ON THE SHARP DISTORTION THEOREMS FOR A SUBCLASS OF STARLIKE MAPPINGS IN SEVERAL COMPLEX VARIABLES

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Abstract. In this article, we first establish the sharp distortion theorems of the Fréchet derivative for a subclass of starlike mappings on the unit ball of complex Banach spaces and the bounded starlike circular domain in $\mathbb{C}^n$. Meanwhile, we also obtain the sharp distortion theorems of the Jacobi determinant for a subclass of starlike mappings on the bounded starlike circular domain in $\mathbb{C}^n$. Our derived conclusions are the generalizations of some known results in several complex variables and the classical results in one complex variable.

1. INTRODUCTION

In the theory of univalent functions, there is a classical and well-known theorem as follows.

Theorem A. [6]. *If f is a normalized biholomorphic function on the unit disk U in $\mathbb{C}$, then*

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, z \in U.$$

However, Cartan[2] pointed out that Theorem A is fail by providing a counter example, and he proposed that people should study the geometric properties for the subclasses of biholomorphic mappings, such as normalized biholomorphic starlike mappings and normalized biholomorphic convex mappings.

In several complex variables, the results concerning the distortion theorems for convex mappings are the best. For example, the estimates of the Jacobi determinant for convex mappings defined on the Euclidean unit ball in $\mathbb{C}^2$ were first established due to Barnard, FitzGerald and Gong [1] in 1994, and consequently Liu and Zhang

[13] obtained a general version of the above result. As to the distortion theorem of
the Frećhet derivative for convex mappings, Gong, Wang and Yu [5] first established
the corresponding estimates for convex mappings, after that Gong and Liu [4], Liu
and Zhang [14], Zhu and Liu [19], Chu, Hamada, Honda and Kohr [3] established
various versions of the distortion theorem for convex mappings on different unit balls
in complex Banach spaces. Hamada and Kohr [8] obtained the stronger upper bounds
estimate of the distortion theorem for convex mappings on the unit ball of a complex
Hilbert space. In contrast to the distortion theorems for convex mappings, the corre-
sponding results of starlike mappings are rather few until now. There is not any work
on this topic except the distortion theorem for starlike mappings on the unit polydisk
along a unit direction was given by Liu, Wang and Lu [12]. At present, more and more
people see that the problems concerning the distortion theorem for starlike mappings
are extreme difficult. A natural question arouse us to pay attention to whether the
distortion theorem for starlike mappings under restricted conditions holds or not. Our
answer is affirmative. That is to say, we establish the sharp distortion theorems of the
Fréchet derivative and the Jacobi determinant for a subclass of starlike mappings in
several complex variables.

Let X be a complex Banach space with the norm $\| . \|$, X^* be the dual space of
X, B be the open unit ball in X, and U be the Euclidean open unit disk in $\mathbb{C}$. Also,
let U^n be the open unit polydisk in $\mathbb{C}^n$, and let $\mathbb{N}$ be the set of all positive integers.
We denote by ∂U^n the boundary of U^n, and $\partial_0 U^n$ the distinguished boundary of U^n.
Let the symbol $'$ mean transpose. For each $x \in X \backslash \{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(B)$ be the set of all holomorphic mappings from B into X. We know that
if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n),$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth-Fréchet derivative
of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)(\underbrace{y-x, \cdots, y-x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

We say that a holomorphic mapping $f : B \rightarrow X$ is biholomorphic if the inverse
f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said
to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for
each $x \in B$. If $f : B \rightarrow X$ is a holomorphic mapping, then we say that f is normalized

if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. A domain Ω in $\mathbb{C}^n$ is said to be circular if $e^{i\theta}z \in \Omega$ for any $z \in \Omega$, and a domain Ω in $\mathbb{C}^n$ is said to be a complete Reinhardt domain if $(\xi_1 z_1, \xi_2 z_2, \cdots, \xi_n z_n)' \in \Omega$ for any $z = (z_1, z_2, \cdots, z_n)' \in \Omega$, where $\xi_k \in \overline{U}(k = 1, 2, \cdots, n)$, $i = \sqrt{-1}$ and θ is a real number.

We say that a normalized biholomorphic mapping $f : B \to X$ is a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Definition 1.1. [9].   Suppose that Ω is a domain (connected open set) in X which contains 0. It is said that $x = 0$ is a zero of order k of $f(x)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.

We denote by $S^*(B)$ (resp. $S^*(\Omega)$) the set of all normalized biholomorphic starlike mappings on B (resp. Ω), and $S^*_{k+1}(B)$ (resp. $S^*_{k+1}(\Omega)$) by the set of all normalized biholomorphic starlike mappings on B (resp. Ω) and $x = 0$ (resp. $z = 0$) is a zero of order $k + 1$ of $f(x) - x$ (resp. $f(z) - z$).

In this article, we shall establish the sharp distortion theorems of the Fréchet derivative for a subclass of starlike mappings on the unit ball of complex Banach spaces and the bounded starlike circular domain in $\mathbb{C}^n$, and the sharp distortion theorems of the Jacobi determinant for a subclass of starlike mappings on the bounded starlike circular domain in $\mathbb{C}^n$. Our obtained conclusions show that some known results in several complex variables are extended, and they both reduce to the classical results in one complex variable.

2. Sharp Distortion Theorems of the Fréchet Derivative for a Subclass of Starlike Mappings

In order to establish the desired results in this section, it is necessary to give some lemmas as follows.

Lemma 2.1. [18]. *Let f be a normalized locally biholomorphic mapping on B. Then $f \in S^*(B)$ if and only if*

$$\Re e \left\{ T_x[(Df(x))^{-1}f(x)] \right\} \geqslant 0 \, , x \in B \, .$$

Lemma 2.2. [10]. *If $F \in S^*_{k+1}(B)$, then*

$$\frac{\|x\|(1 - \|x\|^k)}{1 + \|x\|^k} \leq |T_x[(DF(x))^{-1}F(x)]| \leq \frac{\|x\|(1 + \|x\|^k)}{1 - \|x\|^k} \, , x \in B \, .$$

We now begin to present the following theorems in this section.

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Theorem 2.1. *Let* $f : B \to \mathbb{C} \in H(B)$, $F(x) = xf(x) \in S_{k+1}^*(B)$. *Then*

$$\frac{\|x\|(1 - \|x\|^k)}{(1 + \|x\|^k)^{1+\frac{2}{k}}} \leq \|DF(x)x\| \leq \frac{\|x\|(1 + \|x\|^k)}{(1 - \|x\|^k)^{1+\frac{2}{k}}}, x \in B,$$

and the above estimates are sharp.

Proof. Since $F(x) = xf(x)$, we have

(2.1) $$DF(x)x = xf(x) + (Df(x)x)x, x \in B.$$

Also $F(x) = xf(x) \in S^*(B)$, then we obtain

$$\frac{\|x\|}{(1 + \|x\|^k)^{\frac{2}{k}}} \leq \|F(x)\| \leq \frac{\|x\|}{(1 - \|x\|^k)^{\frac{2}{k}}}, x \in B,$$

from [10, Theorem 1]. Hence

$$|f(x)| \geq \frac{1}{(1 + \|x\|^k)^{\frac{2}{k}}}, x \in B.$$

This implies that $f(x) \neq 0$ for $x \in B$.

Straightforward computation shows that

(2.2) $$[DF(x)]^{-1}F(x) = \frac{f(x)x}{f(x) + Df(x)x}, x \in B.$$

Note that $\Re e\{T_x[(DF(x))^{-1}F(x)]\} > 0, x \in B \setminus \{0\}$ because of $F \in S^*(B)$. Therefore according to (2.2), it yields that

$$f(x) + Df(x)x \neq 0, x \in B.$$

In view of Lemma 2.2, we have

$$\frac{\|x\|(1 - \|x\|^k)}{1 + \|x\|^k} \leq |T_x[(DF(x))^{-1}F(x)]| \leq \frac{\|x\|(1 + \|x\|^k)}{1 - \|x\|^k}, x \in B,$$

On the other hand, by (2.1) and (2.2), we deduce that

$$DF(x)x = xf(x) + (Df(x)x)x = xf(x)\left(1 + \frac{Df(x)x}{f(x)}\right) = F(x)\frac{\|x\|}{T_x[(DF(x))^{-1}F(x)]}.$$

It yields that

$$\frac{\|x\|(1 - \|x\|^k)}{(1 + \|x\|^k)^{1+\frac{2}{k}}} \leq \|DF(x)x\| \leq \frac{\|x\|(1 + \|x\|^k)}{(1 - \|x\|^k)^{1+\frac{2}{k}}}, x \in B$$

from Lemma 2.2 and [10, Theorem 1].

It is easy to verify that

$$F(x) = \frac{x}{(1 - T_u^k(x))^{\frac{2}{k}}}, x \in B$$

satisfies the condition of Theorem 2.1. A short computation shows that

$$DF(x)x = \left(I + \frac{2T_u^{k-1}(x)xT_u(.)}{1 - T_u^k(x)} \right) \frac{x}{(1 - T_u^k(x))^{\frac{2}{k}}} = \frac{x(1 + T_u^k(x))}{(1 - T_u^k(x))^{1+\frac{2}{k}}}, x \in B .$$

We set $x = ru$ or $x = e^{\frac{\pi i}{k}} ru (0 \le r < 1), u \in \partial B$. Then it is shown that the estimates of Theorem 2.1 are sharp. This completes the proof. ∎

Corollary 2.1. *Let* $f : B \to \mathbb{C} \in H(B)$, $F(x) = xf(x) \in S^*(B)$. *Then*

$$\frac{\|x\|(1 - \|x\|)}{(1 + \|x\|)^3} \le \|DF(x)x\| \le \frac{\|x\|(1 + \|x\|)}{(1 - \|x\|)^3}, x \in B,$$

and the above estimates are sharp.

Let Ω be a bounded starlike circular domain in $\mathbb{C}^n$, and its Mikowski functional $\rho(z)$ is a C^1 function except for a lower dimensional manifold in $\overline{\Omega}$.

Lemma 2.3. *If* $F \in S_{k+1}^*(\Omega)$, *then*

$$\frac{\rho(z)(1 - \rho^k(z))}{1 + \rho^k(z)} \le \left| 2\frac{\partial\rho(z)}{\partial z}(DF(z))^{-1}F(z) \right| \le \frac{\rho(z)(1 + \rho^k(z))}{1 - \rho^k(z)}, z \in \Omega .$$

Proof. Fix $z \in \Omega \setminus \{0\}$, and denote $z_0 = \frac{z}{\rho(z)}$. Define

$$p(\xi) = \begin{cases} 2\frac{\partial\rho(z)}{\partial z}\frac{DF(\xi z_0))^{-1}F(\xi z_0)}{\xi}, & \xi \in U \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}$$

Since $F \in S^*(\Omega)$, similar to that in the proof of [15, the case $\alpha = 0$ of Lemma 4], we conclude that $p \in H(U)$, $p(0) = 1$, and $\Re e\, p(\xi) > 0, \xi \in U \setminus \{0\}$, and $\xi = 0$ is at least a zero of order k of $p(\xi) - 1$. According to [10, Lemma 3], it follows the desired result. This completes the proof. ∎

Theorem 2.2. *Let* $f : \Omega \to \mathbb{C} \in H(\Omega)$, $F(z) = zf(z) \in S_{k+1}^*(\Omega)$. *Then*

$$\frac{\rho(z)(1 - \rho^k(z))}{(1 + \rho^k(z))^{1+\frac{2}{k}}} \le \rho(DF(z)z) \le \frac{\rho(z)(1 + \rho^k(z))}{(1 - \rho^k(z))^{1+\frac{2}{k}}}, z \in \Omega,$$

and the above estimates are sharp.

    Proof. Similar to that in the proof of Theorem 2.1, we obtain

$$DF(z)z = F(z)\frac{\rho(z)}{2\frac{\partial \rho(z)}{\partial z}(DF(z))^{-1}F(z)} \ .$$

Therefore

$$\frac{\rho(z)(1-\rho^k(z))}{(1+\rho^k(z))^{1+\frac{2}{k}}} \le \rho(DF(z)z) \le \frac{\rho(z)(1+\rho^k(z))}{(1-\rho^k(z))^{1+\frac{2}{k}}}, z \in \Omega$$

from Lemma 2.3 and [15, the case $\alpha = 0$ of Theorem 5].

    It is not difficult to check that

$$F(z) = \frac{z}{\left(1-\left(\frac{z_1}{r}\right)^k\right)^{\frac{2}{k}}}, z \in \Omega$$

satisfies the condition of Theorem 2.2, where $r = \sup\{|z_1| : z = (z_1, 0, \cdots, 0)' \in \Omega\}$. A direct computation shows that

$$DF(z)z = \frac{z}{\left(1-\left(\frac{z_1}{r}\right)^k\right)^{\frac{2}{k}}} + \frac{2\left(\frac{z_1}{r}\right)^{k-1}\frac{z_1}{r}z}{\left(1-\left(\frac{z_1}{r}\right)^k\right)^{1+\frac{2}{k}}} = \frac{1+\left(\frac{z_1}{r}\right)^k}{\left(1-\left(\frac{z_1}{r}\right)^k\right)^{1+\frac{2}{k}}}z, z \in \Omega \ .$$

Taking $z = Ru$ or $z = e^{\frac{\pi i}{k}}Ru(0 \le R < 1)$, where $u = (u_1, u_2, \cdots, u_n)' \in \partial\Omega$, $u_1 = r$, and $i = \sqrt{-1}$, then the estimates of Theorem 2.2 are sharp. This completes the proof.   ■

    Taking $k = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.2. *Let $f : \Omega \to \mathbb{C} \in H(\Omega)$, $F(z) = zf(z) \in S^*(\Omega)$. Then*

$$\frac{\rho(z)(1-\rho(z))}{(1+\rho(z))^3} \le \rho(DF(z)z) \le \frac{\rho(z)(1+\rho(z))}{(1-\rho(z))^3}, z \in \Omega,$$

and the above estimates are sharp.

    In Lemma 2.4, Theorem 2.3 and Corollary 2.3 as follows, let $m_l(l = 1, 2, \cdots, n)$ be a non-negative integer, $N = m_1 + m_2 + \cdots + m_n \in \mathbb{N}$, and $m_l = 0$ mean the corresponding component in Z and $F(Z)$ is omitted. We denote by Ω^{m_l} (resp. Ω^N) the bounded complete Reinhardt domain of $\mathbb{C}^{m_l}(l = 1, 2, \cdots, n)$ (resp. $\mathbb{C}^N$). Let $\rho_{m_l}(Z_l)$ (resp. $\rho_N(Z)$) be the Minkowski functional of Ω^{m_l} (resp. Ω^N). U^{m_l} (resp. U^N) is denoted by the unit polydisk of $\mathbb{C}^{m_l}(l = 1, 2, \cdots, n)$ (resp. $\mathbb{C}^N$).

Lemma 2.4. *Suppose that $f_l : \Omega^{m_l} \to \mathbb{C} \in H(\Omega^{m_l}), l = 1, 2, \cdots, n,$ $F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))' = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))', Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N.$ Then $F \in S^*(\Omega^N)$ if and only if $F_l \in S^*(\Omega^{m_l}), l = 1, 2, \cdots, n.$*

Proof. By a simple calculation, it yields that

$$
(2.3) \qquad (DF(Z))^{-1}F(Z) = ((DF_1(Z_1))^{-1}F_1(Z_1), (DF_2(Z_2))^{-1}F_2(Z_2), \cdots,
$$
$$
(DF_n(Z_n))^{-1}F_n(Z_n))'
$$

and

$$
(2.4) \qquad (DF_l(Z_l))^{-1}F_l(Z_l) = \frac{Z_l f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l}, l = 1, 2, \cdots, n.
$$

Therefore

$$
(2.5) \qquad 2\frac{\partial \rho_N(Z)}{\partial Z}(DF(Z))^{-1}F(Z) = \sum_{l=1}^{n} \left(2\frac{\partial \rho_N(Z)}{\partial Z_l} Z_l \right) \frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l}
$$

and

$$
2\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_l}(DF_l(Z_l))^{-1}F_l(Z_l)
$$
$$
(2.6)
$$
$$
= \frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \left(\sum_{k=1}^{m_l} 2\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{lk}} Z_{lk} \right), l = 1, 2, \cdots, n
$$

from (2.3) and (2.4), where $\frac{\partial \rho_N(Z)}{\partial Z} = \left(\frac{\partial \rho_N(Z)}{\partial Z_1}, \cdots, \frac{\partial \rho_N(Z)}{\partial Z_n} \right)$, $\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_l} = \left(\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{l1}}, \cdots, \frac{\partial \rho_N(Z_l)}{\partial Z_{lm_l}} \right), l = 1, 2, \cdots, n.$ Note that

$$
(2.7) \qquad 2\frac{\partial \rho_n(z)}{\partial z_l} z_l \mid_{z=(0,\cdots,z_l,\cdots,0)'}
$$
$$
= \rho_n((0, \cdots, z_l, \cdots, 0)') > 0, z_l \neq 0, l = 1, 2, \cdots, n
$$

(see the proof of [20, Theorem 2.1]).

On the one hand, in view of (2.5) and (2.7), we deduce that

$$
\Re e \left(\frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \right) > 0, Z_l \in \Omega^{m_l} \setminus \{0\}, l = 1, 2, \cdots, n
$$

if $F \in S^*(\Omega^N)$ for $Z = (0, \cdots, Z_l, \cdots, 0)'$. Hence

$$
\Re e \left(2\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_l}(DF_l(Z_l))^{-1}F_l(Z_l) \right)
$$
$$
= \Re e \left(\frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \right) \left(\sum_{k=1}^{m_l} 2\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{lk}} Z_{lk} \right) > 0,
$$
$$
Z_l \in \Omega^{m_l} \setminus \{0\}, l = 1, 2, \cdots, n.
$$

This implies that $F_l \in S^*(\Omega^{m_l})(l = 1, 2, \cdots, n)$ (see [11]).

On the other hand, taking into account (2.6) and (2.7), we also conclude that

$$\Re e \left(\frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \right) > 0, Z_l \in \Omega^{m_l} \setminus \{0\}, l = 1, 2, \cdots, n$$

if $F_l \in S^*(\Omega^{m_l})$. Consequently

$$\Re e \left(2\frac{\partial \rho_N(Z)}{\partial Z}(DF(Z))^{-1}F(Z) \right)$$
$$= \sum_{l=1}^{n} \left(2\frac{\partial \rho_N(Z)}{\partial Z_l}Z_l \right) \Re e \left(\frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \right) > 0,$$
$$Z \in \Omega^N \setminus \{0\}.$$

It is shown that $F \in S^*(\Omega^N)$ (see [11]). This completes the proof. ∎

Theorem 2.3. *Let* $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l = 1, 2, \cdots, n, F(Z) = (Z_1f_1(Z_1), Z_2f_2(Z_2), \cdots, Z_nf_n(Z_n))' \in S^*_{k+1}(U^N)$. *Then*

$$\frac{\|Z\|(1 - \|Z\|^k)}{(1 + \|Z\|^k)^{1+\frac{2}{k}}} \leq \|DF(Z)Z\| \leq \frac{\|Z\|(1 + \|Z\|^k)}{(1 - \|Z\|^k)^{1+\frac{2}{k}}}, Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N,$$

and the above estimates are sharp.

Proof. Let $F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))'$. By a simple computation, we have

$$DF(Z)Z = (DF_1(Z_1)Z_1, DF_2(Z_2)Z_2, \cdots, DF_n(Z_n)Z_n)'$$

and

$$(DF(Z))^{-1}F(Z)$$
$$= ((DF_1(Z_1))^{-1}F_1(Z_1), (DF_2(Z_2))^{-1}F_2(Z_2), \cdots, (DF_n(Z_n))^{-1}F_n(Z_n))'$$

Therefore it is shown that

$$F \in S^*(U^N) \Leftrightarrow F_l \in S^*(U^{m_l}), l = 1, 2, \cdots, n$$

from Lemma 2.4, and $Z_l = 0$ is at least a zero of order $k + 1$ of $F_l(Z) - Z_l$ for each $l = 1, 2, \cdots, n$ if $Z = 0$ is a zero of order $k + 1$ of $F(Z) - Z$. We easily know that $\frac{t(1+t^k)}{(1-t^k)^{1+\frac{2}{k}}}$ is an increasing function on interval $[0, 1)$ with respect to t. Also by applying the facts $\|DF(Z)Z\| = \max_{1 \leq l \leq n} \{\|DF_l(Z_l)Z_l\|\}$ and

$$\frac{\|Z_l\|(1 - \|Z_l\|^k)}{(1 + \|Z_l\|^k)^{1+\frac{2}{k}}} \leq \|DF(Z_l)Z_l\| \leq \frac{\|Z_l\|(1 + \|Z_l\|^k)}{(1 - \|Z_l\|^k)^{1+\frac{2}{k}}}, Z_l \in U^{m_l}, l = 1, 2, \cdots, n$$

(the case $B = U^{m_l}$ of Theorem 2.1), we see that

$$\|DF(Z)Z\| = \max_{1 \leq l \leq n}\{\|DF_l(Z_l)Z_l\|\} \leq \frac{\|Z\|(1 + \|Z\|^k)}{(1 - \|Z\|^k)^{1+\frac{2}{k}}}, Z \in U^N$$

and

$$\|DF(Z)Z\| \geq \|DF_j(Z_j)Z_j\| \geq \frac{\|Z_j\|(1 - \|Z_j\|^k)}{(1 + \|Z_j\|^k)^{1+\frac{2}{k}}} = \frac{\|Z\|(1 - \|Z\|^k)}{(1 + \|Z\|^k)^{1+\frac{2}{k}}}, Z \in U^N,$$

here $\|Z_l\|_{m_l}$ (resp. $\|Z\|_N$) is briefly written as $\|Z_l\|$ (resp. $\|Z\|$), and j satisfies $\|Z_j\| = \|Z\| = \max_{1 \leq l \leq n}\{\|Z_l\|\}$. The sharpness of the estimates of Theorem 2.3 is similar to that in the proof of Theorem 2.2, we only pay attention to the right side equality of the estimates of Theorem 2.3 holds for $Z_1 = (r, 0, \cdots, 0)'(0 \leq r < 1), Z_l = (0, 0, \cdots, 0)'(l = 2, \cdots, n)$, the details are omitted here. This completes the proof. ∎

Letting $k = 1$ in Theorem 2.3, we directly obtain

Corollary 2.3. *Let* $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l = 1, 2, \cdots, n,$ $F(Z) = (Z_1 f_1(Z_1),$ $Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S^*(U^N)$. *Then*

$$\frac{\|Z\|(1 - \|Z\|)}{(1 + \|Z\|)^3} \leq \|DF(Z)Z\| \leq \frac{\|Z\|(1 + \|Z\|)}{(1 - \|Z\|)^3}, Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N,$$

and the above estimates are sharp.

Theorem 2.4. *Suppose that* $F(z) = (F_1(z), F_2(z), \cdots, F_n(z)) \in H(U^n)$, *and* $z = 0$ *is a zero of order* $k + 1(k \in \mathbb{N})$ *of* $F(z) - z$. *If* $\Re e \frac{DF_j(z)z}{F_j(z)} > 0, z \in U^n \setminus \{0\}$, *where* j *satisfies the condition* $|z_j| = \|z\| = \max_{1 \leq l \leq n} |z_l|$, *then*

$$\frac{\|z\|(1 - \|z\|^k)}{(1 + \|z\|^k)^{1+\frac{2}{k}}} \leq \|DF(z)z\| \leq \frac{\|z\|(1 + \|z\|^k)}{(1 - \|z\|^k)^{1+\frac{2}{k}}}, z \in U^n,$$

and the above estimates are sharp.

Proof. Fix $z \in U^n \setminus \{0\}$, and denote $z_0 = \frac{z}{\|z\|}$. Let

(2.8) $$h_j(\xi) = \frac{\|z\|}{z_j} F_j(\xi z_0), \xi \in U,$$

where j satisfies the condition $|z_j| = \|z\| = \max_{1 \leq l \leq n}\{|z_l|\}$. Straightforward computation shows that

$$\Re e \left(\frac{h_j'(\xi)\xi}{h_j(\xi)}\right) = \Re e \frac{DF_j(\xi z_0)\xi z_0}{F_j(\xi z_0)} > 0, \xi \in U \setminus \{0\}$$

from the hypothesis of Theorem 2.4 and (2.8). Hence we deduce that $h_j \in S^*(U)$, and $\xi = 0$ is at least a zero of order $k + 1$ of $h_j(\xi) - \xi$.

On the other hand, we conclude that

$$\frac{|\xi|(1 - |\xi|^k)}{(1 + |\xi|^k)^{1 + \frac{2}{k}}} \leq |DF_j(\xi z_0)\xi z_0| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1 + \frac{2}{k}}}, z_0 \in \partial U^n$$

from [7]. It is obvious to deduce that

$$(2.9) \qquad \frac{|\xi|(1 - |\xi|^k)}{(1 + |\xi|^k)^{1 + \frac{2}{k}}} \leq |DF_j(\xi z_0)\xi z_0| \leq \|DF(\xi z_0)\xi z_0\|, z_0 \in \partial U^n .$$

Taking $\xi = \|z\|$, we obtain

$$(2.10) \qquad \|DF(z)z\| \geq \frac{\|z\|(1 - \|z\|^k)}{(1 + \|z\|^k)^{1 + \frac{2}{k}}}, z \in U^n.$$

When $z_0 \in \partial_0 U^n$, it is shown that

$$|DF_l(\xi z_0)\xi z_0| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1 + \frac{2}{k}}}, l = 1, 2, \cdots, n$$

from (2.9). Noticing that $w(z) = DF_l(\xi z)\xi z$ is a holomorphic function on $\overline{U^n}$, in view of the maximum modulus theorem of holomorphic functions on the unit polydisk, it yields that

$$|DF_l(\xi z_0)\xi z_0| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1 + \frac{2}{k}}}, z_0 \in \partial U^n, l = 1, 2, \cdots, n .$$

This implies that

$$\|DF(\xi z_0)\xi z_0\| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1 + \frac{2}{k}}}, z_0 \in \partial U^n .$$

Letting $\xi = \|z\|$, we have

$$(2.11) \qquad \|DF(z)z\| \leq \frac{\|z\|(1 + \|z\|^k)}{(1 - \|z\|^k)^{1 + \frac{2}{k}}}, z \in U^n.$$

It follows the desired results from (2.10) and (2.11).

We set $k = 1$ in Theorem 2.4. It is immediately shown that

Corollary 2.4. *Suppose that* $F(z) = (F_1(z), F_2(z), \cdots, F_n(z)) \in H(U^n)$. *If* $\Re e \frac{DF_j(z)z}{F_j(z)} > 0, z \in U^n \setminus \{0\}$, *where j satisfies the condition* $|z_j| = \|z\| = \max_{1 \leq l \leq n} |z_l|$, *then*

$$\frac{\|z\|(1 - \|z\|)}{(1 + \|z\|)^3} \leq \|DF(z)z\| \leq \frac{\|z\|(1 + \|z\|)}{(1 - \|z\|)^3}, z \in U^n,$$

and the above estimates are sharp.

Remark 2.1. We see that Theorem 2.1 (the case $X = \mathbb{C}^n, B = U^n$) is the special case of Theorem 2.4, and Theorem 2.3 (the case $m_1 = n, m_l = 0, l = 2, \cdots, n$ or $m_l = 1, l = 1, 2, \cdots, n$) is also the special case of Theorem 2.4.

Remark 2.2. In view of Theorems 2.1, 2.2 and 2.3, the forms of distortion theorems of the Fréchet derivative for a subclass of starlike mappings are similar to each other. We may pose the following open problem.

Open problem 2.1. *If $F \in S^*(B)$, then*

$$\frac{\|x\|(1 - \|x\|)}{(1 + \|x\|)^3} \leq \|DF(x)x\| \leq \frac{\|x\|(1 + \|x\|)}{(1 - \|x\|)^3}, x \in B,$$

and the above estimates are sharp.

3. Distortion Theorems of the Jacobi Determinant for a Subclass of Starlike Mappings

In this section, $J_F(z)$ is denoted by the Jacobi matrix of the holomorphic mapping $F(z)$, and $\det J_F(z)$ is denoted by the Jacobi determinant of the holomorphic mapping $F(z)$. Also, let B be the unit ball of $\mathbb{C}^n$ with arbitrary norm $\|.\|$, and I_n be the unit matrix of $\mathbb{C}^n$. Vectors in $\mathbb{C}^n$ are usually written as column vectors in this section.

Suppose that Ω is a bounded starlike circular domain in $\mathbb{C}^n$, and its Mikowski functional $\rho(z)$ is a C^1 function except for a lower dimensional manifold in $\overline{\Omega}$.

Theorem 3.1. *Let $f : \Omega \to \mathbb{C} \in H(\Omega)$, and $F(z) = zf(z) \in S^*_{k+1}(\Omega)$. Then*

$$\frac{1 - \rho^k(z)}{(1 + \rho^k(z))^{\frac{2n}{k}+1}} \leq |\det J_F(z)| \leq \frac{1 + \rho^k(z)}{(1 - \rho^k(z))^{\frac{2n}{k}+1}}, z \in \Omega,$$

and the above estimates are sharp.

Proof. Since $F(z) = zf(z)$, we obtain

$$J_F(z)z = zf(z) + (J_f(z)z)z, z \in \Omega.$$

With the same arguments of the proof in Theorem 2.1, it yields that $f(z) \neq 0$ for $z \in \Omega$, and

$$f(z) + J_f(z)z \neq 0, z \in B.$$

A direct calculation shows that

(3.1) $$[J_F(z)]^{-1}F(z) = \frac{f(z)z}{f(z) + J_f(z)z}, z \in \Omega$$

and

$$J_F(z) = f(z)(I_n + z(J_f(z))' = f(z)\left(I_n + \frac{z(J_f(z))'}{f(z)}\right).$$

Also

$$|\det J_F(z)| = |f(z)|^n \left|1 + \frac{J_f(z)z}{f(z)}\right| = |f(z)|^n \frac{\rho(z)}{2\left|\frac{\partial\rho(z)}{\partial z}((J_F(z))^{-1}F(z))\right|}$$

from (3.1). Therefore, it is shown that

$$\frac{1 - \rho^k(z)}{(1 + \rho^k(z))^{\frac{2n}{k}+1}} \leq |\det J_F(z)| \leq \frac{1 + \rho^k(z)}{(1 - \rho^k(z))^{\frac{2n}{k}+1}}, z \in \Omega$$

from Lemma 2.3 and [15, the case $\alpha = 0$ of Theorem 5].

It is easy to verify that

$$F(z) = \frac{z}{(1 - (\frac{z_1}{r})^k)^{\frac{2}{k}}}, z \in \Omega$$

satisfies the condition of Theorem 3.2, where $r = \sup\{|z_1| : z = (z_1, 0, \cdots, 0)' \in \Omega\}$. A simple calculation shows that

$$J_F(z) = \frac{1}{(1 - (\frac{z_1}{r})^k)^{\frac{2}{k}}}\left(I_n + \frac{2z(\frac{z_1}{r})^{k-1}(\frac{1}{r}, 0, \cdots, 0)}{1 - \frac{z_1}{r}}\right), z \in \Omega.$$

We set $z = Ru$ or $z = Re^{k\pi i}u(0 \leq R < 1)$, where $u = (u_1, u_2, \cdots, u_n)' \in \partial\Omega$, $u_1 = r$, and $i = \sqrt{-1}$. Then the estimates of Theorem 3.1 are sharp. This completes the proof.   ∎

When $\Omega = B, \rho(z) = \|z\|$, we immediately obtain the following corollary.

Corollary 3.1. *Let $f : B \to \mathbb{C} \in H(B)$, and $F(z) = zf(z) \in S^*_{k+1}(B)$. Then*

$$\frac{1 - \|z\|^k}{(1 + \|z\|)^{\frac{2n}{k}+1}} \leq |\det J_F(z)| \leq \frac{1 + \|z\|^k}{(1 - \|z\|)^{\frac{2n}{k}+1}}, z \in B,$$

and the above estimates are sharp.

When $k = 1$, we directly have

Corollary 3.2. *Let $f : \Omega \to \mathbb{C} \in H(\Omega)$, and $F(z) = zf(z) \in S^*(\Omega)$. Then*

$$\frac{1 - \rho(z)}{(1 + \rho(z))^{2n+1}} \leq |\det J_F(z)| \leq \frac{1 + \rho(z)}{(1 - \rho(z))^{2n+1}}, z \in \Omega,$$

and the above estimates are sharp.

Remark 3.1. Let

$$S_\pi^*(B^n) = \left\{ F(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_j)}{z_j} \right)^{\lambda_j} : f_j \in S^*(U), \lambda_j \geq 0, j = 1, 2, \cdots, n, \right.$$

$$\left. \sum_{j=1}^{n} \lambda_j = 1, z = (z_1, z_2, \cdots, z_n)' \in B^n \right\}$$

and

$$SS^*(B^n) = \{ F(z) = z f(z) \in S^*(B^n) : f : B_n \to \mathbb{C} \in H(B^n) \}.$$

It is apparent to see that

$$S_\pi^*(B^n) \subsetneqq SS^*(B^n) \subsetneqq S^*(B^n)$$

from the fact that $F(z) = z + \sum_{m=2}^{\infty} \frac{D^m F(0)(z^m)}{m!} \in S^*(B^n)$ if $\frac{\|D^m F(0)\|}{m!} (m = 2, 3, \cdots)$ are small enough (see [16]). Therefore, Corollary 3.2 is a general version of the corresponding theorem in [17].

In the following theorems and corollaries, let $m_l (l = 1, 2, \cdots, n)$ be a non-negative integer, $N = m_1 + m_2 + \cdots + m_n \in \mathbb{N}$, and $m_l = 0$ means the corresponding component in Z and $F(Z)$ are omitted. Ω^{m_l} (resp. Ω^N) is denoted by the bounded complete Reinhardt domain of $\mathbb{C}^{m_l} (l = 1, 2, \cdots, n)$ (resp. $\mathbb{C}^N$). Let $\rho_{m_l}(Z_l)$ (resp. $\rho_N(Z)$) be the Minkowski functional of Ω^{m_l} (resp. Ω^N). We denote by U^{m_l} (resp. U^N) the unit polydisk of $\mathbb{C}^{m_l} (l = 1, 2, \cdots, n)$ (resp. $\mathbb{C}^N$).

Theorem 3.2. *Let $f_l : \Omega^{m_l} \to \mathbb{C} \in H(\Omega^{m_l}), l = 1, 2, \cdots, n$, $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S_{k+1}^*(\Omega^N)$. Then*

$$\frac{(1 - \rho_N^k(Z))^n}{(1 + \rho_N^k(Z))^{\frac{2N}{k} + n}} \leq |\det J_F(Z)| \leq \frac{(1 + \rho_N^k(Z))^n}{(1 - \rho_N^k(Z))^{\frac{2N}{k} + n}}, Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N.$$

Proof. Let $F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))'$. According to the hypothesis of Theorem 3.2, for any $Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N$, it yields that

$$(3.2) \qquad J_F(Z) = \begin{pmatrix} J_{F_1}(Z_1) & 0 & \cdots & 0 \\ 0 & J_{F_2}(Z_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{F_n}(Z_n) \end{pmatrix}$$

and

$$(J_F(Z))^{-1}F(Z)$$
$$= ((J_{F_1}(Z_1))^{-1}F_1(Z_1), (J_{F_2}(Z_2))^{-1}F_2(Z_2), \cdots, (J_{F_n}(Z_n))^{-1}F_n(Z_n))'.$$

Consequently, we see that

$$F \in S^*(\Omega^N) \Leftrightarrow F_l \in S^*(\Omega^{m_l}), l = 1, 2, \cdots, n$$

from Lemma 2.4. It is not difficult to check that $\frac{1-t^k}{(1+t^k)^{\frac{2m_l}{k}+1}}$ (resp. $\frac{1+t^k}{(1-t^k)^{\frac{2m_l}{k}+1}}(l = 1, 2, \cdots, n)$ is a decreasing function (resp. increasing function) on interval $[0, 1)$ with respect to t. For any $Z = (Z_1, \cdots, Z_l, \cdots, Z_n)' \in \Omega^N$, $Z_l \in \Omega^{m_l}, l = 1, 2, \cdots, n$, we see that $\rho_N(Z) \geq \rho_N((0, \cdots, Z_l, \cdots, 0)') = \rho_{m_l}(Z_l)$ from the definitions of $\rho_N(Z)$ and $\rho_{m_l}(Z_l)$. In view of

$$\frac{1 - \rho_{m_l}^k(Z_l)}{(1 + \rho_{m_l}^k(Z_l))^{\frac{2m_l}{k}+1}} \leq |\det J_{F_l}(Z_l)| \leq \frac{1 + \rho_{m_l}^k(Z_l)}{(1 - \rho_{m_l}^k(Z_l))^{\frac{2m_l}{k}+1}}, Z_l \in \Omega^{m_l}$$

(the case $\Omega = \Omega^{m_l}$ of Theorem 3.1), this implies that

$$\frac{(1 - \rho_N^k(Z))^n}{(1 + \rho_N^k(Z))^{\frac{2N}{k}+n}} \leq |\det J_F(Z)| \leq \frac{(1 + \rho_N^k(Z))^n}{(1 - \rho_N^k(Z))^{\frac{2N}{k}+n}}, Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N$$

from (3.2). It follows the result, as desired. This completes the proof. ∎

When $k = 1$, we immediately obtain

Corollary 3.3. *Let* $f_l : \Omega^{m_l} \to \mathbb{C} \in H(\Omega^{m_l}), l = 1, 2, \cdots, n, F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S^*(\Omega^N).$ *Then*

$$\frac{(1 - \rho_N(Z))^n}{(1 + \rho_N(Z))^{2N+n}} \leq |\det J_F(Z)| \leq \frac{(1 + \rho_N(Z))^n}{(1 - \rho_N(Z))^{2N+n}}, Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N.$$

Remark 3.2. We do not know the sharpness of estimates of Theorem 3.2. However, we obtain the following corollary if $\Omega = U^n$. Moreover, the estimates of Corollary 3.4 are sharp.

Corollary 3.4. *Let* $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l = 1, 2, \cdots, n, F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S^*_{k+1}(U^N).$ *Then*

(3.3)
$$\frac{(1 - \|Z\|^k)^n}{(1 + \|Z\|^k)^{\frac{2N}{k}+n}} \leq |\det J_F(Z)|$$
$$\leq \frac{(1 + \|Z\|^k)^n}{(1 - \|Z\|^k)^{\frac{2N}{k}+n}}, Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N,$$

and the above estimates are sharp.

Proof. Let $\Omega^N = U^N, \Omega^{m_l} = U^{m_l}$, then (3.3) holds from Theorem 3.2 immediately. Consider

$$F(Z) = \left(\frac{Z_1}{(1 - Z_{11}^k)^{\frac{2}{k}}}, \frac{Z_2}{(1 - Z_{21}^k)^{\frac{2}{k}}}, \cdots, \frac{Z_n}{(1 - Z_{n1}^k)^{\frac{2}{k}}} \right)', Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega_N.$$

Then F satisfies the condition of Corollary 3.4, where $Z_l = (Z_{l1}, Z_{l2}, \cdots, Z_{lm_l})' \in U^{m_l}, l = 1, 2, \cdots, n$. Straightforward computation shows that

$$J_F(Z) = \begin{pmatrix} J_{F_1}(Z_1) & 0 & \cdots & 0 \\ 0 & J_{F_2}(Z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{F_n}(Z_n) \end{pmatrix},$$

where

$$J_{F_l}(Z_l) = \begin{pmatrix} \dfrac{1 + Z_{l1}^k}{(1 - Z_{l1})^{\frac{2}{k}+1}} & 0 & \cdots & 0 \\ \dfrac{2Z_{l2}Z_{l1}}{(1 - Z_{l1})^{\frac{2}{k}+1}} & \dfrac{1}{(1 - Z_{l1}^k)^{\frac{2}{k}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{2Z_{lm_l}Z_{l1}}{(1 - Z_{l1})^{\frac{2}{k}} + 1} & 0 & \cdots & \dfrac{1}{(1 - Z_{l1}^k)^{\frac{2}{k}}} \end{pmatrix}, l = 1, 2, \cdots, n.$$

Taking $Z_l = (e^{\frac{\pi i}{k}}r, 0, \cdots, 0)'$ or $(r, 0, \cdots, 0)'(0 \leq r < 1), i = \sqrt{-1}, l = 1, 2, \cdots, n$, then the estimates of Corollary 3.4 are sharp. This completes the proof. ∎

Remark 3.3. According to Theorems 3.1, 3.2 and Corollary 3.4, the most likely form of the sharp distortion theorems of Jacobi determinant for a subclass of starlike mappings is the same as Corollary 3.4 (the case $m_l = 1, l = 1, 2, \cdots, n, k = 1$). Hence we propose the following open problem.

Open problem 3.1. *If $F \in S^*(U^n)$, then*

$$\frac{(1 - \|z\|)^n}{(1 + \|z\|)^{3n}} \leq |\det J_F(z)| \leq \frac{(1 + \|z\|)^n}{(1 - \|z\|)^{3n}}, z \in U^n,$$

and the above estimates are sharp.

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