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BILATERAL CONTACT PROBLEM WITH FRICTION AND WEAR FOR AN ELECTRO ELASTIC-VISCOPLASTIC MATERIALS WITH DAMAGE

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Abstract. We consider a mathematical problem for quasistatic contact between an electro elastic-viscoplastic body and an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. We employ the electro elastic-viscoplastic with damage constitutive law for the material. The evolution of the damage is described by an inclusion of parabolic type. The problem is formulated as a system of an elliptic variational inequality for the displacement, a parabolic variational inequality for the damage and a variational equality for the electric stress. We establish a variational formulation for the model and we give the wear conditions for the existence of a unique weak solution to the problem. The proofs are based on classical results for elliptic variational inequalities, parabolic inequalities and fixed point arguments.

1. Introduction

Scientific research in mechanics are articulated around two main components: one devoted to the laws of behavior and other boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world.

In this paper, we study a problem involving boundary conditions describing real phenomena such as contact and friction and other very important such as the damage and the wear of materials. for the constitutive law we consider an electro elastic-viscoplastic body .The piezoelectric effect is characterized by the coupling between the mechanical and electrical behavior of the materials.

The piezoelectric effect is the apparition of electric charges on surfaces of particular crystals after deformation. Its reverse effect consists of the generation of stress and strain in crystals under the action of the electric field on the boundary. Materials undergoing piezoelectric effects are called piezoelectric materials; their study require

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techniques and results from electromagnetic theory and continuum mechanics. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore, there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [4, 5, 9, 10, 11] . A static frictional contact problem for electric-elastic materials was considered in [5, 11]. A frictional contact problem for electro viscoelastic materials was studied in [10].Contact problems with friction and adhesion for electro elastic-viscoplastic materials were studied recently in [1] . The goal of this paper is to make the coupling of an electro elastic-viscoplastic problem with damage and a frictional contact problem with wear. We study a quasistatic problem of frictional bilateral contact with wear. We model the material behavior with an electro elastic-viscoplastic constitutive law with damage and the contact is frictional and bilateral with a moving rigid foundation. We derive a variational formulation and prove the existence and uniqueness of the weak solution.

The paper is organized as follows. In Section 2 we introduce the notation and give some preliminaries. In Section 3 we present the mechanical problem, list the assumptions on the data, give the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result, Theorem (4.1) The proof of the theorem is based on arguments for elliptic variational inequalities, parabolic inequalities and fixed point arguments.

2. Notations and Preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [6, 7, 15]. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while "$\cdot$" and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with outer Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega = \Gamma$. We shall use the notation

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) : u_i \in L^2(\Omega)\right\},$$
$$\mathcal{H} = \left\{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$
$$H^1(\Omega)^d = \{\mathbf{u} = (u_i) \in H : u_i \in H^1(\Omega)\},$$
$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} : Div\sigma \in H\},$$

Here $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\sigma = (\sigma_{ij,j}) \ .$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, $H^1(\Omega)^d$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by :

$$(\mathbf{u},\mathbf{v})_H=\int_\Omega u_iv_idx,\quad (\sigma,\tau)_{\mathcal{H}}=\int_\Omega \sigma_{ij}\tau_{ij}dx,$$

$$(\sigma,\tau)_{\mathcal{H}}=\int_\Omega \sigma\cdot\tau dx\ \ \forall\sigma,\tau\in\mathcal{H},$$

$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d}=\int_\Omega \mathbf{u}.\mathbf{v}dx+\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}dx\ \ \forall\mathbf{u},\mathbf{v}\in H^1(\Omega)^d,$$

where

$$\nabla\mathbf{v}=(v_{i,j})\quad \forall\mathbf{v}\in H^1(\Omega)^d.$$

$$(\sigma,\tau)_{\mathcal{H}_1}=(\sigma,\tau)_{\mathcal{H}}+(\mathrm{Div}\sigma,\mathrm{Div}\tau)_H\ \forall\sigma,\tau\in\mathcal{H}_1,$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Let $H_\Gamma=(H^{1/2}(\Gamma))^d$ and $\gamma: H^1(\Gamma))^d\to H_\Gamma$ be the trace map. For every element $\mathbf{v}\in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace map $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ, and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu=\mathbf{v}\cdot\nu,\quad \mathbf{v}_\tau=\mathbf{v}-v_\nu\nu. \tag{2.1}$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\sigma:\Omega\to\mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu=(\sigma\nu)\cdot\nu,\quad \mathbf{v}_\tau=\sigma\nu-\sigma_\nu\nu, \tag{2.2}$$

and for all $\sigma\in\mathcal{H}_1$ the following Green's formula holds:

$$(\sigma,\varepsilon(\mathbf{v}))_{\mathcal{H}}+(\mathrm{Div}\sigma,\mathbf{v})_H=\int_\Gamma \sigma\nu\cdot\mathbf{v}da\quad \forall\mathbf{v}\in H^1(\Omega)^d.$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1\leqslant p\leqslant\infty$ and $k>1$. For $T>0$ we denote by $\mathcal{C}(0,T;X)$ and $\mathcal{C}^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\mathbf{f}\|_{\mathcal{C}(0,T;X)}=\max_{t\in[0,T]}\|\mathbf{f}\,(t)\,\|_X,$$

$$\|\mathbf{f}\|_{\mathcal{C}^1(0,T;X)}=\max_{t\in[0,T]}\|\mathbf{f}\|_X+\max_{t\in[0,T]}\|\dot{\mathbf{f}}\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and if X_1 and X_2 are real Hilbert spaces then $X_1\times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

3. The Model and Variational Problem

We describe the model for the process, we present its variational formulation. The physical setting is the following. An electro elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body undergoes the action of body forces of density f_0 and volume electric charges of density q_0. It also undergoes the mechanical and electric constraint on the boundary.We consider a partition of Γ into three disjoint parts Γ_1, Γ_2 and Γ_3, on one hand, and into two measurable parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_1) > 0$, $meas(\Gamma_a) > 0$, and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ acts in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach to the process. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use an electro elastic-viscoplastic constitutive law with damage given by

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{B}(\varepsilon(\mathbf{u}(t)), \beta) \\ &\quad + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s))\Big) ds - \mathcal{E}^* E(\varphi), \\ \mathbf{D} &= \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{B}E(\varphi), \end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor $\mathcal{E}^*$ is its transpose and $\boldsymbol{B}$ denotes the electric permittivity tensor, and $\mathcal{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material, where β is an internal variable describing the damage of the material caused by elastic deformations. The differential inclusion used for the evolution of the damage field is

$$\dot{\beta} - k\Delta\beta + \partial\varphi_k(\beta) \ni S(\varepsilon(\mathbf{u}), \beta),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in V : 0 \leqslant \xi(x) \leqslant 1 \text{ a.e. } x \in \Omega\},$$

where k is a positive coefficient, $\partial\varphi_k$ denotes the subdifferential of the indicator function of the set K and S is a given constitutive function which describes the sources of the damage in the system. When $\beta = 1$ the material is undamaged, when $\beta = 0$ the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [16] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of one-dimensional damage models can be found in [12].

We now briefly describe the boundary conditions on the contact surface Γ_3 , based on the model derived in [16]. We introduce the wear function $w : \Gamma_3 \times [0, T] \to \mathbb{R}^+$ which measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Since the body is in bilateral contact with the foundation, it follows that

(3.1) $$\mathbf{u}_\nu = -w \text{ on } \Gamma_3.$$

Thus the location of the contact evolves with the wear. We point out that the effect of the wear is the recession on Γ_3 and therefore, it is natural to expect that $\mathbf{u}_\nu \leqslant 0$ on Γ_3 , which implies $w \geqslant 0$ on Γ_3 .

The evolution of the wear of the contacting surface is governed by a simplified version of Archard's law (see [16]) which we now describe. The rate form of

Archard's law is

$$\dot{w} = -k_1 \sigma_\nu \left| \dot{\mathbf{u}}_\tau - \mathbf{v}^* \right|,$$

where $k_1 > 0$ is a wear coefficient, $\mathbf{v}^*$ is the tangential velocity of the foundation and $|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|$ represents the slip speed between the contact surface and the foundation.

We see that the rate of wear is assumed to be proportional to the contact stress and the slip speed. For the sake of simplicity we assume in the rest of the section that the motion of the foundation is uniform, i.e., $\mathbf{v}^*$ does not vary in time. Denote $v^* = |\mathbf{v}^*| > 0$.

We assume that v^* is large so that we can neglect in the sequel $\dot{\mathbf{u}}_\tau$ ascompared with $\mathbf{v}^*$ to obtain the following version of Archard's law

(3.2) $$\dot{w} = -k_1 v^* \sigma_\nu,$$

The use of the simplified law (3.2) for the evolution of the wear avoids some mathematical difficulties in the study of the quasistatic electro-viscoplastic contact problem.

We can now eliminate the unknown function w from the problem. In this manner, the problem decouples, and once the solution of the frictional contact problem has been obtained, the wear of the surface can be obtained by integration of (3.2).Let $\zeta = k_1 v^*$ and $\alpha = \frac{1}{\zeta}$. Using (3.1) and (3.2) we have

(3.3) $$\sigma_\nu = \alpha \dot{\mathbf{u}}_\nu.$$

We model the frictional contact between the electro-viscoplastic body and the foundation with Coulomb's law of dry friction. Since there is only sliding contact, it

$$|\sigma_\tau| = \mu\,|\sigma_\nu|\,, \quad \sigma_\tau = -\lambda\,(\dot{\mathbf{u}}_\tau - \mathbf{v}^*)\,, \quad \lambda \geqslant 0, \tag{3.4}$$

where $\mu > 0$ is the coefficient of friction. These relations set constraints on the evolution of the tangential stress; in particular, the tangential stress is in the direction opposite to the relative sliding velocity $\dot{\mathbf{u}}_\tau - \mathbf{v}^*$.

Naturally, the wear increases in time, i.e. $\dot{w} \geqslant 0$. Hence, it follows from (3.1) and (3.2) that $\dot{\mathbf{u}}_\nu \leqslant 0$ and $\sigma_\nu \leqslant 0$ on Γ_3 . Thus, the conditions (3.3) and (3.4) imply

$$-\sigma_\nu = \alpha\,|\dot{\mathbf{u}}_\nu|\,,\ \ |\sigma_\tau| = -\mu\sigma_\nu,\ \ \sigma_\tau = -\lambda\,(\dot{\mathbf{u}}_\tau - \mathbf{v}^*)\,, \quad \lambda \geqslant 0\ . \tag{3.5}$$

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0,T]$. Then, the classical formulation of the mechanical problem of a frictional bilateral contact with wear may be stated as follows.

Problem $\mathcal{P}$

Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$, and a damage field $\beta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}(\varepsilon(\mathbf{u}(t)),\beta) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s))\Big)ds \\ + \mathcal{E}^*\nabla\varphi(t) \quad \text{in } \Omega \times (0,T),\end{aligned} \tag{3.6}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{B}\nabla(\varphi) \quad \text{in } \Omega \times (0,T), \tag{3.7}$$

$$\dot{\beta} - k\Delta\beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}),\beta), \tag{3.8}$$

$$\text{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.9}$$

$$\text{div}\mathbf{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.10}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{3.11}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.12}$$

$$\left\{\begin{array}{l} \sigma_\nu = -\alpha\,|\dot{\mathbf{u}}_\nu|\,, \quad |\sigma_\tau| = -\mu\sigma_\nu, \\ \\ \sigma_\tau = -\lambda\,(\dot{\mathbf{u}}_\tau - \mathbf{v}^*)\,, \quad \lambda \geqslant 0, \end{array}\right. \quad \text{on } \Gamma_3 \times (0,T), \tag{3.13}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.14}$$

(3.15) $$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T),$$

(3.16) $$\mathbf{D} \cdot \boldsymbol{\nu} = q_2 \quad \text{on } \Gamma_b \times (0,T),$$

(3.17) $$\mathbf{u}(0) = \mathbf{u}_0, \quad \beta(0) = \beta_0 \quad \text{in } \Omega.$$

We now describe problem (3.6)-(3.17) and provide explanation of the equations and the boundary conditions.

Equations(3.6) and (3.7) represent the electro elastic-viscoplastic constitutive law with damage,the evolution of the damage field is governed by the inclusion of parabolic type given by the relation (3.8) where S is the mechanical source of the damage growth, assumed to be rather general function of the strains and damage itself, $\partial\varphi_k$ is the subdifferential of the indicator function of the admissible damage functions set K. Next equations(3.9) and (3.10) are the steady equations for the stress and electric-displacement field, respectively, in which “Div” and “div” denote the divergence operator for tensor and vector valued functions, i.e.,

$$\mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}), \quad \mathrm{div}\mathbf{D} = (D_{i,i}).$$

We use these equations since the process is assumed to be mechanically quasistatic and electrically static.

Conditions (3.11) and (3.12) are the displacement and traction boundary conditions, where as (3.15) and (3.16) represent the electric boundary conditions; the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while the forces and free electric charges are prescribed on Γ_2 and Γ_b, respectively.

We turn to the boundary condition (3.13) describe the frictional bilateral contact with wear described above on the potential contact surface Γ_3.

The relation (3.14) describes a homogeneous Neumann boundary condition where $\partial\beta/\partial\nu$ is the normal derivative of β.

Next, (3.17) represents the initial displacement field and the initial damage field where $\mathbf{u}_0$ is the initial displacement, and β_0 is the initial damage.

To obtain the variational formulation of problem (3.6)-(3.17), we introduce the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_1 \right\}.$$

Since $meas(\Gamma_1) > 0$, Korn’s inequality holds and there exists a constant $C_k > 0$, depending only on Ω and Γ_1 , such that

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}} \geqslant C_k \|\mathbf{v}\|_{H^1(\Omega)^d} \ \forall \mathbf{v} \in V.$$

A proof of Korn’s inequality may be found in ([15], p. 79). On the space V we consider the inner product and the associated norm given by

(3.18) $$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \ \|\mathbf{v}\|_V = \|\varepsilon(\mathbf{v})\|_{\mathcal{H}} \ \forall \mathbf{u},\mathbf{v} \in V.$$

It follows that the norms $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent on V and, therefore, the space $(V,(\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.7) , there exists a constant $C_0>0$, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leqslant C_0\|\mathbf{v}\|_V, \quad \forall \mathbf{v}\in V. \tag{3.19}$$

We also introduce the spaces

$$\begin{aligned} W &= \left\{\phi\in H^1(\Omega)^d : \phi=0 \text{ on } \Gamma_a\right\},\\ \mathcal{W} &= \{\ \mathbf{D}=(D_i) : D_i\in L^2(\Omega), \mathrm{div}\mathbf{D}\in L^2(\Omega)\}, \end{aligned}$$

The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$\begin{aligned} (\varphi,\phi)_W &= \int_\Omega \nabla\varphi\cdot\nabla\phi\,dx,\\ (\mathbf{D},\mathbf{E})_{\mathcal{W}} &= \int_\Omega \mathbf{D}\cdot\mathbf{E}\,dx + \int_\Omega \mathrm{div}\mathbf{D}\cdot\mathrm{div}\mathbf{E}dx\ . \end{aligned}$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Moreover, when $\mathbf{D}\in\mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\mathbf{D},\nabla\phi)_H + (\mathrm{div}\mathbf{D},\phi)_{L^2(\Omega)} = \int_\Omega \mathbf{D}\cdot\nu\phi da,\ \ \forall\phi\in H^1(\Omega).$$

Since $\mathrm{meas}\Gamma_a>0$, the Friedrichs-Poincaré inequality holds, thus,

$$\|\nabla\phi\|_{\mathcal{W}} \geqslant C_F\,\|\phi\|_{H^1(\Omega)}, \quad \forall\,\phi\in W, \tag{3.20}$$

where $C_F>0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product

$$(\varphi,\zeta)_W = (\nabla\varphi,\nabla\zeta)_{\mathcal{W}},$$

We now list the assumptions on the problem's data.
The *viscosity operator* $\mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\mathcal{A}}>0 \text{ such that}\\ \quad \|\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)\|\leqslant L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\| \text{ for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, a.e.\mathbf{x}\in\Omega.\\ \text{(b) There exists } m_{\mathcal{A}}>0 \text{ such that}\\ \quad (\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geqslant m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \quad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, a.e.\mathbf{x}\in\Omega.\\ \text{(c) The mapping } \mathbf{x}\mapsto\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\\ \quad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(d) The mapping } \mathbf{x}\mapsto\mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{3.21}$$

The *elasticity operator* $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$
(3.22) \quad \begin{cases} \text{(a) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1, \alpha_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2, \alpha_2)\| \leqslant L_{\mathcal{B}} \left(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\alpha_1 - \alpha_2\|\right) \\ \quad \forall \ \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R} \quad , \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}, \alpha) \text{ is Lebesque measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R} \ . \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases}
$$

The *plasticity operator* $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$
(3.23) \quad \begin{cases} \text{(a) There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \quad \text{for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d, \ \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{is Lebesgue measurable on } \Omega \ , \\ \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(c) The mapping } \mathbf{x} \to \mathcal{G}(x, 0, 0) \in \mathcal{H}. \end{cases}
$$

The *damage source function* $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$
(3.24) \quad \begin{cases} \text{(a) There exists a constant } M_S > 0 \text{ such that} \\ \quad \|S(\mathbf{x}, \boldsymbol{\varepsilon}_1, \alpha_1) - S(\mathbf{x}, \boldsymbol{\varepsilon}_2, \alpha_2)\| \leqslant M_S(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\alpha_1 - \alpha_2\|) \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{for all } \alpha_1, \alpha_2 \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \alpha \in \mathbb{R}, \mathbf{x} \mapsto S(\mathbf{x}, \boldsymbol{\varepsilon}, \alpha) \text{ is Lebesgue measurable on } \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto S(\mathbf{x}, 0, 0) \text{ belongs to } L^2(\Omega). \end{cases}
$$

The *electric permittivity operator* $\boldsymbol{B} = (\boldsymbol{B}_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$
(3.25) \quad \begin{cases} \text{(a) } \boldsymbol{B}(x, E) = (\boldsymbol{B}_{ij}(x) E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{a.e. } x \in \Omega. \\ \text{(b) } \boldsymbol{B}_{ij} = \boldsymbol{B}_{ji} \in L^\infty(\Omega), 1 \leqslant i, j \leqslant d. \\ \text{(c) There exists a constant } M_{\boldsymbol{B}} > 0 \text{ such that } \boldsymbol{B}E.E \geqslant M_{\boldsymbol{B}}|E|^2 \\ \quad \text{for all } E = (E_i) \in \mathbb{R}^d, \text{a.e. in } \Omega. \end{cases}
$$

The *piezoelectric operator* $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$
(3.26) \quad \begin{cases} \text{(a) } \mathcal{E}\left(x, \tau\right) = \left(e_{ijk} \tau_{jk}\right), \forall \tau = \left(\tau_{jk}\right) \in \mathbb{S}^d, \text{a.e. } x \text{ in } \Omega. \\ \text{(b) } e_{ijk} = e_{ikj} \in L^\infty(\Omega) \ , 1 \leqslant i, j, k \leqslant d. \end{cases}
$$

We also suppose that the body forces and surface tractions have the regularity

$$
(3.27) \quad \mathbf{f}_0 \in \mathcal{C}(0, T; H), \quad \mathbf{f}_2 \in \mathcal{C}(0, T; L^2(\Gamma_2)^d),
$$

$$
(3.28) \quad q_0 \in \mathcal{C}(0, T; L^2(\Omega)), \quad q_2 \in \mathcal{C}(0, T; L^2(\Gamma_b)),
$$

$$
(3.29) \quad q_2\left(t\right) = 0 \text{ on } \Gamma_3, \forall t \in [0, T] \, .
$$

The functions α and μ have the following properties:

(3.30) $$\alpha \in L^{\infty}(\Gamma_3)\ ,\ \alpha\,(x) \geqslant \alpha^* > 0, \text{ a.e. on } \Gamma_3,$$

(3.31) $$\mu \in L^{\infty}(\Gamma_3)\ ,\ \mu\,(x) > 0, \text{ a.e. on } \Gamma_3.$$

Note that we need to impose the assumption (3.29) for physical reasons, indeed the foundation is assumed to be an insulator and therefore, the electric charges (which are prescribed on $\Gamma_b \supset \Gamma_3$) have to vanish on the potential contact surface.

The initial displacement field satisfies

(3.32) $$\mathbf{u}_0 \in V,$$

and the initial damage field satisfies

(3.33) $$\beta_0 \in K.$$

The forces, tractions, volume and surface free charge densities satisfy Here, $1 \leqslant p \leqslant \infty$. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$

(3.34) $$a(\xi, \varphi) = k \int_{\Omega} \nabla\xi \cdot \nabla\varphi dx.$$

Next, we define the three mappings $j : V \times V \to \mathbb{R}$, $\mathbf{f} : [0, T] \to V$ and $q : [0, T] \to W$, respectively, by

(3.35) $$(\mathbf{f}(t), \mathbf{v})_V = \int_{\Omega} \mathbf{f}_0(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v}\, da,$$

(3.36) $$(q(t), \phi)_W = \int_{\Omega} q_0(t)\phi\, dx - \int_{\Gamma_b} q_2(t)\phi\, da,$$

(3.37) $$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \alpha\|u_\nu\| \left(\mu \left\|\mathbf{v}_\tau - \mathbf{v}^*\right\|\right) + v_\nu)\ da.$$

for all $\mathbf{u}, \mathbf{v} \in V$, $\phi \in W$ and $t \in [0, T]$. We note that the definitions of h, $\mathbf{f}$ and q are based on the Riesz representation theorem,Moreover, the conditions (3.27) and (3.28) imply that

(3.38) $$\mathbf{f} \in \mathcal{C}(0, T; V) \qquad , q \in \mathcal{C}(0, T; W)\ .$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.6)-(3.17).

Problem $\mathcal{P}_V$.

Find a displacement field $\mathbf{u} : [0, T] \to V$, $\sigma : [0, T] \to \mathcal{H}_1$ and an electric potential $\varphi : [0, T] \to W$, an electric displacement field $\mathbf{D}$:$[0, T] \to H$,and damage field $\beta : [0, T] \to H^1(\Omega)$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}(\varepsilon(\mathbf{u}(t)), \beta) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s))\Big)ds + \mathcal{E}^*\nabla\varphi(t) \quad \text{in } \Omega \times (0,T), \tag{3.39}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{3.40}$$

for all $\mathbf{v} \in V$ and $t \in (0,T)$,

$$(\dot{\beta}(t), \zeta - \beta(t))_{L^2(\Omega)} + a(\beta(t), \zeta - \beta(t)) \geqslant (S(\varepsilon(\mathbf{u}(t)), \beta(t)), \zeta - \beta(t))_{L^2(\Omega)}, \tag{3.41}$$

for all $\beta(t) \in K$, $\zeta \in K$ and $t \in (0,T)$,

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \boldsymbol{B}\nabla\varphi(t), \tag{3.42}$$

$$(\mathbf{D}(t), \nabla\phi)_H = (q(t), \phi)_W, \tag{3.43}$$

for all $\phi \in W$ and $t \in (0,T)$, and

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \beta(0) = \beta_0. \tag{3.44}$$

Remark 3.1. *We remark that if v^* is large enough then $\alpha = 1/(k_1 v^*)$ is sufficiently small and therefore, the condition* (4.1) *for the unique solvability of Problem P_V is satisfied. We conclude that the mechanical problem* (3.6)-(3.17) *has a unique weak solution if the tangential velocity of the foundation is large enough. Moreover, having solved the problem* (3.6)-(3.17) *, we can find the wear function by integrating (*3.2*) and using the initial condition $w(0) = 0$ which means that at the initial moment the body is not subject to any prior wear.*

Remark 3.2. *The functions $\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D}$ and β which satisfy (*3.39*)-(*3.44*) are called a weak solution of the contact problem $\mathcal{P}$. We conclude that, under the assumptions (*3.21*)-(*3.33*) and if (*4.1*), the mechanical problem (*3.6*)-(*3.17*) has a unique weak solution satisfying (*4.2*)-(*4.6*).*

4. Existence and Uniqueness of a Solution

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that* (3.21)-(3.33) *hold.Then there exists a constant $\alpha_0$which depends only on $\Omega, \Gamma_1, \Gamma_3$ and $\mathcal{A}$ such that if*

$$\|\alpha\|_{L^\infty(\Gamma_3)} \Big(\|\mu\|_{L^\infty(\Gamma_3)} + 1\Big) < \alpha_0, \tag{4.1}$$

Then there exists a unique solution $\{\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D}, \beta\}$ *to problem* $\mathcal{PV}$*. Moreover, the solution satisfies*

$$\mathbf{u} \in \mathcal{C}^1(0, T; V), \tag{4.2}$$

$$\boldsymbol{\sigma} \in \mathcal{C}(0, T; \mathcal{H}_1), \quad Div\sigma \in \mathcal{C}(0, T; H), \tag{4.3}$$

$$\varphi \in \mathcal{C}(0, T; W), \tag{4.4}$$

$$\mathbf{D} \in \mathcal{C}(0, T; \mathcal{W}), \tag{4.5}$$

$$\beta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \tag{4.6}$$

Remark 4.2. *We conclude that the mechanical problem (*3.6*)-(*3.17*) has a unique weak solution if the tangential velocity of the foundation is large enough. This is neglecting the term* $\dot{\mathbf{u}}_\tau$ *in the wear conditions(*3.2*) ascompared with* $\mathbf{v}^*$ *.Moreover, having solved the problem (*3.6*)-(*3.17*), we can find the wear function by integrating (*3.2*) and using the initial condition* $w(0) = 0$.

The proof of Theorem 4.1 is carried in several steps .It is based on results of elliptic variational inequalities, parabolic inequalities and fixed point arguments.

First step

Let $\eta \in \mathcal{C}(0, T; X)$ and $\mathbf{g} \in \mathcal{C}(0, T; V)$.
We consider the following variational problem

Problem $\mathcal{PV}_{\eta,\mathbf{g}}$

Find a displacement field $\mathbf{v}_{\eta,\mathbf{g}} : [0, T] \to V$ and a stress field $\boldsymbol{\sigma}_{\eta,\mathbf{g}} : [0, T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t) = \mathcal{A}(\varepsilon(\mathbf{v}_{\eta,\mathbf{g}}(t))) + \eta\,(t)\,, \forall t \in [0, T]\,, \tag{4.7}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t), \varepsilon(\mathbf{v} - \mathbf{v}_{\eta,\mathbf{g}}(t)))_{\mathcal{H}} + j(\mathbf{g}(t), \mathbf{v}) - j(\mathbf{g}(t), \mathbf{v}_{\eta,\mathbf{g}}(t)) \\ &\geqslant (\mathbf{f}(t), \mathbf{v} - \mathbf{v}_{\eta,\mathbf{g}}(t))_V, \end{aligned} \tag{4.8}$$

for all $\mathbf{v} \in V$ and $t \in (0, T)$,
We have the following result for $\mathcal{PV}_{\eta,\mathbf{g}}$.

Lemma 4.3. $\mathcal{PV}_{\eta,\mathbf{g}}$ *has a unique weak solution such that*

$$\mathbf{v}_{\eta,\mathbf{g}} \in \mathcal{C}(0, T; V) \ \ , \boldsymbol{\sigma}_{\eta,\mathbf{g}} \in \mathcal{C}(0, T; \mathcal{H}_1). \tag{4.9}$$

Proof of Lemma 4.3. We define the operator $A : V \to V$ such that

$$(A\mathbf{u}, \mathbf{v}) = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.10}$$

It follows from (4.10) and $(3.21)(a)$ that

$$\|A\mathbf{u} - \mathbf{A}\mathbf{v}\|_V \leqslant L_{\mathcal{A}} \|\mathbf{u} - \mathbf{v}\|_V, \tag{4.11}$$

which shows that $A : V \to V$ is Lipschitz continuous. Now, by (4.10) and $(3.21)(b)$ we find

$$(A\mathbf{u} - \mathbf{A}\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geqslant m_{\mathcal{A}} \|\mathbf{u} - \mathbf{v}\|_V^2, \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.12}$$

i.e., that $A : V \to V$ is a strongly monotone operator on V . Moreover, using Riesz Representation Theorem, we may define an element $\mathbf{F} \in \mathcal{C}(0, T; V)$ by

$$(\mathbf{F}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\eta(t), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \forall \mathbf{u}, \mathbf{v} \in V.$$

Since A is a strongly monotone and Lipschitz continuous operator on V and since $\mathbf{v} \to j(g(t), \mathbf{v})$ is a proper convex lower semicontinuous functional, it follows from classical result on elliptic inequalities (see for example [6]) that there exists a unique function $\mathbf{v}_{\eta,\mathbf{g}} \in V$ which satisfies

$$\begin{aligned} &(A\mathbf{v}_{\eta,\mathbf{g}}(t), \mathbf{v} - \mathbf{v}_{\eta,\mathbf{g}}(t))_{\mathcal{H}} + j(\mathbf{g}(t), \mathbf{v}) - j(\mathbf{g}(t), \mathbf{v}_{\eta,\mathbf{g}}(t)) \\ &\geqslant (\mathbf{F}(t), \mathbf{v} - \mathbf{v}_{\eta,\mathbf{g}}(t))_V, \forall \mathbf{v} \in V. \end{aligned} \tag{4.13}$$

We use the relation (4.7), the assumption (3.21), and the properties of the deformation tensor to obtain that $\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t) \in \mathcal{H}$. Since $\mathbf{v} = \mathbf{v}_{\eta,\mathbf{g}}(t) \pm \psi$ satisfies (4.8), where $\psi \in \mathcal{D}(\Omega)^d$ is arbitrary, using the definition (3.35) for $\mathbf{f}(t)$, we find

$$Div\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t) + \mathbf{f}_0(t) = 0, \ t \in (0, T), \tag{4.14}$$

With the regularity assumption (3.27) on $\mathbf{f}_0$ we see that $Div\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t) \in H$. Therefore, $\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t) \in \mathcal{H}_1$. Let $t_1, t_2 \in [0, T]$ and denote $\eta(t_i) = \eta_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $\mathbf{g}(\mathbf{t_i}) = \mathbf{g}_i$, $\mathbf{v}_{\eta,\mathbf{g}}(t_i) = \mathbf{v}_i$, $\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t_i) = \boldsymbol{\sigma}_i$ for $i = 1, 2$. Using the relation (4.8), we find that

$$\begin{aligned} &(\mathcal{A}\varepsilon(\mathbf{v}_1) - \mathcal{A}\varepsilon(\mathbf{v}_2), \varepsilon(\mathbf{v_1} - \mathbf{v_2}))_{\mathcal{H}} \\ &\leqslant (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{v}_1 - \mathbf{v}_2)_V - (\eta_1 - \eta_2, \varepsilon(\mathbf{v_1} - \mathbf{v_2}))_{\mathcal{H}} \\ &\quad + j(\mathbf{g}_1, \mathbf{v}_2) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_1, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2). \end{aligned} \tag{4.15}$$

From the definition of the functional j given by (3.37) we have

$$\begin{aligned} &j(\mathbf{g}_1, \mathbf{v}_2) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_1, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2) \\ &= \int_{\Gamma_3} (\alpha\|g_{1\nu}\| - \alpha\|g_{2\nu}\|)\,(\mu\,\|\mathbf{v}_{2\tau} - \mathbf{v}^*\| - \mu\,\|\mathbf{v}_{1\tau} - \mathbf{v}^*\|) + v_{2\nu} - v_{1\nu}\, da. \end{aligned}$$

The relation (3.19) and the assumptions (3.30) and (3.31) imply

$$\begin{aligned}&\|j(\mathbf{g}_1,\mathbf{v}_2)+j(\mathbf{g}_2,\mathbf{v}_1)-j(\mathbf{g}_1,\mathbf{v}_1)-j(\mathbf{g}_2,\mathbf{v}_2)\|\\ &\leqslant C_0^2\,\|\alpha\|_{L^\infty(\Gamma_3)}\left(\|\mu\|_{L^\infty(\Gamma_3)}+1\right)\|\mathbf{g}_1-\mathbf{g}_2\|_V\,\|\mathbf{v}_1-\mathbf{v}_2\|_V\,.\end{aligned}\tag{4.16}$$

The relation (3.18), the assumption (3.21), and the inequality (4.16) combined with (4.15) give us

$$\begin{aligned}&m_{\mathcal{A}}\|\mathbf{u}-\mathbf{v}\|_V\\ &\leqslant C_0^2\,\|\alpha\|_{L^\infty(\Gamma_3)}\left(\|\mu\|_{L^\infty(\Gamma_3)}+1\right)\|\mathbf{g}_1-\mathbf{g}_2\|_V+\|\mathbf{f}_1-\mathbf{f}_2\|_V+\|\eta_1-\eta_2\|_{\mathcal{H}}\,.\end{aligned}\tag{4.17}$$

The inequality (4.17) and the regularity of the functions $\mathbf{f},\mathbf{g}$, and η show that

$$\mathbf{v}_{\eta,\mathbf{g}}\in\mathcal{C}(0,T;V).$$

From the assumption (3.21) and the relation (4.7) we have

$$\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|_{\mathcal{H}}\leqslant L_{\mathcal{A}}\,\|\mathbf{v}_1-\mathbf{v}_2\|_V+\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_{\mathcal{H}}\,,\tag{4.18}$$

and from (4.14) we have

$$\mathrm{Div}\boldsymbol{\sigma}(t_i)+\mathbf{f}_0(t_i)=0, i=1,2.\tag{4.19}$$

The regularity of the function η , $\mathbf{v}$, $\mathbf{f}_0$ and the relations (4.18)-(4.19) show that

$$\boldsymbol{\sigma}_{\eta,\mathbf{g}}\in\mathcal{C}(0,T;\mathcal{H}_1).$$ ■

We consider the following operator

$$\Lambda_{\boldsymbol{\eta}}:\mathcal{C}(0,T;V)\to\mathcal{C}(0,T;V),$$

defined by

$$\Lambda_{\boldsymbol{\eta}}\mathbf{g}=\mathbf{v}_{\eta,\mathbf{g}}\ \ ,\forall\mathbf{g}\in\mathcal{C}(0,T;V).\tag{4.20}$$

Lemma 4.4. *Assume that* (3.21)-(3.33) *hold. Then there exists a real* $\alpha_0>0$ *which depends only on* Ω, Γ_1 ,Γ_3, *and* $\mathcal{A}$ *such that if* (4.1) *is satisfied then the operator* $\Lambda_{\boldsymbol{\eta}}$ *has a unique fixed point* $\mathbf{g}^*_{\boldsymbol{\eta}}\in\mathcal{C}(0,T;V)$.

Proof. Let $\mathbf{g}_1,\mathbf{g}_2\in\mathcal{C}(0,T;V)$ and let $\eta\in\mathcal{C}(0,T;X)$. We use the notation $\mathbf{v}_{\eta,\mathbf{g}}(t_i)=\mathbf{v}_i$ and $\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t_i)=\boldsymbol{\sigma}_i$ for $i=1,2$. Using similar arguments as those in (4.17), we find

$$\begin{aligned}&m_{\mathcal{A}}\|\mathbf{v}_1(t)-\mathbf{v}_2(t)\|_V\\ &\leqslant C_0^2\,\|\alpha\|_{L^\infty(\Gamma_3)}\left(\|\mu\|_{L^\infty(\Gamma_3)}+1\right)\|\mathbf{g}_1(t)-\mathbf{g}_2(t)\|_V\,,\forall t\in[0,T]\,,\end{aligned}\tag{4.21}$$

From (4.20) and (4.21) we find that

$$\begin{aligned} & \|\Lambda_{\boldsymbol{\eta}}\mathbf{g}_1(t) - \Lambda_{\boldsymbol{\eta}}\mathbf{g}_2(t)\|_V \\ & \leqslant \frac{C_0^2}{m_{\mathcal{A}}}\|\alpha\|_{L^\infty(\Gamma_3)}\Big(\|\mu\|_{L^\infty(\Gamma_3)}+1\Big)\|\mathbf{g}_1(t)-\mathbf{g}_2(t)\|_V, \ \forall t\in[0,T]. \end{aligned} \tag{4.22}$$

Let

$$\alpha_0 = \frac{C_0^2}{m_{\mathcal{A}}},$$

where α_0 is a positive constant which depends on Ω, Γ_1 ,Γ_3, and on the operator $\mathcal{A}$ If (4.1) is satisfied we deduce from (4.22) that the operator $\Lambda_{\boldsymbol{\eta}}$ is a contraction. From Banach's Fixed Point Theorem we conclude that the operator $\Lambda_{\boldsymbol{\eta}}$ has a unique fixed point $\mathbf{g}^*_{\boldsymbol{\eta}} \in \mathcal{C}(0,T;V)$. ∎

Remark 4.5. *If the condition of the wear* $\|\alpha\|_{L^\infty(\Gamma_3)}\Big(\|\mu\|_{L^\infty(\Gamma_3)}+1\Big) < \alpha_0 = \frac{C_0^2}{m_{\mathcal{A}}}$*, then the problem* $\mathcal{P}_{\mathcal{V}}$ *has unique weak solution*

Second step

Denote

$$\mathbf{v}_\eta = \mathbf{v}_{\eta,\mathbf{g}^*} \ , \ \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_{\eta,\mathbf{g}^*}, \tag{4.23}$$

and let $\mathbf{u}_\eta : [0,T] \to V$ be the function defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\,ds + \mathbf{u}_0 \ \ \forall t \in [0,T]. \tag{4.24}$$

Using (4.9) we find that $\mathbf{u}_\eta$ satisfies the regularity expressed in (4.2). In the second step, let $\eta \in \mathcal{C}(0,T;\mathcal{H})$; we use the displacement field $\mathbf{u}_\eta$ obtained in (4.24) and consider the following variational problem.

Problem $\mathcal{PV}^2_{\boldsymbol{\eta}}$

Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$(\boldsymbol{B}\nabla\varphi_\eta(t), \nabla\phi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \nabla\phi)_H = (q(t),\phi)_W. \tag{4.25}$$

for all $\phi \in W$, $t \in (0,T)$.

The well-posedness of problem $\mathcal{PV}^2_\eta$ follows.

Lemma 4.6. $\mathcal{PV}^2_\eta$ *has a unique solution* ϕ_η *which satisfies the regularity* (4.4).

Proof. We define a bilinear form $b(\cdot,\cdot) : W \times W \to \mathbb{R}$ such that

$$b(\varphi,\phi) = (\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H, \forall\varphi,\phi \in W. \tag{4.26}$$

We use (3.25) to show that the bilinear form b is continuous, symmetric, and coercive on W . Moreover using the Riesz Representation Theorem we may define an element $q_{\boldsymbol{\eta}} : [0, T] \to W$ such that

$$(q_{\boldsymbol{\eta}}\,(t)\,,\phi)_W = (q\,(t)\,,\phi)_W - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \nabla\phi)_H\,; \forall\phi \in W, \forall t \in (0, T)\,.$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta \in W$ such that

$$b\,(\varphi_\eta\,(t)\,, \phi) = (q_{\boldsymbol{\eta}}\,(t)\,,\phi)_W\,; \forall\phi \in W. \tag{4.27}$$

We conclude that $\varphi_\eta\,(t)$ is a solution of $\mathcal{PV}_\eta^2$. Let $t_1, t_2 \in [0, T]$. It follows from (3.20),(3.25),(3.26), (4.26), and (4.27) that

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leqslant c\,\left(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V + \|q\,(t_1) - q\,(t_2)\|_W\right),$$

and the previous inequality and the regularity of $\mathbf{u}_\eta$ and q imply that $\varphi_\eta \in \mathcal{C}\,(0, T; W)$. ∎

Third step

Let $\theta \in \mathcal{C}\left(0, T; L^2\,(\Omega)\right)$ be given and consider the following variational problem for the damage filed.

Problem $\mathcal{PV}_\theta$

Find the damage field $\beta_\theta : [0, T] \to H^1(\Omega)$ such that $\beta_\theta(t) \in K$ and

$$\begin{aligned} &(\dot\beta_\theta(t), \xi - \beta_\theta)_{L^2(\Omega)} + a(\beta_\theta(t), \xi - \beta_\theta(t)) \\ &\qquad\qquad \geqslant (\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)} \quad \forall\xi \in K,\ \text{a.e. } t \in (0, T), \end{aligned} \tag{4.28}$$

$$\beta_\theta(0) = \beta_0. \tag{4.29}$$

To solve $\mathcal{PV}_\theta$, we recall the following standard result for parabolic variational inequalities (see [[3], p. 124]). Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V . We write

$$V \subset H \subset V'.$$

and we say that the inclusions above define a Gelfand triple. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$, and $\|\cdot\|_{V'}$, the norms on the spaces V , H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V .Note that if $f \in H$ then

$$(f, v)_{V'\times V} = (f, v)_H\,, \forall v \in H.$$

Theorem 4.7. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V . Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) = c_0 \|v\|_H^2 \geqslant \zeta \|v\|_V^2 \ , \forall v \in H.$$

Then, for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$(\dot{u}(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geqslant (f(t), v - u(t))_H \, , \ \forall v \in K.$$

We apply this theorem to Problem $\mathcal{PV}_\theta$.

Lemma 4.8. *There exists a unique solution β_θ to the auxiliary problem $\mathcal{PV}_\theta$ such that:*

$$\beta_\theta \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{4.30}$$

The above lemma follows from a standard result for parabolic variational inequalities, see [3, p. 124].

Proof. The inclusion mapping of $\left(H^1(\Omega), \|\ .\|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega), \|\ .\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset \left(H^1(\Omega)\right)'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$. we have

$$(\beta,\xi)_{(H^1(\Omega))' \times H^1(\Omega)} = (\beta,\xi)_{L^2(\Omega)} \, , \forall \beta \in L^2(\Omega) \, , \xi \in H^1(\Omega)$$

and we note that K is a closed convex set in $H^1(\Omega)$. Then, using the definition (3.34) of the bilinear form a , and the fact that $\beta_\theta \in K$ in (3.33), it is easy to see that Lemma (4.8) is a consequence of Theorem (4.7). ∎

Finally, as a consequence of these results and using the properties of the operator $\mathcal{B}$, the operator $\mathcal{G}$ and the function S for $t \in [0,T]$, we consider the element

$$\Lambda(\boldsymbol{\eta},\theta)(t) = (\Lambda_1(\boldsymbol{\eta},\theta)(t), \Lambda_2(\boldsymbol{\eta},\theta)(t)) \in \mathcal{H} \times L^2(\Omega), \tag{4.31}$$

defined by

$$\Lambda_1(\boldsymbol{\eta},\theta)(t) = \mathcal{B}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_\eta\left(t\right)\right),\beta_\theta\left(t\right)\right)+\Big(\int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta,\theta}(s),\boldsymbol{\varepsilon}(\mathbf{u}_\eta(s))\Big)ds\Big) + \mathcal{E}^*\nabla\varphi_\eta(t),\forall t\in[0,T], \tag{4.32}$$

$$\Lambda_2(\boldsymbol{\eta},\theta)(t) = S\big(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)),\beta_\theta(t)\big),\forall t\in[0,T]. \tag{4.33}$$

We have the following result.

Lemma 4.9. *Let* (4.1) *be satisfied. Then for* $(\boldsymbol{\eta},\theta)\in\mathcal{C}(0,T;\mathcal{H}\times L^2(\Omega))$ *, the maping* $\Lambda(\boldsymbol{\eta},\theta):[0,T]\to\mathcal{H}\times L^2(\Omega)$ *has a unique element* $(\boldsymbol{\eta}^*,\theta^*)\in\mathcal{C}(0,T;\mathcal{H}\times L^2(\Omega))$ *such that* $\Lambda(\boldsymbol{\eta}^*,\theta^*)=(\boldsymbol{\eta}^*,\theta^*)$

Proof. Let $(\boldsymbol{\eta}_1,\theta_1)$, $(\boldsymbol{\eta}_2,\theta_2)\in\mathcal{C}(0,T;\mathcal{H}\times L^2(\Omega))$, and $t\in[0,T]$. We use the notation $\mathbf{u}_{\eta_i}=\mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i}=\dot{\mathbf{u}}_i$, $\beta_{\theta_i}=\beta_i$, $\varphi_{\eta_i}=\varphi_i$, $\mathbf{g}^*_{\eta_i}=\mathbf{g}_i$ and $\boldsymbol{\sigma}_{\eta_i,\mathbf{g}_i}=\boldsymbol{\sigma}_i$, for $i=1,2$. From the notation used in (4.20) and (4.23), we deduce that $\mathbf{v}_i=\mathbf{g}_i$, Using (3.18), (3.22), (3.23) and (3.26) we obtain

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\\ &\leqslant C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}\\ &\quad+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}ds+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds\\ &\quad+\|\varphi_1(t)-\varphi_2(t)\|^2_W\Big).\end{aligned} \tag{4.34}$$

Since

$$\mathbf{u}_i(t)=\int_0^t\mathbf{v}_i(s)\,ds+\mathbf{u}_0,\forall t\in[0,T],$$

we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V\leqslant\int_0^t\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2_V ds. \tag{4.35}$$

It follows now from $\mathcal{PV}_{\eta,\mathbf{g}}$ for $\boldsymbol{\eta}=\boldsymbol{\eta}_i$, $i=1,2$, that

$$\boldsymbol{\sigma}_i(t)=\mathcal{A}(\varepsilon(\mathbf{v}_i(t)))+\eta_i(t),\forall t\in[0,T], \tag{4.36}$$

$$(\boldsymbol{\sigma}_i(t),\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{v}_i(t)))_{\mathcal{H}}+j(\mathbf{g}_i(t),\mathbf{v})-j(\mathbf{g}_i(t),\mathbf{v}_i(t))\geqslant(\mathbf{f}(t),\mathbf{v}-\mathbf{v}_i(t))_V \tag{4.37}$$

for all $\mathbf{v}\in\mathbf{V}$, and all $t\in[0,T]$.

Using the relation (4.37) we obtain that

$$\begin{aligned}&(\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t),\varepsilon(\mathbf{v_1(t)}-\mathbf{v_2(t)}))_{\mathcal{H}}\\ &\leqslant j(\mathbf{g}_1(t),\mathbf{v}_2(t))+j(\mathbf{g}_2(t),\mathbf{v}_1(t))-j(\mathbf{g}_1(t),\mathbf{v}_1(t))-j(\mathbf{g}_2(t),\mathbf{v}_2(t))\end{aligned}$$

for all $t \in [0,T]$. and similar arguments to those used in (4.16) on the functional j yield

$$(\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{v_1(t)} - \mathbf{v_2(t)}))_{\mathcal{H}}$$
$$\leqslant C_0^2 \|\alpha\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)}+1\right) \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|_V \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V, \ \forall t \in [0,T].$$

Keeping in mind that $\mathbf{v}_i = \mathbf{g}_i$ for $i = 1, 2$, it follows that

$$(\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{v_1(t)} - \mathbf{v_2(t)}))_{\mathcal{H}}$$
$$\leqslant C_0^2 \|\alpha\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2, \ \forall t \in [0,T].$$

We substitute (4.36) into the previous inequality and use (3.18) and (3.21) to deduce that

$$\left(m_{\mathcal{A}} - C_0^2 \|\alpha\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right)\right) \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V$$
$$\leqslant \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{\mathcal{H}}, \forall t \in [0,T].$$

It follows from (4.1) that

$$\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2 \leqslant C\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{\mathcal{H}}^2, \ \forall t \in [0,T]. \tag{4.38}$$

For the electric potential field, we use (4.25), (4.20), (3.25), and (3.26) to obtain

$$\|\varphi_1(t) - \varphi_2(t)\|_W^2 \leqslant C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathcal{H}}^2, \ \forall t \in [0,T].$$

From (4.28) we deduce that

$$(\dot\beta_1 - \dot\beta_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2) \leqslant (\theta_1 - \theta_2, \beta_1 - \beta_2)_{L^2(\Omega)}, \quad \text{a.e } t \in (0,T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and inequality $a(\beta_1 - \beta_2, \beta_1 - \beta_2) \geqslant 0$ to find

$$\frac{1}{2}\|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}^2 \leqslant \int_0^t (\theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} ds, \tag{4.39}$$

which implies

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}^2 \leqslant \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}^2 ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}^2 \leqslant C \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds, \quad \forall t \in [0,T]. \tag{4.40}$$

We substitute (4.39) into (4.34) and use (4.35) to obtain

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\\ &\leqslant C\left(\int_0^t\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2_V ds+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}\right).\end{aligned}$$

It follows now from the previous inequality, the estimates (4.38) and (4.40) that

$$\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\leqslant C\int_0^t\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)}ds.$$

Reiterating this inequality n times leads to

$$\begin{aligned}&\|\Lambda^n(\boldsymbol{\eta}_1,\theta_1)-\Lambda^n(\boldsymbol{\eta}_2,\theta_2)\|^2_{W^{1,p}(0,T;\mathcal{H}\times L^2(\Omega))}\\ &\leqslant\frac{C^nT^n}{n!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|^2_{W^{1,p}(0,T;\mathcal{H}\times L^2(\Omega))}.\end{aligned}$$

Thus, for n sufficiently large, Λ^n is a contraction on $\mathcal{C}(0,T;\mathcal{H}\times L^2(\Omega))$, and so Λ has a unique fixed point in this Banach space. ■

Now, we have all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1.

Existence

Let $(\boldsymbol{\eta}^*,\theta^*)\in\mathcal{C}(0,T;\mathcal{H}\times L^2(\Omega))$ be the fixed point of Λ defined by (4.31)-(4.33) and and let $(\mathbf{v},\boldsymbol{\sigma})$ be the solution of $\mathcal{PV}_{\eta,\mathbf{g}}$ for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$, $\mathbf{g}=\mathbf{g}^*_{\boldsymbol{\eta}^*}$ obtained in Lemma 4.1.denote $\mathbf{u}=\mathbf{u}_{\boldsymbol{\eta}^*}$. Let now $\varphi_{\boldsymbol{\eta}^*}=\varphi$ and $\beta_{\theta^*}=\beta$ be the solutions of $\mathcal{PV}^2_{\boldsymbol{\eta}}$ and $\mathcal{PV}_\theta$ for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and $\theta=\theta^*$ obtained in Lemmas 4.3 and 4.4. The equalities $\Lambda_1(\boldsymbol{\eta}^*,\theta^*)=\boldsymbol{\eta}^*$ and $\Lambda_2(\boldsymbol{\eta}^*,\theta^*)=\theta^*$ combined with (4.32), (4.33) show that (3.39)-(3.43) are satisfied. Next, (3.44) and the regularity (4.2)-(4.6) follow from Lemmas 4.1, 4.3, 4.4, and (3.43).

Uniqueness

The uniqueness part of solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.31)-(4.33) and the unique solvability of the Problems $\mathcal{PV}_{\eta,\mathbf{g}}$, $\mathcal{PV}^2_{\boldsymbol{\eta}}$ and $\mathcal{PV}_\theta$ which completes the proof. ■

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