TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 1, pp. 191–203, February 2016
DOI: 10.11650/tjm.20.2016.5559
This paper is available online at http://journal.tms.org.tw

An Approach to the Log-Euclidean Mean via the Karcher Mean on Symmetric Cones

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Abstract. In a general symmetric cone, we show that certain sequence of the Karcher means converges to the Log-Euclidean mean by using the fact that the Karcher mean is the limit of inductive means. One can see this as a generalization of the Lie-Trotter formula of positive definite matrices into a symmetric cone setting via the least squares mean.

1. Introduction

The geometric mean of positive real numbers is a type of mean or average, which indicates the central tendency of a set of positive real numbers by using the product of their values. Although positive real numbers can be naturally generalized to positive definite matrices having positive eigenvalues, it was not obvious to define the geometric mean of positive definite matrices due to the non-commutativity of matrices. M. Moakher [18] and R. Bhatia and J. Holbrook [5] observed independently that the geometric mean of two positive definite matrices A and B can be defined by (1.1) as the midpoint of the unique Riemannian geodesic curve connecting from A to B:

$$t \mapsto A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}. \tag{1.1}$$

The Riemannian distance between A and B is given by $\delta(A,B) = \left\|\log A^{-1/2}BA^{-1/2}\right\|_2$. As a suitable extension of the geometric mean of two positive definite matrices to n-variables, the least squares mean (or the Karcher mean) $\Lambda(A_1,\dots,A_n)$ of positive definite matrices $A_1,\dots,A_n$ is proposed. It is defined to be the unique minimizer of the sum of squares of the Riemannian distances to each of the A_i, i.e.,

$$\Lambda(A_1,\dots,A_n) = \underset{X\in\mathbb{P}}{\arg\min}\sum_{i=1}^n \delta^2(X,A_i).$$

Received November 12, 2014, accepted May 28, 2015.
Communicated by Ruey-Lin Sheu.
2010 *Mathematics Subject Classification.* 47A64, 17C50, 15B48, 53C20.
Key words and phrases. Lie-Trotter formula, Least squares mean, Symmetric cone, Hadamard space.
Kum was supported by Basic Science Research Program through NRF Grant No. 2012R1A1A2041136. This work was accomplished while the author was visiting to KIAS. He is thankful for the hospitality of KIAS
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Here $\mathbb{P}$ denotes the open convex cone of positive definite matrices with fixed dimension. Since its appearance, the least squares mean of positive definite matrices for the Riemannian distance has played a significant role in many applied areas. Also, there have been studied a variety of constructions for the weighted geometric means of n positive definite matrices (see [2, 6, 9, 10]). Among them is the *Log-Euclidean mean* given by

$$\exp\left(\sum_{i=1}^{n} w_i \log A_i\right) \tag{1.2}$$

where $\omega = (w_1, \dots, w_n)$ is a positive probability vector. It may not satisfy all of the Ando-Li-Mathias properties, but we can see in [3] that it has been used in the field of diffusion tensor magnetic resonance imaging (DT - MRI).

In this article, we find a new theoretical connection between the Karcher mean and the Log-Euclidean mean. In fact, it will be shown in Theorem 4.1 that a certain sequence of the Karcher means converges to the Log-Euclidean mean. This result can be regarded as a generalization of the Lie-Trotter formula related with the least squares mean. We do the task in the general framework of symmetric cones which contain the convex cone of positive definite matrices as a special case. In addition, we raise a corresponding question concerning positive definite operators in Remark 4.4. So the main result may have its own theoretical interest concerned with matrix mean theory.

2. Euclidean Jordan algebras and symmetric cones

In this section, we briefly describe (following mostly [7]) some Jordan-algebraic concepts pertinent to our purpose. A *Jordan algebra* V over $\mathbb{R}$ is a commutative algebra satisfying $x^2(xy) = x(x^2y)$ for all $x, y \in V$. For $x \in V$, let $L(x)$ be the linear operator defined by $L(x)y = xy$, and let $P(x) = 2L(x)^2 - L(x^2)$. The map P is called the quadratic representation of V. An element $x \in V$ is said to be invertible if there exists an element y (denoted by $y = x^{-1}$) in the subalgebra generated by x and e (the Jordan identity) such that $xy = e$. The following appears at Propositions II.3.1 and II.3.3 of [7].

Proposition 2.1. *Let V be a Jordan algebra with an identity element e.*

(i) *An element x in V is invertible if and only if $P(x)$ is invertible. In this case: $P(x)^{-1} = P(x^{-1})$.*

(ii) *If x and y are invertible, then $P(x)y$ is invertible and $(P(x)y)^{-1} = P(x^{-1})y^{-1}$.*

(iii) *For any elements x and y, $P(P(x)y) = P(x)P(y)P(x)$. In particular, $P(x^m) = P(x)^m$ for $m \in \mathbb{N}$.*

(iv) *If V is finite-dimensional, then* $P(\exp x) = \exp 2L(x)$ *for any* $x \in V$*, where*

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

A finite-dimensional Jordan algebra V with an identity element e is said to be *Euclidean* if there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\langle xy, z \rangle = \langle y, xz \rangle$ for all $x, y, z \in V$. An element $c \in V$ is called an idempotent if $c^2 = c \neq 0$. We say that $c_1, \ldots, c_k$ is a complete system of orthogonal idempotents if $c_i^2 = c_i$, $c_i c_j = 0$, $i \neq j$, $c_1 + \cdots + c_k = e$. An idempotent is primitive if it is non-zero and cannot be written as the sum of two non-zero idempotents. A Jordan frame is a complete system of orthogonal primitive idempotents.

Theorem 2.2 (Spectral theorem, first version)**.** [7, Theorem III.1.1] *Let V be a Euclidean Jordan algebra. Given $x \in V$, there exist real numbers $\lambda_1, \ldots, \lambda_k$ all distinct and a unique complete system of orthogonal idempotents $c_1, \ldots, c_k$ such that*

$$x = \sum_{i=1}^{k} \lambda_i c_i. \tag{2.1}$$

The numbers λ_i are called the eigenvalues and (2.1) *is called the spectral decomposition of x.*

Theorem 2.3 (Spectral theorem, second version)**.** [7, Theorem III.1.2] *Any two Jordan frames in a Euclidean Jordan algebra V have the same number of elements (called the rank of V, denoted by* $\operatorname{rank}(V)$*). Given $x \in V$, there exists a Jordan frame $c_1, \ldots, c_r$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that $x = \sum_{i=1}^{r} \lambda_i c_i$. The numbers λ_i (with their multiplicities) are uniquely determined by x.*

Definition 2.4. Let V be a Euclidean Jordan algebra of $\operatorname{rank}(V) = r$. The spectral mapping $\lambda \colon V \to \mathbb{R}^r$ is defined by $\lambda(x) = (\lambda_1(x), \ldots, \lambda_r(x))$, where $\lambda_i(x)$'s are eigenvalues of x (with multiplicities) as in Theorem 2.3 in non-increasing order $\lambda_{\max}(x) = \lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x) = \lambda_{\min}(x)$. Furthermore, $\det(x) = \prod_{i=1}^{r} \lambda_i(x)$ and $\operatorname{tr}(x) = \sum_{i=1}^{r} \lambda_i(x)$.

Let Q be the set of all square elements of V. Then Q is a closed convex cone of V with $Q \cap -Q = \{0\}$, and is the set of element $x \in V$ such that $L(x)$ is positive semi-definite. It turns out that Q has non-empty interior Ω, and Ω is a symmetric cone, that is, the group $G(\Omega) = \{g \in \mathrm{GL}(V) \mid g(\Omega) = \Omega\}$ acts transitively on it and Ω is a self-dual cone with respect to the inner product $\langle \cdot, \cdot \rangle$ (see [7]). Furthermore, for any a in Ω, $P(a) \in G(\Omega)$ and is positive definite. Note that $\overline{\Omega} = \{x \in V \mid \lambda_i(x) \geq 0, i = 1, \ldots, r\}$. For $x, y \in V$, we define

$$x \leq y \quad \text{if } y - x \in \overline{\Omega}$$

and $x < y$ if $y - x \in \Omega$. Clearly $\overline{\Omega} = \{x \in V \mid x \geq 0\}$ and $\Omega = \{x \in V \mid x > 0\} = \{x \in V \mid \lambda_{\min}(x) > 0\}$.

On the other hand, the symmetric cone Ω in a Euclidean Jordan algebra V has an important geometric feature. That is, it admits a $G(\Omega)$-invariant Riemannian metric defined by

$$\langle u, v \rangle_x = \left\langle P(x)^{-1}u, v \right\rangle, \quad x \in \Omega,\ u, v \in V. \tag{2.2}$$

For this, refer to [7]. So Ω is a symmetric Riemannian space of non-compact type with respect to its distance metric. In this case, it is shown in [13, Proposition 2.6] that the unique geodesic curve joining a and b is $t \mapsto a\#_t b := P(a^{1/2})(P(a^{-1/2})b)^t$ and the Riemannian distance $\delta(a, b)$ is given by

$$\delta(a, b) = \left(\sum_{i=1}^{r} \log^2 \lambda_i(P(a^{-1/2})b) \right)^{1/2}.$$

Basically the trace is an inner product on V, and the Jordan algebra V endowed with the trace inner product $\langle x, y \rangle = \mathrm{tr}(xy)$ is still Euclidean [7]. Hence, *throughout this paper, we assume that V is a Euclidean Jordan algebra of rank r equipped with the trace inner product. Also V is always assumed to be simple.*

For square matrices (or bounded operators) X and Y on a Hilbert space the *Lie-Trotter formula* is given by

$$\exp(X + Y) = \lim_{m \to \infty} \left[\exp\left(\frac{X}{m}\right) \exp\left(\frac{Y}{m}\right) \right]^m. \tag{2.3}$$

It has been fundamental in the Lie theory and is of great interest in many research areas including quantum relative entropy. In [1, Theorem 2.7], the Lie-Trotter formula for n positive definite matrices has been derived in terms of weighted geometric means called the Sagae-Tanabe means. For the purpose to establish the main result, we extend this result into a general symmetric cone setting according to the argument in [14].

Theorem 2.5. *Let $a_1, \ldots, a_n$ be elements in Ω and let $(w_1, \ldots, w_n)$ be a positive probability vector. Then*

$$\exp\left(\sum_{i=1}^{n} w_i \log a_i \right) = \lim_{m \to \infty} \left[a_n^{1/m} \#_{\alpha_{n-1}} \cdots \#_{\alpha_2} \left(a_2^{1/m} \#_{\alpha_1} a_1^{1/m} \right) \right]^m,$$

where

$$\alpha_k = 1 - w_{k+1} \left(\sum_{i=1}^{k+1} w_i \right)^{-1} = \left(\sum_{i=1}^{k} w_i \right) \left(\sum_{i=1}^{k+1} w_i \right)^{-1}$$

for all $k = 1, \ldots, n-1$.

Proof. By Proposition 2.1(iii), (iv) and [1, Theorem 2.7], we get

$$\begin{aligned}
P\left(\exp\left(\sum_{i=1}^n w_i \log a_i\right)\right) &= \exp 2L\left(\sum_{i=1}^n w_i \log a_i\right)\\
&= \exp\left(\sum_{i=1}^n w_i L(2\log a_i)\right)\\
&= \lim_{m\to\infty}\left[\exp\frac{1}{m}L(2\log a_n)\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}\exp\frac{1}{m}L(2\log a_1)\right]^m\\
&= \lim_{m\to\infty}\left[P\left(\exp\frac{\log a_n}{m}\right)\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}P\left(\exp\frac{\log a_1}{m}\right)\right]^m\\
&= \lim_{m\to\infty}\left[P\left(\exp\frac{\log a_n}{m}\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}\exp\frac{\log a_1}{m}\right)\right]^m\\
&= \lim_{m\to\infty}P\left(\left[\exp\frac{\log a_n}{m}\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}\exp\frac{\log a_1}{m}\right]^m\right)
\end{aligned}$$

where the fifth equality comes from the formula $P(a\#_\alpha b) = P(a)\#_\alpha P(b)$ for any $a, b \in \Omega$ in [14]. Hence

$$\begin{aligned}
\left[\exp\left(\sum_{i=1}^n w_i \log a_i\right)\right]^2 &= P\left(\exp\left(\sum_{i=1}^n w_i \log a_i\right)\right)e\\
&= \lim_{m\to\infty}P\left(\left[\exp\frac{\log a_n}{m}\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}\exp\frac{\log a_1}{m}\right]^m\right)e\\
&= \lim_{m\to\infty}\left[\exp\frac{\log a_n}{m}\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}\exp\frac{\log a_1}{m}\right]^{2m}.
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
\exp\left(\sum_{i=1}^n w_i \log a_i\right) &= \lim_{m\to\infty}\left[\exp\frac{\log a_n}{m}\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}\exp\frac{\log a_1}{m}\right]^m\\
&= \lim_{m\to\infty}\left[a_n^{1/m}\#_{\alpha_{n-1}}\cdots\#_{\alpha_1}a_1^{1/m}\right]^m.
\end{aligned}$$

This completes the proof. □

3. Hadamard space and least squares mean

A complete metric space (X, d) is called a *Hadamard space* if it satisfies the semiparallelogram law: for any $x, y \in X$ there exists $m \in X$

$$d^2(m,z) + \frac{1}{4}d^2(x,y) \leq \frac{1}{2}\left[d^2(x,z) + d^2(y,z)\right] \tag{3.1}$$

for any $z \in X$. The point m appeared in the equation (3.1) is the unique metric midpoint between x and y, and it gives rise to a unique minimal geodesic $\gamma\colon [0,1] \to X$ joining

$\gamma(0) = x$ and $\gamma(1) = y$. One can see that for any $s, t \in [0, 1]$,

$$d(\gamma(s), \gamma(t)) = |s - t|\, d(x, y).$$

The unique geodesic γ connecting x and y is denoted by $\gamma(t) = x\#_t y$ for $t \in [0, 1]$. We call $x\#_t y$ the *t-weighted geometric mean* of x and y.

Definition 3.1. Let (X, d) be a Hadamard space. The *least squares mean* $\Lambda_d(\omega; \boldsymbol{a})$ of the n-tuple $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in X^n$ and a positive probability vector $\omega = (w_1, w_2, \ldots, w_n)$ is defined by the minimizer of the function $\sum_{i=1}^{n} w_i d^2(x, a_i)$. In other words,

$$\Lambda_d(\omega; \boldsymbol{a}) = \operatorname*{arg\,min}_{x \in X} \sum_{i=1}^{n} w_i d^2(x, a_i). \tag{3.2}$$

On Hadamard spaces the least squares mean of any n-tuple of points always exists and is unique. Moreover, it is well-known (cf. [19]) that Λ_d is contractive for d, that is,

$$d(\Lambda_d(\omega; \boldsymbol{a}), \Lambda_d(\omega; \boldsymbol{b})) \leq \sum_{i=1}^{n} w_i d(a_i, b_i) \tag{3.3}$$

for all $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n) \in X^n$ and a positive probability vector $\omega = (w_1, \ldots, w_n)$. We see two typical examples of Hadamard spaces and the least squares means.

Example 3.2. Let us consider a metric d on $\mathbb{P}$ defined by

$$d(A, B) = \|\log A - \log B\|_2$$

for any $A, B \in \mathbb{P}$. Then $(\mathbb{P}, d)$ is a Hadamard space, since the exponential map exp from the space $(\mathbb{H}, \|\cdot\|_2)$ of all Hermitian matrices to $(\mathbb{P}, d)$ is an isometry. One can easily see that the least squares mean $\Lambda_d(\omega; \mathbb{A})$ of the n-tuple $\mathbb{A} = (A_1, A_2, \ldots, A_n) \in \mathbb{P}^n$ is the Log-Euclidean mean, that is,

$$\Lambda_d(\omega; \mathbb{A}) = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right).$$

Example 3.3. The aforementioned metric space (Ω, δ) consisting of a symmetric cone Ω equipped with the Riemannian distance δ is a Hadamard space which contains $(\mathbb{P}, \delta)$ as a particular case [5]. Recently, many scholars have been studying the method to find the Karcher mean of positive definite matrices with its properties, see [8, 11, 16, 17].

Now we introduce a recent result by Y. Lim and M. Pálfia [17] which is concerned with so called 'no dice theorem'. Motivated by the beginning work of J. Holbrook [8] in $(\mathbb{P}, \delta)$, they have constructed a sequence of weighted inductive means and have shown that it converges to the weighted Karcher mean [17] in a general Hadamard space. To be more specific, let (X, δ) be a Hadamard space. For a positive probability vector $\omega = (w_1, w_2, \ldots, w_n)$, we denote

$$\overline{\omega} := (w_1, \ldots, w_n, w_1, \ldots, w_n, \ldots),$$

and $s(N) := \sum_{i=1}^{N} \overline{\omega}_i$ for each $N \in \mathbb{N}$, where $\overline{\omega}_i$ is the ith component of the infinite-dimensional vector $\overline{\omega}$. For an n-tuple $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in X^n$,

$$\Delta \boldsymbol{a} := \max \left\{ \delta(a_i, a_j) : 1 \leq i, j \leq n \right\},$$

which is called a diameter of $\boldsymbol{a}$.

Definition 3.4. Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in X^n$ and $\omega = (w_1, w_2, \ldots, w_n)$ a positive probability vector. Then the sequence of weighted inductive means is defined by

$$S_1(\omega; \boldsymbol{a}) = a_1, \quad S_N(\omega; \boldsymbol{a}) = a_k \#_{\frac{s(N-1)}{s(N)}} S_{N-1}(\omega; \boldsymbol{a}) \tag{3.4}$$

for natural numbers $N \geq 2$, where $k \in \{1, \ldots, n\}$ is chosen so that $k \equiv N \pmod{n}$.

Theorem 3.5. [17, Theorem 3.4] *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in X^n$ and $\omega = (w_1, w_2, \ldots, w_n)$ a positive probability vector. Then*

$$\delta^2(\Lambda_\delta(\omega; \boldsymbol{a}), S_N(\omega; \boldsymbol{a})) \leq \frac{1}{s(N)} \left[3(\Delta \boldsymbol{a})^2 + \sum_{i=1}^{n} w_i \delta^2(\Lambda_\delta(\omega; \boldsymbol{a}), a_i) \right] \tag{3.5}$$

for all $N \in \mathbb{N}$. That is,

$$\lim_{N \to \infty} S_N(\omega; \boldsymbol{a}) = \Lambda_\delta(\omega; \boldsymbol{a}).$$

4. Main results

From now on, we restrict our attention to the Hadamard space (Ω, δ) where Ω is the symmetric cone in a simple Euclidean Jordan algebra $(V, \|\cdot\|)$. Here $\|\cdot\|$ denotes the norm induced by the trace inner product on V. Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \Omega^n$ and let $\omega = (w_1, w_2, \ldots, w_n)$ be a positive probability vector. We consider a double sequence $\{S_{m,N}(\boldsymbol{a})\}$

$$S_{m,1}(\omega; \boldsymbol{a}) = a_1, \quad S_{m,N}(\omega; \boldsymbol{a}) = \left[a_k^{1/m} \#_{\frac{s(N-1)}{s(N)}} S_{m,N-1}^{1/m}(\omega; \boldsymbol{a}) \right]^m \tag{4.1}$$

for natural numbers $N \geq 2$, where $k \in \{1, \ldots, n\}$ is chosen so that $k \equiv N \pmod{n}$. The main result is that the sequence of Karcher means converges to the Log-Euclidean mean, which can be seen as the generalization of Lie-Trotter formula with the least squares means in a symmetric cone.

Theorem 4.1. *Let $\boldsymbol{a} = (a_1, a_2, \dots, a_n) \in \Omega^n$ and $\omega = (w_1, w_2, \dots, w_n)$ a positive probability vector. Then*

$$\lim_{m\to\infty} \Lambda_\delta(\omega; \boldsymbol{a}^{1/m})^m = \exp\left(\sum_{i=1}^n w_i \log a_i\right),$$

where $\boldsymbol{a}^{1/m} = (a_1^{1/m}, a_2^{1/m}, \dots, a_n^{1/m})$ for any $m \in \mathbb{N}$.

Since the exponential map is a diffeomorphism from V onto Ω, we obtain the following.

Corollary 4.2. *Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in V^n$ and $\omega = (w_1, w_2, \dots, w_n)$ a positive probability vector. Then*

$$\lim_{m\to\infty} \Lambda_\delta(\omega; e^{\boldsymbol{x}/m})^m = \exp\left(\sum_{i=1}^n w_i x_i\right),$$

where $e^{\boldsymbol{x}} = (e^{x_1}, e^{x_2}, \dots, e^{x_n}) \in \Omega^n$.

One may be able to show the following by direct computation of the Log-Euclidean mean, but it is also possible to prove using the main result. On the other hand, the properties related with monotonicity are not satisfied because, in general, $a^m \not\leq b^m$ for $m \geq 2$ even if $a \leq b$.

Corollary 4.3. *Let $\boldsymbol{a} = (a_1, a_2, \dots, a_n) \in \Omega^n$ and $\omega = (w_1, w_2, \dots, w_n)$ a positive probability vector. Then the Log-Euclidean mean $G(\omega; \boldsymbol{a}) = \exp\left(\sum_{i=1}^n w_i \log a_i\right)$ satisfies the following.*

(L1) $G(\omega; \boldsymbol{a}) = \prod_{i=1}^n a_i^{w_i}$ *if a_i's operator commute.*

(L2) $G(\omega; \alpha_1 a_1, \dots, \alpha_n a_n) = [\prod_{i=1}^n \alpha_i^{w_i}]\, G(\omega; \boldsymbol{a})$ *for all $\alpha_i > 0$.*

(L3) $G(\omega_\sigma; \boldsymbol{a}_\sigma) = G(\omega; \boldsymbol{a})$, *where $\omega_\sigma = (w_{\sigma(1)}, \dots, w_{\sigma(n)})$.*

(L4) $\delta(G(\omega; \boldsymbol{a}), G(\omega; \boldsymbol{b})) \leq \sum_{i=1}^n w_i \delta(a_i, b_i)$.

(L5) $G(\omega; \boldsymbol{a}^{-1})^{-1} = G(\omega; \boldsymbol{a})$.

(L6) $\det G(\omega; \boldsymbol{a}) = \prod_{i=1}^n (\det a_i)^{w_i}$.

Remark 4.4. The main result Theorem 4.1 shows a remarkable relation between the least squares mean and the Log-Euclidean mean in a symmetric cone. On the other hand, J. Lawson and Y. Lim [12] have defined the Karcher mean Λ for positive definite operators $A_1, \dots, A_n$ as the limit of the monotonically decreasing family of power means. From this idea, it would be interesting to establish the same relation for positive definite operators $A_1, \dots, A_n$, that is,

$$\lim_{m\to\infty} \Lambda(\omega; A_1^{1/m}, \dots, A_n^{1/m})^m = \exp\left(\sum_{i=1}^n w_i \log A_i\right). \tag{4.2}$$

5. Proof of Theorem 4.1

In order to prove Theorem 4.1, we need the following two lemmas. The first one is a generalization of the well-known exponential metric increasing property (EMI) due to Bhatia [4] into a symmetric cone case. Actually, this fact is pointed out by Lim [15].

Lemma 5.1 (Exponential Metric Increasing Property)**.** *For all $x, y \in V$, we have*

$$\|x - y\| \leq \delta(\exp x, \exp y).$$

Lemma 5.2. *Let (X, d) be a complete metric space. Let $\{a_{m,n}\}_{m,n=1}^{\infty}$ be a double sequence in X. Suppose that*

$$\lim_{n\to\infty} \sup_{m} d(a_{m,n}, a_m) = 0,$$

where $a_m = \lim_{n\to\infty} a_{m,n}$ for each $m \in \mathbb{N}$, and $\lim_{m\to\infty} a_{m,n} = b_n$ for each $n \in \mathbb{N}$. Then $\{b_n\}$ converges, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} = \lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{m\to\infty} a_m.$$

Proof. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} a_{m,n}$ exists for each $m \in \mathbb{N}$, $\{a_{m,n}\}$ is a Cauchy sequence in (X, d) for each $m \in \mathbb{N}$. There exists an $N \in \mathbb{N}$ such that for any $n \geq N$ and $l \geq N$, and any $m \in \mathbb{N}$,

$$d(a_{m,n}, a_{m,l}) < \frac{\epsilon}{3}.$$

So we have

$$\begin{aligned} d(b_n, b_l) &\leq d(b_n, a_{m,n}) + d(a_{m,n}, a_{m,l}) + d(a_{m,l}, b_l) \\ &< \frac{\epsilon}{3} + d(b_n, a_{m,n}) + d(a_{m,l}, b_l). \end{aligned}$$

Taking $m \to \infty$ implies that $d(b_n, b_l) \leq \epsilon$, and so $\{b_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{b_n\}$ converges to an element $b \in X$. On the other hand, for any $m, n \in \mathbb{N}$ we have

$$d(a_m, b) \leq d(a_m, a_{m,n}) + d(a_{m,n}, b_n) + d(b_n, b).$$

Then by the uniform convergence, we first choose an $n \in \mathbb{N}$ such that for any $m \in \mathbb{N}$

$$d(a_{m,n}, a_m) < \frac{\epsilon}{3} \quad \text{and} \quad d(b_n, b) < \frac{\epsilon}{3}.$$

For chosen n, since $\lim_{m\to\infty} a_{m,n} = b_n$, there exists an $N \in \mathbb{N}$ such that for any $m \geq N$

$$d(a_{m,n}, b_n) < \frac{\epsilon}{3},$$

which implies that for any $m \geq N$, $d(a_m, b) < \epsilon$. Therefore,

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{m\to\infty} a_m = b = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}. \qquad \square$$

We are now in a position to prove the main result.

Proof of Theorem 4.1. Note from Example 3.3 that (Ω, δ) is a complete metric space. By Lemma 5.1, we have, for any $x, y \in V$,

$$\|x - y\| \le \delta(\exp x, \exp y).$$

So

$$\begin{aligned} &\left\|\log(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m})^m - \log S_{m,N}(\omega; \boldsymbol{a})\right\| \\ &= m\left\|\log \Lambda_\delta(\omega; \boldsymbol{a}^{1/m}) - \log S_{m,N}(\omega; \boldsymbol{a})^{1/m}\right\| \\ &\le m\delta(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), S_{m,N}(\omega; \boldsymbol{a})^{1/m}). \end{aligned} \tag{5.1}$$

Furthermore, by (3.5), we have

$$\begin{aligned} &\delta^2(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), S_{m,N}(\omega; \boldsymbol{a})^{1/m}) \\ &\le \frac{1}{s(N)}\left[3(\Delta(\boldsymbol{a}^{1/m}))^2 + \sum_{i=1}^n w_i \delta^2(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), a_i^{1/m})\right]. \end{aligned} \tag{5.2}$$

Here, we compute $\Delta(\boldsymbol{a}^{1/m})$ and $\delta(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), a_i^{1/m})$. By the contractive property of geometric means

$$\Delta(\boldsymbol{a}^{1/m}) = \max\left\{\delta(a_i^{1/m}, a_j^{1/m}) : 1 \le i, j \le n\right\} \le \frac{1}{m}\Delta\boldsymbol{a},$$

and by (3.3)

$$\begin{aligned} \delta(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), a_i^{1/m}) &= \delta(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), \Lambda_\delta(\omega; a_i^{1/m}, \dots, a_i^{1/m})) \\ &\le \sum_{j=1}^n w_j \delta(a_j^{1/m}, a_i^{1/m}) \le \frac{1}{m}\sum_{j=1}^n w_j \delta(a_j, a_i) \le \frac{1}{m}\Delta\boldsymbol{a}. \end{aligned}$$

So (5.2) can be simplified to

$$\delta^2(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), S_{m,N}(\omega; \boldsymbol{a})^{1/m}) \le \frac{4}{m^2 s(N)}(\Delta\boldsymbol{a})^2,$$

and equivalently,

$$\delta(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), S_{m,N}(\omega; \boldsymbol{a})^{1/m}) \le \frac{2}{m\sqrt{s(N)}}\Delta\boldsymbol{a}.$$

Then the equation (5.1) becomes

$$\begin{aligned} &\left\|\log \Lambda_\delta(\omega; \boldsymbol{a}^{1/m})^m - \log S_{m,N}(\omega; \boldsymbol{a})\right\| \\ &\le m\delta(\Lambda_\delta(\omega; \boldsymbol{a}^{1/m}), S_{m,N}(\omega; \boldsymbol{a})^{1/m}) \le \frac{2}{\sqrt{s(N)}}\Delta\boldsymbol{a}. \end{aligned}$$

Therefore, we have

$$\lim_{N\to\infty} \sup_m \left\| \log \Lambda_\delta(\omega; \boldsymbol{a}^{1/m})^m - \log S_{m,N}(\omega; \boldsymbol{a}) \right\| = 0,$$

since $s(N) \to \infty$ as $N \to \infty$. For each $N \in \mathbb{N}$, there exists a $q \in \mathbb{N}$ such that $N = n \cdot q + k$ for some $k = 1, \ldots, n$. Then Theorem 2.5 implies that

$$\lim_{m\to\infty} S_{m,N}(\omega; \boldsymbol{a}) = \exp\left\{ \frac{1}{s(N)} \left(r \sum_{i=1}^{m} w_i \log a_i + \sum_{j=1}^{k} w_j \log a_j \right) \right\}.$$

By Lemma 5.2, the right-hand side of the above equation converges to the Log-Euclidean mean. That is,

$$\lim_{N\to\infty} \exp\left\{ \frac{1}{s(N)} \left(q \sum_{i=1}^{n} w_i \log a_i + \sum_{j=1}^{k} w_j \log a_j \right) \right\} = \exp\left(\sum_{i=1}^{n} w_i \log a_i \right),$$

since $s(N) = q \sum_{i=1}^{m} w_i + \sum_{j=1}^{k} w_j = q + \sum_{j=1}^{k} w_j$,

$$\lim_{N\to\infty} \frac{q}{s(N)} = \lim_{q\to\infty} \frac{q}{q + \sum_{j=1}^{k} w_j} = 1.$$

The conclusion of Lemma 5.2 tells us that

$$\begin{aligned} \lim_{m\to\infty} \Lambda_\delta(\omega; \boldsymbol{a}^{1/m})^m &= \lim_{m\to\infty} \lim_{N\to\infty} S_{m,N}(\omega; \boldsymbol{a}) \\ &= \lim_{N\to\infty} \lim_{m\to\infty} S_{m,N}(\omega; \boldsymbol{a}) = \exp\left(\sum_{i=1}^{n} w_i \log a_i \right). \end{aligned}$$

This completes the proof. □

Acknowledgments

The authors thank Professor Jimmie Lawson for his comments and help in carrying out the research appearing in this paper.

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