or Minkowski space are the meridian surfaces. Meridian surfaces in the Euclidean 4-space $\mathbb{R}^4$ are defined in [13] as special class of surfaces, which are one-parameter systems of meridians of the standard rotational hypersurface in $\mathbb{R}^4$. In [2] we studied the meridian surfaces with pointwise 1-type Gauss map. We showed that a meridian surface in $\mathbb{R}^4$ has a harmonic Gauss map if and only if it is part of a plane. We gave necessary and sufficient conditions for a meridian surface to have pointwise 1-type Gauss map and found all meridian surfaces with pointwise 1-type Gauss map of first and second kind. The meridian surfaces of Weingarten type are described in [1].

The meridian surfaces of elliptic or hyperbolic type in the Minkowski 4-space $\mathbb{R}^4_1$ are constructed in [14] similarly to the Euclidean case. They are two-dimensional spacelike surfaces in $\mathbb{R}^4_1$ which are one-parameter systems of meridians of the rotational hypersurface with timelike or spacelike axis, respectively. Recently, some special classes of meridian surfaces of elliptic or hyperbolic type have been classified. For example, marginally trapped meridian surfaces of elliptic or hyperbolic type are described in [14]. The complete classification of meridian surfaces of elliptic or hyperbolic type with constant Gauss curvature or with constant mean curvature is given in [16]. The Chen meridian surfaces and the meridian surfaces with parallel normal bundle are also classified in [16].

In the present paper we study meridian surfaces of elliptic or hyperbolic type in $\mathbb{R}^4_1$ with respect to their Gauss map. In Theorem 4.1 and Theorem 4.2 we describe all meridian surfaces of elliptic or hyperbolic type with harmonic Gauss map. We give the complete classification of meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of first kind in Theorem 5.1 and Theorem 5.2, respectively. The meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of second kind are classified in Theorem 6.1 and Theorem 6.2, respectively.

2. Preliminaries

Let $\mathbb{R}^4_1$ be the four-dimensional Minkowski space endowed with the metric $\langle \, , \rangle$ of signature $(3,1)$ and $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system such that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, $\langle e_4, e_4\rangle = -1$. The standard flat metric is given in local coordinates by $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$.

A surface M in $\mathbb{R}^4_1$ is said to be *spacelike* if $\langle \, , \rangle$ induces a Riemannian metric g on M. Thus at each point p of a spacelike surface M we have the following decomposition:

$$\mathbb{R}^4_1 = T_pM \oplus N_pM$$

with the property that the restriction of the metric $\langle \, , \rangle$ onto the tangent space T_pM is of signature $(2,0)$, and the restriction of the metric $\langle \, , \rangle$ onto the normal space N_pM is of signature $(1,1)$.

We denote by ∇' and ∇ the Levi Civita connections on $\mathbb{R}^4_1$ and M, respectively. Let x and y be vector fields tangent to M and ξ be a normal vector field. The formulas of Gauss and Weingarten giving the decompositions of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into tangent and normal components are given, respectively, by

$$\begin{aligned} \nabla'_x y &= \nabla_x y + \sigma(x,y); \\ \nabla'_x \xi &= -A_\xi x + D_x \xi, \end{aligned}$$

where σ is the second fundamental tensor, D is the normal connection, and A_ξ is the shape operator with respect to ξ.

The mean curvature vector field H of M is defined as $H = \frac{1}{2}\operatorname{tr}\sigma$. A submanifold M is said to be minimal (respectively, totally geodesic) if $H = 0$ (respectively, $\sigma = 0$). A surface M in the Minkowski 4-space is called *marginally trapped* [7], if its mean curvature vector field H is lightlike at each point, i.e., $H \neq 0$, $\langle H, H\rangle = 0$.

The Gauss map G of a submanifold M of $\mathbb{E}^m$ is defined as follows. Let $G(n,m)$ be the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ and $\wedge^n\mathbb{E}^m$ be the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n\mathbb{E}^m$ with the Euclidean space $\mathbb{E}^N$, where $N = \binom{m}{n}$. Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ be a local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, \dots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ are normal to M. The map $G\colon M \to G(n,m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \dots \wedge e_n)(p)$ is called the *Gauss map* of M. It is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}^m$ obtained by the parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

In a similar way one can consider the Gauss map of a submanifold M of pseudo-Euclidean space $\mathbb{E}^m_s$.

If M is a spacelike submanifold of $\mathbb{E}^m_s$, then for any function f on M the Laplacian of f is given by the formula

$$\Delta f = -\sum_i \left(\nabla'_{e_i} \nabla'_{e_i} f - \nabla'_{\nabla_{e_i} e_i} f \right),$$

where ∇' is the Levi-Civita connection of $\mathbb{E}^m_s$ and ∇ is the induced connection on M.

3. Meridian surfaces of elliptic or hyperbolic type

Meridian surfaces in the Minkowski space $\mathbb{E}^4_1$ are special families of two-dimensional spacelike surfaces lying on rotational hypersurfaces in $\mathbb{R}^4_1$ with timelike or spacelike axis, which are constructed as follows.

Let $f = f(u)$, $g = g(u)$ be smooth functions, defined on an interval $I \subset \mathbb{R}$, such that $(f'(u))^2 - (g'(u))^2 > 0$, $u \in I$. We assume that $f(u) > 0$, $u \in I$. The standard rotational hypersurface $\mathcal{M}'$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m\colon u \to (f(u), g(u))$ about the Oe_4-axis, is parameterized as follows:

$$\begin{aligned} \mathcal{M}' : Z(u, w^1, w^2) = {} & f(u) \cos w^1 \cos w^2 \, e_1 + f(u) \cos w^1 \sin w^2 \, e_2 \\ & + f(u) \sin w^1 \, e_3 + g(u) \, e_4. \end{aligned}$$

The rotational hypersurface $\mathcal{M}'$ is a two-parameter system of meridians. If $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$, we can consider the two-dimensional surface $\mathcal{M}'_m$ lying on $\mathcal{M}'$, constructed in the following way:

$$\mathcal{M}'_m : z(u,v) = Z(u, w^1(v), w^2(v)), \quad u \in I, \, v \in J.$$

Since $\mathcal{M}'_m$ is a one-parameter system of meridians of $\mathcal{M}'$, it is called a *meridian surface of elliptic type* [14].

If we denote $l(w^1, w^2) = \cos w^1 \cos w^2\, e_1 + \cos w^1 \sin w^2\, e_2 + \sin w^1\, e_3$, then the surface $\mathcal{M}'_m$ is parameterized by

$$\mathcal{M}'_m : z(u,v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\ v \in J. \tag{3.1}$$

Note that $l(w^1, w^2)$ is the unit position vector of the 2-dimensional sphere $S^2(1)$ lying in the Euclidean space $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$ and centered at the origin O.

We assume that the smooth curve $c : l = l(v) = l(w^1(v), w^2(v)),\ v \in J$ on $S^2(1)$ is parameterized by the arc-length, i.e., $\langle l'(v), l'(v)\rangle = 1$. Let $t(v) = l'(v)$ be the tangent vector field of c. Since $\langle t(v), t(v)\rangle = 1$, $\langle l(v), l(v)\rangle = 1$, and $\langle t(v), l(v)\rangle = 0$, there exists a unique (up to a sign) vector field $n(v)$, such that $\{l(v), t(v), n(v)\}$ is an orthonormal frame field in $\mathbb{R}^3$. With respect to this frame field we have the following Frenet formulas of c on $S^2(1)$:

$$\begin{aligned} l' &= t; \\ t' &= \kappa\, n - l; \\ n' &= -\kappa\, t, \end{aligned} \tag{3.2}$$

where $\kappa(v) = \langle t'(v), n(v)\rangle$ is the spherical curvature of c.

Without loss of generality we assume that $(f'(u))^2 - (g'(u))^2 = 1$. The tangent space of $\mathcal{M}'_m$ is spanned by the vector fields:

$$z_u = f'\, l + g'\, e_4; \quad z_v = f\, t,$$

so, the coefficients of the first fundamental form of $\mathcal{M}'_m$ are $E = 1$; $F = 0$; $G = f^2(u) > 0$. Hence, the first fundamental form is positive definite, i.e., $\mathcal{M}'_m$ is a spacelike surface.

Denote $x = z_u$, $y = \frac{z_v}{f} = t$ and consider the following orthonormal normal frame field:

$$n_1 = n(v); \quad n_2 = g'(u)\, l(v) + f'(u)\, e_4.$$

Thus we obtain a frame field $\{x, y, n_1, n_2\}$ of $\mathcal{M}'_m$, such that $\langle n_1, n_1\rangle = 1$, $\langle n_2, n_2\rangle = -1$, $\langle n_1, n_2\rangle = 0$.

Taking into account (3.2) we get the following derivative formulas [16]:

$$\begin{aligned} &\nabla'_x x = \kappa_m\, n_2; && \nabla'_x n_1 = 0; \\ &\nabla'_x y = 0; && \nabla'_y n_1 = -\frac{\kappa}{f} y; \\ &\nabla'_y x = \frac{f'}{f} y; && \nabla'_x n_2 = \kappa_m\, x; \\ &\nabla'_y y = -\frac{f'}{f} x + \frac{\kappa}{f} n_1 + \frac{g'}{f} n_2; && \nabla'_y n_2 = \frac{g'}{f} y, \end{aligned} \tag{3.3}$$

where $\kappa_m(u) = f'(u)g''(u) - g'(u)f''(u)$ is the curvature of the meridian curve m, and $\kappa = \kappa(v)$ is the spherical curvature of c.

In a similar way one can consider meridian surfaces lying on the rotational hypersurface in $\mathbb{R}^4_1$ with spacelike axis. Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $(f'(u))^2 + (g'(u))^2 > 0$, $f(u) > 0$, $u \in I$. The rotational hypersurface $\mathcal{M}''$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m\colon u \to (f(u), g(u))$ about the Oe_1-axis is parameterized as follows:

$$\begin{aligned}\mathcal{M}'' : Z(u, w^1, w^2) = g(u)\, e_1 + f(u) \cosh w^1 \cos w^2\, e_2 \\ + f(u) \cosh w^1 \sin w^2\, e_3 + f(u) \sinh w^1\, e_4.\end{aligned}$$

If $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$, we consider the surface $\mathcal{M}''_m$ in $\mathbb{R}^4_1$ defined by

$$\mathcal{M}''_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I,\, v \in J.$$

$\mathcal{M}''_m$ is a one-parameter system of meridians of $\mathcal{M}''$ and is called a *meridian surface of hyperbolic type* [14].

If we denote $l(w^1, w^2) = \cosh w^1 \cos w^2\, e_2 + \cosh w^1 \sin w^2\, e_3 + \sinh w^1\, e_4$, then the surface $\mathcal{M}''_m$ is given by

$$\mathcal{M}''_m : z(u, v) = f(u)\, l(v) + g(u)\, e_1, \quad u \in I,\, v \in J, \tag{3.4}$$

$l(w^1, w^2)$ being the unit position vector of the timelike sphere $S^2_1(1)$ in the Minkowski space $\mathbb{R}^3_1 = \operatorname{span}\{e_2, e_3, e_4\}$, i.e., $S^2_1(1) = \left\{V \in \mathbb{R}^3_1 : \langle V, V\rangle = 1\right\}$. $S^2_1(1)$ is a timelike surface in $\mathbb{R}^3_1$ known also as the de Sitter space.

Assume that the curve $c : l = l(v) = l(w^1(v), w^2(v))$, $v \in J$ on $S^2_1(1)$ is parameterized by the arc-length, i.e., $\langle l'(v), l'(v)\rangle = 1$. Similarly to the elliptic case we consider an orthonormal frame field $\{l(v), t(v), n(v)\}$ in $\mathbb{R}^3_1$, such that $t(v) = l'(v)$ and $\langle n(v), n(v)\rangle = -1$. With respect to this frame field we have the following decompositions of the vector fields $l'(v)$, $t'(v)$, $n'(v)$:

$$\begin{aligned} l' &= t; \\ t' &= -\kappa\, n - l; \\ n' &= -\kappa\, t, \end{aligned} \tag{3.5}$$

which can be considered as Frenet formulas of c on $S^2_1(1)$. The function $\kappa(v) = \langle t'(v), n(v)\rangle$ is the spherical curvature of c on $S^2_1(1)$.

We assume that $(f'(u))^2 + (g'(u))^2 = 1$. Denote $x = z_u = f'\, l + g'\, e_1$, $y = \frac{z_v}{f} = t$ and consider the orthonormal normal frame field defined by:

$$n_1 = g'(u)\, l(v) - f'(u)\, e_1; \quad n_2 = n(v).$$

Thus we obtain a frame field $\{x, y, n_1, n_2\}$ of $\mathcal{M}''_m$, such that $\langle n_1, n_1\rangle = 1$, $\langle n_2, n_2\rangle = -1$, $\langle n_1, n_2\rangle = 0$.

Using (3.5) we get the following derivative formulas [16]:

$$\begin{array}{ll}
\nabla'_x x = -\kappa_m\, n_1; & \nabla'_x n_1 = \kappa_m\, x; \\
\nabla'_x y = 0; & \nabla'_y n_1 = \dfrac{g'}{f} y; \\
\nabla'_y x = \dfrac{f'}{f} y; & \nabla'_x n_2 = 0; \\
\nabla'_y y = -\dfrac{f'}{f} x - \dfrac{g'}{f} n_1 - \dfrac{\kappa}{f} n_2; & \nabla'_y n_2 = -\dfrac{\kappa}{f} y,
\end{array} \tag{3.6}$$

where $\kappa_m(u) = f'(u)g''(u) - g'(u)f''(u)$ is the curvature of the meridian curve m, and $\kappa = \kappa(v)$ is the spherical curvature of c.

4. Meridian surfaces of elliptic or hyperbolic type with harmonic Gauss map

In the present section we give the classification of the meridian surfaces of elliptic or hyperbolic type with harmonic Gauss map.

Let $\mathcal{M}'_m$ and $\mathcal{M}''_m$ be meridian surfaces of elliptic and hyperbolic type, respectively, and $\{x, y, n_1, n_2\}$ be the frame field of $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) defined in Section 3. This frame field generates the following frame of the Grassmannian manifold:

$$\{x \wedge y, x \wedge n_1, x \wedge n_2, y \wedge n_1, y \wedge n_2, n_1 \wedge n_2\}.$$

The indefinite inner product on the Grassmannian manifold is given by

$$\langle e_{i_1} \wedge e_{i_2}, f_{j_1} \wedge f_{j_2}\rangle = \det\left(\langle e_{i_k}, f_{j_l}\rangle\right).$$

Thus we have

$$\begin{array}{lll}
\langle x \wedge y, x \wedge y\rangle = 1; & \langle x \wedge n_1, x \wedge n_1\rangle = 1; & \langle x \wedge n_2, x \wedge n_2\rangle = -1; \\
\langle y \wedge n_1, y \wedge n_1\rangle = 1; & \langle y \wedge n_2, y \wedge n_2\rangle = -1; & \langle n_1 \wedge n_2, n_1 \wedge n_2\rangle = -1,
\end{array}$$

and all other scalar products are equal to zero.

The Gauss map G of $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) is defined by $G(p) = (x \wedge y)(p)$, $p \in \mathcal{M}'_m$ (resp. $p \in \mathcal{M}''_m$). Then the Laplacian of the Gauss map is given by the formula

$$\Delta G = -\nabla'_x \nabla'_x G + \nabla'_{\nabla_x x} G - \nabla'_y \nabla'_y G + \nabla'_{\nabla_y y} G. \tag{4.1}$$

Using (3.3), (3.6) and (4.1), we obtain that in the elliptic case the Laplacian of the Gauss map is expressed as

$$\Delta G = \frac{\kappa^2 - g'^2 - f^2\kappa_m^2}{f^2}\, x \wedge y - \frac{\kappa'}{f^2}\, x \wedge n_1 - \frac{\kappa f'}{f^2}\, y \wedge n_1 + \frac{f(f\kappa_m)' - f'g'}{f^2}\, y \wedge n_2, \tag{4.2}$$

and in the hyperbolic case the Laplacian of the Gauss map is given by

$$\Delta G = \frac{-\kappa^2 + g'^2 + f^2\kappa_m^2}{f^2}\, x \wedge y + \frac{\kappa'}{f^2}\, x \wedge n_2 + \frac{f'g' - f(f\kappa_m)'}{f^2}\, y \wedge n_1 + \frac{\kappa f'}{f^2}\, y \wedge n_2, \tag{4.3}$$

where $\kappa' = \dfrac{d}{dv}(\kappa)$.

Theorem 4.1. *Let $\mathcal{M}'_m$ be a meridian surface of elliptic type, defined by* (3.1). *The Gauss map of $\mathcal{M}'_m$ is harmonic if and only if $\mathcal{M}'_m$ is part of a plane.*

Proof. First, we suppose that the Gauss map of $\mathcal{M}'_m$ is harmonic, i.e., $\Delta G = 0$. Then, from (4.2) it follows that

$$\begin{aligned} \kappa^2 - g'^2 - f^2\kappa_m^2 &= 0;\\ \kappa' &= 0;\\ \kappa f' &= 0;\\ f(f\kappa_m)' - f'g' &= 0. \end{aligned}$$

In the elliptic case we have $f'^2 \geq 1$, since $f'^2 - g'^2 = 1$. Hence, the above equalities imply

$$\kappa = 0; \quad g' = 0; \quad \kappa_m = 0.$$

Using (3.3) we get that $\mathcal{M}'_m$ is totally geodesic, i.e., $\mathcal{M}'_m$ is part of a plane.

Conversely, if $\mathcal{M}'_m$ is totally geodesic, then $\Delta G = 0$. □

Theorem 4.2. *Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type, defined by* (3.4). *The Gauss map of $\mathcal{M}''_m$ is harmonic if and only if one of the following cases holds:*

(i) *$\mathcal{M}''_m$ is part of a plane;*

(ii) *the curve c has spherical curvature $\kappa = \pm 1$ and the meridian curve m is determined by $f(u) = a$; $g(u) = \pm u + b$, where $a = \text{const}$, $b = \text{const}$. In this case $\mathcal{M}''_m$ is a marginally trapped developable ruled surface in $\mathbb{E}^4_1$.*

Proof. Suppose that the Gauss map of $\mathcal{M}''_m$ is harmonic, i.e., $\Delta G = 0$. Then, from (4.3) it follows that

$$\begin{aligned} \kappa^2 - g'^2 - f^2\kappa_m^2 &= 0;\\ \kappa' &= 0;\\ \kappa f' &= 0;\\ f(f\kappa_m)' - f'g' &= 0. \end{aligned} \tag{4.4}$$

In the hyperbolic case we have $f'^2 \leq 1$, since $f'^2 + g'^2 = 1$. Hence, from the third equality of (4.4) we get the following two cases:

Case (i): $\kappa = 0$. Then, the first equality of (4.4) implies $g' = 0$; $\kappa_m = 0$. Using (3.6) we get that $\mathcal{M}''_m$ is totally geodesic, i.e., $\mathcal{M}''_m$ is part of a plane.

Case (ii): $\kappa \neq 0$. Then $f' = 0$, i.e., $f(u) = a = \text{const}$, and $g'^2 = 1$, i.e., $g(u) = \pm u + b$, $b = \text{const}$. In this case $\kappa_m = 0$. The second equality of (4.4) implies $\kappa = \text{const}$. From the first equality of (4.4) we obtain $\kappa^2 = g'^2$. Hence, $\kappa = \varepsilon g'$, where $\varepsilon = \pm 1$. In this case derivative formulas (3.6) take the form:

$$
\begin{array}{ll}
\nabla'_x x = 0; & \nabla'_x n_1 = 0; \\
\nabla'_x y = 0; & \nabla'_y n_1 = \pm\dfrac{1}{a} y; \\
\nabla'_y x = 0; & \nabla'_x n_2 = 0; \\
\nabla'_y y = \mp\dfrac{1}{a}(n_1 + \varepsilon n_2); & \nabla'_y n_2 = \mp\dfrac{\varepsilon}{a} y.
\end{array}
\tag{4.5}
$$

$\mathcal{M}''_m$ is a ruled surface, since the meridian curve m is a straight line. From (4.5) we have that $\nabla'_x n_1 = 0$; $\nabla'_x n_2 = 0$, i.e., the normal space is constant at the points of each generator. Hence, $\mathcal{M}''_m$ is developable. Moreover, the mean curvature vector field H is given by

$$H = \mp\frac{1}{2a}(n_1 + \varepsilon n_2),$$

which implies that $\langle H, H \rangle = 0$. Hence, $\mathcal{M}''_m$ is a marginally trapped surface.

Conversely, if one of the cases (i) or (ii) holds, then by straightforward calculations we get $\Delta G = 0$, i.e., $\mathcal{M}''_m$ has harmonic Gauss map. □

Remark 4.3. In the Euclidean space $\mathbb{E}^4$ planes are the only surfaces with harmonic Gauss map. However, in the Minkowski space $\mathbb{E}^4_1$ there are surfaces with harmonic Gauss map which are not planes. Theorems 4.1 and 4.2 show that in the class of the meridian surfaces of elliptic type there are no surfaces with harmonic Gauss map other than planes, while in the class of the meridian surfaces of hyperbolic type we obtain surfaces with harmonic Gauss map, which are not planes.

5. Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of first kind

In this section we classify the meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of first kind, i.e., the Gauss map G satisfies the condition

$$\Delta G = \lambda G$$

for some non-zero smooth function λ.

First, let us consider the meridian surface of elliptic type $\mathcal{M}'_m$, defined by (3.1). So, the Laplacian of the Gauss map is given by formula (4.2).

Theorem 5.1. *Let $\mathcal{M}'_m$ be a meridian surface of elliptic type, defined by* (3.1). *Then $\mathcal{M}'_m$ has pointwise* 1*-type Gauss map of first kind if and only if the curve c has zero spherical curvature and the meridian curve m is determined by a solution $f(u)$ of the following differential equation*

$$f\left(\frac{ff''}{\sqrt{f'^2-1}}\right)' - f'\sqrt{f'^2-1} = 0, \tag{5.1}$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2(u)-1}$.

Proof. From (4.2) it follows that $\Delta G = \lambda G$ if and only if

$$\begin{aligned} \kappa' &= 0; \\ \kappa f' &= 0; \\ f(f\kappa_m)' - f'g' &= 0. \end{aligned} \tag{5.2}$$

Let $\mathcal{M}'_m$ be of pointwise 1-type Gauss map of first kind. Since $f' \neq 0$, from the second equality of (5.2) we get that $\kappa = 0$. If we suppose that $g' = 0$, then $\kappa_m = 0$ and (4.2) implies $\Delta G = 0$, which contradicts the assumption $\lambda \neq 0$. Hence, $g' \neq 0$. Then using that $f'^2 - g'^2 = 1$ we obtain $\kappa_m = \frac{f''}{\sqrt{f'^2-1}}$. So, the third equality of (5.2) takes the form (5.1). Conversely, if $\kappa = 0$ and $f(u)$ is a solution of (5.1), then $\Delta G = \lambda G$. □

In the next theorem we give the classification of the meridian surfaces of hyperbolic type with pointwise 1-type Gauss map of first kind.

Theorem 5.2. *Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type, defined by* (3.4). *Then $\mathcal{M}''_m$ has pointwise* 1*-type Gauss map of first kind if and only if one of the following cases holds:*

(i) *the curve c has zero spherical curvature and the meridian curve m is determined by a solution $f(u)$ of the following differential equation*

$$f\left(\frac{ff''}{\sqrt{1-f'^2}}\right)' + f'\sqrt{1-f'^2} = 0, \tag{5.3}$$

$g(u)$ is defined by $g'(u) = \sqrt{1-f'^2(u)}$;

(ii) *the curve c has non-zero constant spherical curvature k ($\kappa \neq \pm 1$) and the meridian curve m is determined by $f(u) = a$; $g(u) = \pm u + b$, where $a =$ const, $b =$ const. Moreover, $\mathcal{M}''_m$ is a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3_1$ (if $\kappa^2 - 1 > 0$) or $\mathbb{E}^3$ (if $\kappa^2 - 1 < 0$) of $\mathbb{E}^4_1$.*

Proof. Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type, defined by (3.4). So, the Laplacian of the Gauss map is given by formula (4.3). From (4.3) it follows that $\Delta G = \lambda G$ if and only if

$$\begin{aligned} \kappa' &= 0; \\ \kappa f' &= 0; \\ f'g' - f(f\kappa_m)' &= 0. \end{aligned} \tag{5.4}$$

Let $\mathcal{M}''_m$ be of pointwise 1-type Gauss map of first kind. From the second equality of (5.4) we get the following two cases:

Case (i): $\kappa = 0$. If we suppose that $g' = 0$, then $\kappa_m = 0$ and (4.3) implies $\Delta G = 0$, which contradicts the assumption $\lambda \neq 0$. Hence, $g' \neq 0$. Then using that $f'^2 + g'^2 = 1$ we obtain $\kappa_m = -\frac{f''}{\sqrt{1-f'^2}}$. Thus, the third equality of (5.4) takes the form (5.3).

Case (ii): $\kappa \neq 0$. Then $f' = 0$, i.e., $f(u) = a = \text{const}$, and $g' = \pm 1$, i.e., $g(u) = \pm u + b$, $b = \text{const}$. In this case $\kappa_m = 0$. The first equality of (5.4) implies $\kappa = \text{const}$. Then the Laplacian of the Gauss map takes the form:

$$\Delta G = \frac{1-\kappa^2}{a^2}\, x \wedge y,$$

which implies that the surface $\mathcal{M}''_m$ has 1-type Gauss map, since $\lambda = \frac{1-\kappa^2}{a^2} = \text{const}$. If $\kappa^2 = 1$ then $\Delta G = 0$, which contradicts the assumption $\lambda \neq 0$. Hence, $\kappa^2 \neq 1$, i.e., $\kappa \neq \pm 1$. In this case derivative formulas (3.6) take the form:

$$\begin{array}{ll} \nabla'_x x = 0; & \nabla'_x n_1 = 0; \\ \nabla'_x y = 0; & \nabla'_y n_1 = \pm \frac{1}{a} y; \\ \nabla'_y x = 0; & \nabla'_x n_2 = 0; \\ \nabla'_y y = \mp \frac{1}{a} n_1 - \frac{\kappa}{a} n_2; & \nabla'_y n_2 = -\frac{\kappa}{a} y. \end{array} \tag{5.5}$$

$\mathcal{M}''_m$ is a developable ruled surface, since the meridian curve m is a straight line and $\nabla'_x n_1 = 0$; $\nabla'_x n_2 = 0$. Now we shall prove that $\mathcal{M}''_m$ lies in a constant hyperplane of $\mathbb{E}^4_1$.

An arbitrary orthonormal frame $\{n, n^\perp\}$ of the normal bundle is determined by

$$\begin{aligned} n &= \cosh\theta\, n_1 + \sinh\theta\, n_2, \\ n^\perp &= \sinh\theta\, n_1 + \cosh\theta\, n_2, \end{aligned} \tag{5.6}$$

for some smooth function θ. Note that n is spacelike and $n^\perp$ is timelike. Using (5.5) and (5.6) we get

$$\begin{array}{ll} \nabla'_x n = \theta'_u\, n^\perp; & \nabla'_y n = \dfrac{\theta'_v}{a}\, n^\perp - \dfrac{1}{a}(\mp\cosh\theta + \kappa\sinh\theta) y; \\ \nabla'_x n^\perp = \theta'_u\, n; & \nabla'_y n^\perp = \dfrac{\theta'_v}{a}\, n - \dfrac{1}{a}(\mp\sinh\theta + \kappa\cosh\theta) y. \end{array} \tag{5.7}$$

In the case $\kappa^2 - 1 > 0$ we choose $\theta = \pm\frac{1}{2}\ln\left(\frac{\kappa+1}{\kappa-1}\right)$. Then $\mp\cosh\theta + \kappa\sinh\theta = 0$. Hence, from (5.5) and (5.7) it follows that

$$\begin{aligned} &\nabla'_x x = 0; && \nabla'_x n = 0;\\ &\nabla'_x y = 0; && \nabla'_y n = 0;\\ &\nabla'_y x = 0; && \nabla'_x n^\perp = 0;\\ &\nabla'_y y = -\frac{\kappa^2-1}{a\kappa}\cosh\ln\left(\frac{\kappa+1}{\kappa-1}\right)^{\pm\frac{1}{2}} n^\perp; && \nabla'_y n^\perp = -\frac{\kappa^2-1}{a\kappa}\cosh\ln\left(\frac{\kappa+1}{\kappa-1}\right)^{\pm\frac{1}{2}} y. \end{aligned}$$

The last equalities imply that $n = \mathrm{const}$ and the surface $\mathcal{M}''_m$ lies in the constant hyperplane $\mathbb{E}^3_1 = \operatorname{span}\left\{x, y, n^\perp\right\}$ of $\mathbb{E}^4_1$.

In the case $\kappa^2 - 1 < 0$ we choose $\theta = \pm\frac{1}{2}\ln\left(\frac{\kappa+1}{1-\kappa}\right)$. Then $\mp\sinh\theta + \kappa\cosh\theta = 0$. Hence, formulas (5.5) and (5.7) imply that

$$\begin{aligned} &\nabla'_x x = 0; && \nabla'_x n = 0;\\ &\nabla'_x y = 0; && \nabla'_y n = -\frac{\kappa^2-1}{a\kappa}\sinh\ln\left(\frac{\kappa+1}{1-\kappa}\right)^{\pm\frac{1}{2}} y;\\ &\nabla'_y x = 0; && \nabla'_x n^\perp = 0;\\ &\nabla'_y y = \frac{\kappa^2-1}{a\kappa}\sinh\ln\left(\frac{\kappa+1}{1-\kappa}\right)^{\pm\frac{1}{2}} n; && \nabla'_y n^\perp = 0. \end{aligned}$$

From the last equalities we get that $n^\perp = \mathrm{const}$ and the surface $\mathcal{M}''_m$ lies in the constant hyperplane $\mathbb{E}^3 = \operatorname{span}\{x, y, n\}$ of $\mathbb{E}^4_1$.

Conversely, if one of the cases (i) or (ii) holds, then by straightforward calculations it can be seen that $\Delta G = \lambda G$, i.e., $\mathcal{M}''_m$ has pointwise 1-type Gauss map of first kind. $\square$

6. Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of second kind

In this section we give the classification of the meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of second kind, i.e., the Gauss map G satisfies the condition

$$\Delta G = \lambda(G + C) \tag{6.1}$$

for some non-zero smooth function λ and a constant vector $C \neq 0$.

First we consider meridian surfaces of elliptic type with pointwise 1-type Gauss map of second kind. They are classified by the following theorem.

Theorem 6.1. *Let $\mathcal{M}'_m$ be a meridian surface of elliptic type, defined by* (3.1)*. Then $\mathcal{M}'_m$ has pointwise* 1*-type Gauss map of second kind if and only if one of the following cases holds:*

(i) *the curve c has non-zero constant spherical curvature κ and the meridian curve m is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = \mathrm{const}$, $b = \mathrm{const}$. In this case $\mathcal{M}'_m$ is a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3$ of $\mathbb{E}^4_1$.*

(ii) *the curve c has constant spherical curvature κ and the meridian curve m is determined by $f(u) = au+a_1$; $g(u) = bu+b_1$, where a, a_1, b and b_1 are constants, $a^2 \geq 1$, $a^2 - b^2 = 1$. In this case $\mathcal{M}'_m$ is either a marginally trapped developable ruled surface (if $\kappa^2 = b^2$) or a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3$ (if $\kappa^2 - b^2 > 0$) or $\mathbb{E}^3_1$ (if $\kappa^2 - b^2 < 0$) of $\mathbb{E}^4_1$.*

(iii) *the curve c has zero spherical curvature and the meridian curve m is determined by the solutions of the following differential equation*

$$\left(\ln \frac{\sqrt{f'^2-1}\left(f(f'^2-1)(ff'')' - f^2f'f''^2 - f'(f'^2-1)^2\right)}{(f'^2-1)^2 + f^2f''^2 - ff'(f'^2-1)(ff'')'}\right)' = \frac{f'f''}{f'^2-1}.$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2(u) - 1}$.

Proof. Let $\mathcal{M}'_m$ be a meridian surface of elliptic type, defined by (3.1). Suppose that $\mathcal{M}'_m$ has pointwise 1-type Gauss map of second kind. Then equations (4.2) and (6.1) imply

$$\left(\frac{\kappa^2 - g'^2 - f^2\kappa_m^2}{f^2} - \lambda\right) x\wedge y - \frac{\kappa'}{f^2}\, x\wedge n_1 - \frac{\kappa f'}{f^2}\, y\wedge n_1 + \frac{f(f\kappa_m)' - f'g'}{f^2}\, y\wedge n_2 = \lambda C. \tag{6.2}$$

Since $\lambda \neq 0$, from (6.2) we get

$$\begin{array}{lll} \langle C, x\wedge y\rangle = \dfrac{\kappa^2 - g'^2 - f^2\kappa_m^2}{\lambda f^2} - 1; & \langle C, x\wedge n_1\rangle = -\dfrac{\kappa'}{\lambda f^2}; & \langle C, y\wedge n_1\rangle = -\dfrac{\kappa f'}{\lambda f^2}; \\ \langle C, y\wedge n_2\rangle = -\dfrac{f(f\kappa_m)' - f'g'}{\lambda f^2}; & \langle C, x\wedge n_2\rangle = 0; & \langle C, n_1\wedge n_2\rangle = 0. \end{array} \tag{6.3}$$

Differentiating the last two equalities of (6.3) with respect to u and v we obtain

$$\begin{aligned} \kappa_m \langle C, x\wedge n_1\rangle &= 0; \\ g'\langle C, x\wedge y\rangle + f'\langle C, y\wedge n_2\rangle &= 0; \\ g'\langle C, y\wedge n_1\rangle + \kappa\langle C, y\wedge n_2\rangle &= 0. \end{aligned} \tag{6.4}$$

Hence, equalities (6.3) and (6.4) imply

$$\begin{aligned} \kappa'\kappa_m &= 0; \\ \kappa(f\kappa_m)' &= 0; \\ g'(1+\kappa^2 - f^2\kappa_m^2) - ff'(f\kappa_m)' &= \lambda f^2 g'. \end{aligned} \tag{6.5}$$

We distinguish the following cases.

Case I: $g' = 0$. Then $\kappa \neq 0$ (otherwise the Gauss map is harmonic). From (6.2) we get

$$C = \left(\frac{\kappa^2}{\lambda f^2} - 1\right) x \wedge y - \frac{\kappa'}{\lambda f^2}\, x \wedge n_1 - \frac{\kappa f'}{\lambda f^2}\, y \wedge n_1. \tag{6.6}$$

Using (3.3) and (6.6) we obtain

$$\begin{aligned}
\nabla'_x C &= \kappa^2 \left(\frac{1}{\lambda f^2}\right)'_u x \wedge y - \kappa' \left(\frac{1}{\lambda f^2}\right)'_u x \wedge n_1 - \kappa \left(\frac{f'}{\lambda f^2}\right)'_u y \wedge n_1;\\
\nabla'_y C &= \frac{\kappa}{\lambda^2 f^3}\left(3\kappa'\lambda - \kappa\lambda'_v\right) x \wedge y + \frac{1}{\lambda^2 f^3}\left(-\kappa''\lambda + k'\lambda'_v + \kappa^3\lambda + \kappa\lambda - \kappa\lambda^2 f^2\right) x \wedge n_1\\
&\quad + \frac{f'}{\lambda^2 f^3}\left(-2\kappa'\lambda + \kappa\lambda'_v\right) y \wedge n_1.
\end{aligned}$$

The last formulas imply that $C = \text{const}$ if and only if $\kappa = \text{const}$ and $\lambda = \frac{\kappa^2+1}{f^2}$. In this case

$$\Delta G = \frac{\kappa^2+1}{f^2}(G + C),$$

where $C = -\frac{1}{\kappa^2+1}(x \wedge y + \kappa f'\, y \wedge n_1)$. From $g' = 0$ it follows that $\kappa_m = 0$ and the meridian curve m is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = \text{const}$, $b = \text{const}$. The surface $\mathcal{M}'_m$ is a developable ruled surface lying in the hyperplane $\mathbb{E}^3 = \text{span}\,\{x, y, n_1\}$, since $\nabla'_x n_2 = 0$; $\nabla'_y n_2 = 0$.

Case II: $g' \neq 0$. Then the last equality of (6.5) implies

$$\lambda = \frac{1}{g' f^2}\left(g'(1 + \kappa^2 - f^2\kappa_m^2) - f f'(f\kappa_m)'\right).$$

It follows from the first two equalities of (6.5) that there are three subcases.

1. $\kappa_m = 0$. In this subcase the Laplacian of G is given by

$$\Delta G = \frac{\kappa^2 - g'^2}{f^2}\, x \wedge y - \frac{\kappa'}{f^2}\, x \wedge n_1 - \frac{\kappa f'}{f^2}\, y \wedge n_1 - \frac{f'g'}{f^2}\, y \wedge n_2 \tag{6.7}$$

and the function λ is expressed as $\lambda = \frac{\kappa^2+1}{f^2}$. Now, equalities (6.1) and (6.7) imply

$$C = -\frac{1}{1+\kappa^2}\left(f'^2\, x \wedge y + \kappa'\, x \wedge n_1 + \kappa f'\, y \wedge n_1 + f'g'\, y \wedge n_2\right). \tag{6.8}$$

Using formulas (3.3) in the case $\kappa_m = 0$ and (6.8) we obtain

$$\begin{aligned}
\nabla'_x C &= -\frac{1}{1+\kappa^2}\left(2f'f''\, x \wedge y + \kappa f''\, y \wedge n_1 + (f'g'' + g'f'')\, y \wedge n_2\right);\\
\nabla'_y C &= \frac{\kappa\kappa'}{f(1+\kappa^2)^2}(2f'^2 + 1 + \kappa^2)\, x \wedge y + \frac{1}{f(1+\kappa^2)^2}\left(2\kappa\kappa'^2 - \kappa''(1+\kappa^2)\right) x \wedge n_1\\
&\quad - \frac{2\kappa' f'}{f(1+\kappa^2)^2}\, y \wedge n_1 + \frac{2\kappa\kappa' f' g'}{f(1+\kappa^2)^2}\, y \wedge n_2.
\end{aligned}$$

The last formulas imply that $C = \mathrm{const}$ if and only if $\kappa = \mathrm{const}$ and $f'' = 0$. Hence, the meridian curve m is determined by $f(u) = au + a_1$; $g(u) = bu + b_1$, where a, a_1, b and b_1 are constants, $a^2 \geq 1$, $a^2 - b^2 = 1$. Hence, $\mathcal{M}'_m$ is a developable ruled surface, since $\nabla'_x n_1 = 0$; $\nabla'_x n_2 = 0$. We shall prove that in the case $\kappa^2 = b^2$ the surface $\mathcal{M}'_m$ is marginally trapped and in the case $\kappa^2 \neq b^2$ the surface $\mathcal{M}'_m$ lies in a constant hyperplane $\mathbb{E}^3$ or $\mathbb{E}^3_1$ of $\mathbb{E}^4_1$. Indeed, if $\kappa = \varepsilon b$, $\varepsilon = \pm 1$, then from (3.3) we get that $H = \frac{b}{2f}(\varepsilon n_1 + n_2)$ and hence, $\langle H, H \rangle = 0$, which implies that $\mathcal{M}'_m$ is a marginally trapped surface. In the case $\kappa^2 - b^2 \neq 0$ we consider an orthonormal frame $\{n, n^\perp\}$ of the normal bundle which is determined by equalities (5.6) for some function θ. Hence, the derivatives of n and $n^\perp$ satisfy

$$\nabla'_x n = \theta'_u \, n^\perp; \qquad \nabla'_y n = \frac{\theta'_v}{f} \, n^\perp + \frac{1}{f}(b \sinh \theta - \kappa \cosh \theta) y;$$
$$\nabla'_x n^\perp = \theta'_u \, n; \qquad \nabla'_y n^\perp = \frac{\theta'_v}{f} \, n + \frac{1}{f}(b \cosh \theta - \kappa \sinh \theta) y.$$

In the case $\kappa^2 - b^2 > 0$ we choose $\theta = \frac{1}{2} \ln \left(\frac{\kappa + b}{\kappa - b} \right)$. Then $b \cosh \theta - \kappa \sinh \theta = 0$ and $\nabla'_x n^\perp = 0$, $\nabla'_y n^\perp = 0$. In this case the surface $\mathcal{M}'_m$ lies in the constant hyperplane $\mathbb{E}^3 = \mathrm{span}\,\{x, y, n\}$ of $\mathbb{E}^4_1$. In the case $\kappa^2 - b^2 < 0$ we choose $\theta = \frac{1}{2} \ln \left(\frac{b + \kappa}{b - \kappa} \right)$. Then $b \sinh \theta - \kappa \cosh \theta = 0$ and $\nabla'_x n = 0$, $\nabla'_y n = 0$. In this case the surface $\mathcal{M}'_m$ lies in the constant hyperplane $\mathbb{E}^3_1 = \mathrm{span}\,\{x, y, n^\perp\}$ of $\mathbb{E}^4_1$.

2. $\kappa = 0$. In this subcase the Laplacian of G is given by

$$\Delta G = -\frac{g'^2 + f^2 \kappa_m^2}{f^2} \, x \wedge y + \frac{f(f\kappa_m)' - f'g'}{f^2} \, y \wedge n_2 \tag{6.9}$$

and the function λ is expressed as $\lambda = \frac{1}{g' f^2} \left(g'(1 - f^2 \kappa_m^2) - f f' (f \kappa_m)' \right)$. Hence, from equalities (6.1) and (6.9) we get

$$C = -\left(\frac{g'^2 + f^2 \kappa_m^2}{\lambda f^2} + 1 \right) x \wedge y + \frac{f(f\kappa_m)' - f'g'}{\lambda f^2} \, y \wedge n_2. \tag{6.10}$$

Denote $\psi = -\frac{g'^2 + f^2 \kappa_m^2}{\lambda f^2} - 1$; $\varphi = \frac{f(f\kappa_m)' - f'g'}{\lambda f^2}$. Then $C = \psi \, x \wedge y + \varphi \, y \wedge n_2$. Using formulas (3.3) in the case $\kappa = 0$ and (6.10) we obtain

$$\begin{aligned} &\nabla'_x C = (\psi' - \varphi \kappa_m) \, x \wedge y + (\varphi' - \psi \kappa_m) \, y \wedge n_2; \\ &\nabla'_y C = 0. \end{aligned} \tag{6.11}$$

Using the expression of λ we calculate that $\psi = \varphi \frac{f'}{g'}$; $\psi' - \varphi \kappa_m = \frac{f'}{g'}(\varphi' - \psi \kappa_m)$. Hence, formulas (6.11) take the form

$$\nabla'_x C = \frac{f'}{g'}(\varphi' - \psi \kappa_m) \, x \wedge y + (\varphi' - \psi \kappa_m) \, y \wedge n_2;$$
$$\nabla'_y C = 0.$$

The last formulas imply that $C = \text{const}$ if and only if

$$(\ln \varphi)' = \frac{f'}{g'} \kappa_m. \tag{6.12}$$

Using that $f\kappa_m = \frac{ff''}{\sqrt{f'^2-1}}$, we get

$$\varphi = \frac{\sqrt{f'^2-1}\left(f(f'^2-1)(ff'')' - f^2 f' f''^2 - f'(f'^2-1)^2\right)}{(f'^2-1)^2 + f^2 f''^2 - ff'(f'^2-1)(ff'')'}. \tag{6.13}$$

Now, formulas (6.12) and (6.13) imply that $C = \text{const}$ if and only if the function $f(u)$ is a solution of the following differential equation

$$\left(\ln \frac{\sqrt{f'^2-1}\left(f(f'^2-1)(ff'')' - f^2 f' f''^2 - f'(f'^2-1)^2\right)}{(f'^2-1)^2 + f^2 f''^2 - ff'(f'^2-1)(ff'')'}\right)' = \frac{f'f''}{f'^2-1}.$$

3. $\kappa = \text{const} \neq 0$ and $f\kappa_m = a = \text{const}$, $a \neq 0$. In this subcase the Laplacian of G is given by

$$\Delta G = \frac{\kappa^2 - a^2 - g'^2}{f^2}\, x \wedge y - \frac{\kappa f'}{f^2}\, y \wedge n_1 - \frac{f'g'}{f^2}\, y \wedge n_2 \tag{6.14}$$

and the function λ is expressed as $\lambda = \frac{1+\kappa^2-a^2}{f^2}$. Since $\lambda \neq 0$, we get $a^2 \neq 1 + \kappa^2$. Now, equalities (6.1) and (6.14) imply

$$C = \frac{1}{1+\kappa^2-a^2}\left(-f'^2\, x \wedge y - \kappa f'\, y \wedge n_1 - f'g'\, y \wedge n_2\right).$$

Then the derivatives of C are expressed as

$$\begin{aligned} &\nabla'_x C = -\frac{1}{1+\kappa^2-a^2}\left(f'f''\, x \wedge y + \kappa f''\, y \wedge n_1 + g'f''\, y \wedge n_2\right); \\ &\nabla'_y C = 0. \end{aligned} \tag{6.15}$$

It follows from formulas (6.15) that $C = \text{const}$ if and only if $f'' = 0$. But the condition $f'' = 0$ implies $\kappa_m = 0$, which contradicts the assumption that $f\kappa_m \neq 0$.

Consequently, in this subcase there are no meridian surfaces of elliptic type with pointwise 1-type Gauss map of second kind.

Conversely, if one of the cases (i), (ii) or (iii) holds, then it can easily be seen that $\Delta G = \lambda(G + C)$, i.e., $\mathcal{M}'_m$ has pointwise 1-type Gauss map of second kind. □

The next theorem classifies the meridian surfaces of hyperbolic type with pointwise 1-type Gauss map of second kind.

Theorem 6.2. *Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type, defined by* (3.4)*. Then $\mathcal{M}''_m$ has pointwise* 1*-type Gauss map of second kind if and only if one of the following cases holds:*

(i) *the curve c has non-zero constant spherical curvature $\kappa \neq \pm 1$ and the meridian curve m is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = \mathrm{const}$, $b = \mathrm{const}$. In this case $\mathcal{M}''_m$ is a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3_1$ of $\mathbb{E}^4_1$.*

(ii) *the curve c has constant spherical curvature κ and the meridian curve m is determined by $f(u) = au + a_1$; $g(u) = bu + b_1$, where a, a_1, b and b_1 are constants, $a^2 + b^2 = 1$. In this case $\mathcal{M}''_m$ is either a marginally trapped developable ruled surface (if $\kappa^2 = b^2$) or a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3_1$ (if $\kappa^2 - b^2 > 0$) or $\mathbb{E}^3$ (if $\kappa^2 - b^2 < 0$) of $\mathbb{E}^4_1$.*

(iii) *the curve c has zero spherical curvature and the meridian curve m is determined by the solutions of the following differential equation*

$$\left(\ln \frac{\sqrt{1-f'^2}\,\left(f(1-f'^2)(ff'')' + f^2 f' f''^2 + f'(1-f'^2)^2\right)}{(1-f'^2)^2 + f^2 f''^2 + ff'(1-f'^2)(ff'')'}\right)' = -\frac{f'f''}{1-f'^2}.$$

$g(u)$ is defined by $g'(u) = \sqrt{1 - f'^2(u)}$.

Proof. Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type, defined by (3.4). Suppose that $\mathcal{M}''_m$ has pointwise 1-type Gauss map of second kind. Then equations (4.3) and (6.1) imply

$$\left(\frac{g'^2 - \kappa^2 + f^2\kappa_m^2}{f^2} - \lambda\right) x\wedge y + \frac{\kappa'}{f^2}\, x\wedge n_2 + \frac{f'g' - f(f\kappa_m)'}{f^2}\, y\wedge n_1 + \frac{\kappa f'}{f^2}\, y\wedge n_2 = \lambda C. \tag{6.16}$$

Since $\lambda \neq 0$, from (6.16) we get

$$\begin{aligned}
&\langle C, x\wedge y\rangle = \frac{g'^2 - \kappa^2 + f^2\kappa_m^2}{\lambda f^2} - 1; &\qquad &\langle C, x\wedge n_2\rangle = -\frac{\kappa'}{\lambda f^2};\\
&\langle C, y\wedge n_1\rangle = \frac{f'g' - f(f\kappa_m)'}{\lambda f^2}; & &\langle C, y\wedge n_2\rangle = -\frac{\kappa f'}{\lambda f^2};\\
&\langle C, x\wedge n_1\rangle = 0; & &\langle C, n_1\wedge n_2\rangle = 0.
\end{aligned} \tag{6.17}$$

Differentiating the last two equalities of (6.17) with respect to u and v we obtain

$$\begin{aligned}
\kappa'\kappa_m &= 0;\\
\kappa(f\kappa_m)' &= 0;\\
g'(1 - \kappa^2 + f^2\kappa_m^2) - ff'(f\kappa_m)' &= \lambda f^2 g'.
\end{aligned} \tag{6.18}$$

Similarly to the elliptic case, we distinguish the following cases.

Case I: $g' = 0$. Then $\kappa \neq 0$ (otherwise the Gauss map is harmonic). From (6.16) we get

$$C = -\left(\frac{\kappa^2}{\lambda f^2} + 1\right) x\wedge y + \frac{\kappa'}{\lambda f^2}\, x\wedge n_2 + \frac{\kappa f'}{\lambda f^2}\, y\wedge n_2,$$

which implies

$$\begin{aligned}\nabla'_x C &= -\kappa^2 \left(\frac{1}{\lambda f^2}\right)'_u x \wedge y + \kappa' \left(\frac{1}{\lambda f^2}\right)'_u x \wedge n_2 + \kappa \left(\frac{f'}{\lambda f^2}\right)'_u y \wedge n_2;\\ \nabla'_y C &= -\frac{\kappa}{\lambda^2 f^3}\left(3\kappa'\lambda - \kappa\lambda'_v\right) x \wedge y + \frac{1}{\lambda^2 f^3}\left(\kappa''\lambda - \kappa'\lambda'_v + \kappa^3\lambda - \kappa\lambda + \kappa\lambda^2 f^2\right) x \wedge n_2\\ &\quad + \frac{f'}{\lambda^2 f^3}\left(2\kappa'\lambda - \kappa\lambda'_v\right) y \wedge n_2.\end{aligned}$$

It follows from the last formulas that $C = \text{const}$ if and only if $\kappa = \text{const}$ and $\lambda = \frac{1-\kappa^2}{f^2}$. Since $\lambda \neq 0$ we get $\kappa \neq \pm 1$. The Laplacian of the Gauss map is expressed as

$$\Delta G = \frac{1-\kappa^2}{f^2}(G + C),$$

where $C = \frac{1}{\kappa^2-1}(x \wedge y - \kappa f'\, y \wedge n_2)$. The condition $g' = 0$ implies that $\kappa_m = 0$ and the meridian curve m is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = \text{const}$, $b = \text{const}$. The surface $\mathcal{M}''_m$ is a developable ruled surface lying in the hyperplane $\mathbb{E}^3_1 = \operatorname{span}\{x, y, n_2\}$.

Case II: $g' \neq 0$. Then the last equality of (6.18) implies

$$\lambda = \frac{1}{g' f^2}\left(g'(1 - \kappa^2 + f^2\kappa_m^2) - f f'(f\kappa_m)'\right).$$

Similarly to the elliptic case we have to consider the following three subcases.

1. $\kappa_m = 0$. In this subcase $\lambda = \frac{1-\kappa^2}{f^2}$, $\kappa \neq \pm 1$ and the Laplacian of G is given by

$$\Delta G = \frac{g'^2 - \kappa^2}{f^2}\, x \wedge y + \frac{\kappa'}{f^2}\, x \wedge n_2 + \frac{f'g'}{f^2}\, y \wedge n_1 + \frac{\kappa f'}{f^2}\, y \wedge n_2. \tag{6.19}$$

So, equalities (6.1) and (6.19) imply

$$C = \frac{1}{1-\kappa^2}\left(-f'^2\, x \wedge y + \kappa'\, x \wedge n_2 + f'g'\, y \wedge n_1 + \kappa f'\, y \wedge n_2\right). \tag{6.20}$$

Using (3.3) and (6.20) we obtain

$$\begin{aligned}\nabla'_x C &= \frac{1}{1-\kappa^2}\left(-2f'f''\, x \wedge y + (f'g'' + g'f'')\, y \wedge n_1 + \kappa f''\, y \wedge n_2\right);\\ \nabla'_y C &= \frac{-\kappa\kappa'}{f(1-\kappa^2)^2}\left(2f'^2 + 1 - \kappa^2\right) x \wedge y + \frac{1}{f(1-\kappa^2)^2}\left(2\kappa\kappa'^2 + \kappa''(1-\kappa^2)\right) x \wedge n_2\\ &\quad + \frac{2\kappa\kappa' f'g'}{f(1-\kappa^2)^2}\, y \wedge n_1 + \frac{2\kappa' f'}{f(1-\kappa^2)^2}\, y \wedge n_2.\end{aligned}$$

From the last formulas we get that $C = \text{const}$ if and only if $\kappa = \text{const}$ and $f'' = 0$. Then the meridian curve m is determined by $f(u) = au + a_1$; $g(u) = bu + b_1$, where a, a_1, b and b_1 are constants, $a^2 + b^2 = 1$. Hence, $\mathcal{M}''_m$ is a developable ruled surface, since $\nabla'_x n_1 = 0$; $\nabla'_x n_2 = 0$. Analogously to the elliptic case we prove that if $\kappa^2 = b^2$ then $\mathcal{M}''_m$ is a marginally trapped surface, and if $\kappa^2 - b^2 \neq 0$, then $\mathcal{M}''_m$ lies in a constant hyperplane $\mathbb{E}^3$ or $\mathbb{E}^3_1$ of $\mathbb{E}^4_1$. Indeed, in the case $\kappa = \varepsilon b$, $\varepsilon = \pm 1$, from (3.6) we obtain $H = -\frac{b}{2f}(n_1 + \varepsilon n_2)$, which implies that $\langle H, H\rangle = 0$, i.e., $\mathcal{M}''_m$ is a marginally trapped surface. If $\kappa^2 - b^2 > 0$ we choose $\theta = \frac{1}{2}\ln\left(\frac{\kappa+b}{\kappa-b}\right)$ and find a suitable normal frame field $\left\{n, n^{\perp}\right\}$ such that $\mathcal{M}''_m$ lies in the constant hyperplane $\mathbb{E}^3_1 = \operatorname{span}\left\{x, y, n^{\perp}\right\}$ of $\mathbb{E}^4_1$. If $\kappa^2 - b^2 < 0$ we choose $\theta = \frac{1}{2}\ln\left(\frac{b+\kappa}{b-\kappa}\right)$ and get that $\mathcal{M}''_m$ lies in the constant hyperplane $\mathbb{E}^3 = \operatorname{span}\{x, y, n\}$.

2. $\kappa = 0$. In this subcase the Laplacian of G is given by

$$\Delta G = \frac{g'^2 + f^2\kappa_m^2}{f^2}\, x \wedge y + \frac{f'g' - f(f\kappa_m)'}{f^2}\, y \wedge n_1 \tag{6.21}$$

and the function λ is expressed as $\lambda = \frac{1}{g'f^2}\left(g'(1 + f^2\kappa_m^2) - ff'(f\kappa_m)'\right)$. Hence, equalities (6.1) and (6.21) imply that

$$C = \left(\frac{g'^2 + f^2\kappa_m^2}{\lambda f^2} - 1\right) x \wedge y + \frac{f'g' - f(f\kappa_m)'}{\lambda f^2}\, y \wedge n_1.$$

Denoting $\psi = \frac{g'^2 + f^2\kappa_m^2}{\lambda f^2} - 1$; $\varphi = \frac{f'g' - f(f\kappa_m)'}{\lambda f^2}$, as in the elliptic case we obtain that $C = \text{const}$ if and only if

$$(\ln\varphi)' = \frac{f'}{g'}\kappa_m.$$

In the hyperbolic case we have $f\kappa_m = -\frac{ff''}{\sqrt{1-f'^2}}$ and the function φ is expressed as follows:

$$\varphi = \frac{\sqrt{1-f'^2}\left(f(1-f'^2)(ff'')' + f^2f'f''^2 + f'(1-f'^2)^2\right)}{(1-f'^2)^2 + f^2f''^2 + ff'(1-f'^2)(ff'')'}.$$

Consequently, $C = \text{const}$ if and only if the function $f(u)$ is a solution of the following differential equation

$$\left(\ln\frac{\sqrt{1-f'^2}\left(f(1-f'^2)(ff'')' + f^2f'f''^2 + f'(1-f'^2)^2\right)}{(1-f'^2)^2 + f^2f''^2 + ff'(1-f'^2)(ff'')'}\right)' = -\frac{f'f''}{1-f'^2}.$$

3. $\kappa = \text{const} \neq 0$ and $f\kappa_m = a = \text{const}$, $a \neq 0$. In this subcase we have

$$\Delta G = \frac{g'^2 - \kappa^2 + a^2}{f^2}\, x \wedge y + \frac{f'g'}{f^2}\, y \wedge n_1 + \frac{\kappa f'}{f^2}\, y \wedge n_2 \tag{6.22}$$

and $\lambda = \frac{1-\kappa^2+a^2}{f^2}$. Since $\lambda \neq 0$ we get $a^2 \neq \kappa^2 - 1$. Equalities (6.1) and (6.22) imply

$$C = \frac{1}{1-\kappa^2+a^2}\left(-f'^2\, x \wedge y + f'g'\, y \wedge n_1 + \kappa f'\, y \wedge n_2\right).$$

Then the derivatives of C are expressed as

$$\begin{aligned} &\nabla'_x C = \frac{1}{1-\kappa^2+a^2}\left(-f'f''\, x \wedge y + g'f''\, y \wedge n_1 + \kappa f''\, y \wedge n_2\right); \\ &\nabla'_y C = 0. \end{aligned} \tag{6.23}$$

Formulas (6.23) imply that $C = \text{const}$ if and only if $f'' = 0$. But the condition $f'' = 0$ implies $\kappa_m = 0$, which contradicts the assumption $f\kappa_m \neq 0$.

Consequently, in this subcase there are no meridian surfaces of hyperbolic type with pointwise 1-type Gauss map of second kind.

Conversely, if one of the cases (i), (ii) or (iii) holds, then it can be seen that $\Delta G = \lambda(G + C)$, i.e., $\mathcal{M}''_m$ has pointwise 1-type Gauss map of second kind. □

Meridian surfaces of parabolic type in $\mathbb{E}^4_1$ are defined as one-parameter systems of meridians of the rotational hypersurface with lightlike axis analogously to the meridian surfaces of elliptic and hyperbolic type [15]. Similarly to the elliptic and hyperbolic type one can classify the meridian surfaces of parabolic type with pointwise 1-type Gauss map.

Acknowledgments

This paper is prepared during the first author's visit at the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences, Sofia, Bulgaria in September 2014.

The second author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under contract DFNI-I 02/14.

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