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APPROXIMATE FIXED POINT THEOREMS FOR PARTIAL GENERALIZED CONVEX CONTRACTION MAPPINGS IN α-COMPLETE METRIC SPACES

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Abstract. In this paper, we introduce the new concept called partial generalized convex contractions and partial generalized convex contractions of order 2. Also, we establish some approximate fixed point theorems for such mappings in α-complete metric spaces. Our results extend and unify the results of Miandaragh et al. [M. A. Miandaragh, M. Postolache, S. Rezapour, *Approximate fixed points of generalized convex contractions*, Fixed Point Theory and Applications 2013, **2013**:255] and several well-known results in literature. We give some examples of a nonlinear contraction mapping, which is not applied to the existence of approximate fixed point and fixed point by using the results of Miandaragh et al. We also consider approximate fixed point results in metric space endowed with an arbitrary binary relation and approximate fixed point results in metric space endowed with graph.

1. INTRODUCTION

Fixed point theory is an important tool for solving various problems in nonlinear functional analysis since it has many useful for proving the existence solutions for nonlinear differential and integral equations. However, in several practical situations, the conditions in the fixed point theorems are too strong and so the existence of a fixed point is not guaranteed. In this situation, we can consider nearly fixed points what we call as approximate fixed points. For self mapping T on a nonempty set X, the study of approximate fixed point $x \in X$ of T we mean in a sense that Tx is "near to" x. The study of approximate fixed point theorems is equally interesting to that of fixed point theorems.

In 2006, inspired and motivated by the work of Tijs et al. [14], Berinde [2] studied and gave some fundamental approximate fixed point theorems in metric space. In 2013,

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Dey and Saha [4] established the existence of approximate fixed point for the Reich operator [11] which in turn generalizes approximate fixed point theorems of Berinde [2]. There have appeared many works on approximate fixed point results (see, for example, [5, 9, 12] and the references therein).

On the other hand, in 1982, Istratescu [7] introduce the concept of convex contractions and proved that each convex contraction mapping has a unique fixed point on a complete metric space. In 2013, Miandaragh et al. [10] extend the concept of convex contractions to generalized convex contractions and generalized convex contractions of ordered 2. They also established some approximate fixed point theorems for continuous mappings satisfy such contractive conditions in complete metric spaces.

Question 1. Is it possible to extend the concepts of generalized convex contractions and generalized convex contractions of ordered 2 to another convex contractive conditions?

Question 2. Is it possible to prove approximate fixed point theorems for new mappings under the weak condition?

It is our purpose in this paper to give affirmative answers to Questions 1 and 2. We will define the concept of partial generalized convex contraction mappings and partial generalized convex contraction mappings of order 2. Under weaker than condition, we study and obtain approximate fixed point theorems in metric spaces. These results extends, unifies and generalizes the main results of Miandaragh et al. [10] and various well known results in the existing literature. Furthermore, we give nontrivial example of a nonlinear contraction mapping to show that the results of Miandaragh et al. [10] can not applied to the existence of approximate fixed point and fixed point. We also obtain approximate fixed point results in metric space endowed with an arbitrary binary relation and approximate fixed point results in metric space endowed with graph.

2. Preliminaries

In this section, we give some definitions, examples and remarks which are useful for main results in this paper. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}$ denotes the set of real numbers.

Definition 2.1. ([12]). Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\varepsilon > 0$ be a given real number. A point $x_0 \in X$ is said to be an ε *-fixed point* (*approximate fixed point*) of T if

$$d(x_0, Tx_0) < \varepsilon.$$

Remark 2.2. We observe that fixed point is ε-fixed points for all $\varepsilon > 0$. However, the converse is not true.

For a metric space (X, d) and a given $\varepsilon > 0$, the set of all ε-fixed points of $T : X \to X$ is denote by

$$F_\varepsilon(T) := \{x \in X | d(x, Tx) < \varepsilon\}.$$

Definition 2.3. ([9]). Let (X, d) be a metric space and $T : X \to X$ be a mapping. We say that T has the *approximate fixed point property* if for all $\varepsilon > 0$, there exists an ε-fixed point of T, that is,

$$\forall \varepsilon > 0, \qquad F_\varepsilon(T) \neq \emptyset$$

or, equivalently,

$$\inf_{x \in X} d(x, Tx) = 0.$$

In 1996, Browder and Petryshyn [3] defined the following notions.

Definition 2.4. ([3]). A self mapping T on a metric space (X, d) is said to be *asymptotically regular* at a point $x \in X$ if

$$d(T^n x, T^{n+1} x) \to 0 \text{ as } n \to \infty,$$

where $T^n x$ denotes the $n - th$ iterate of T at x.

It is not hard to prove the following results.

Lemma 2.5. *Let (X, d) be a metric space and $T : X \to X$ be an asymptotically regular at a point $z \in X$, then T has the approximate fixed point property.*

Proof. For $n \in \mathbb{N}$, we have

$$\inf_{x \in X} d(x, Tx) \leq d(T^n z, T(T^n z)). \tag{2.1}$$

Since T is an asymptotically regular at a point $z \in X$, we get $d(T^n z, T(T^n z)) = d(T^n z, T^{n+1} z)) \to 0$ as $n \to 0$. From (2.1), we get $\inf_{x \in X} d(x, Tx) = 0$. This implies that T has the approximate fixed point property. ∎

In 2012, Samet et al. [13] introduced the concept of α-admissible mapping as follows:

Definition 2.6. ([13]). Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that T is *α-adminssible* if

$$x, y \in X, \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.7. ([6]). Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. The metric space X is said to be *α-complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.

Remark 2.8. If X is complete metric space, then X is α-complete metric space. But the converse is not true.

Example 2.9. Let $X = (0, \infty)$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} \dfrac{x + y}{2}, & x, y \in [1, 2], \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that (X, d) is not complete metric space, but (X, d) is an α-complete metric space. In deed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [1, 2]$ for all $n \in \mathbb{N}$. Since $[1, 2]$ is a closed subset of $\mathbb{R}$, we get $([1, 2], d)$ is a complete metric space and then there exists $x^* \in [1, 2]$ such that $x_n \to x^*$ as $n \to \infty$.

Definition 2.10. ([6]). Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be two mappings. We say that T is an *α-continuous mapping* on (X, d) if for each sequence $\{x_n\}$ in X with

$$\begin{gathered} x_n \to x \text{ as } n \to \infty \text{ for some } x \in X \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all} \\ n \in \mathbb{N} \quad \Longrightarrow \quad Tx_n \to Tx \text{ as } n \to \infty. \end{gathered}$$

Remark 2.11. If T is a continuous mapping, then T is an α-continuous mapping, where $\alpha : X \times X \to [0, \infty)$ is an arbitrary mappings.

Example 2.12. Let $X = (0, \infty)$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define mappings $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [1, 2], \\ 0, & \text{otherwise} \end{cases}$$

and

$$Tx = \begin{cases} \frac{x}{2}, & x \in [1, 2], \\ x^2 - 6x + 12, & x \in (0, 1) \cup (2, \infty). \end{cases}$$

It is easy to see that T is not continuous at $x = 1$. Therefore, T is not continuous. Next, we show that T is α-continuous. Let $\{x_n\}$ be a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we have $x_n \in [1, 2]$ and then $Tx_n = \frac{x_n}{2}$. If $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $Tx_n = \frac{x_n}{2} \to \frac{x}{2} = Tx$ as $n \to \infty$. Therefore, T is α-continuous.

Definition 2.13. Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that X has the *property (H)* whenever for each $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

3. Approximate Fixed Point Theorems for Partial Generalized Convex Contraction Mappings

In this section, we introduce concepts of partial generalized convex contraction and partial generalized convex contraction of order 2 and prove the approximate fixed point theorems for such mappings.

Definition 3.1. Let (X, d) be a metric space. The mapping $T : X \to X$ is called a *partial generalized convex contraction* if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and $a, b \in [0, \infty)$, with $a + b < 1$, satisfies the following condition:

(3.1) for all $x, y \in X$, $\quad \alpha(x, y) \geq 1 \implies d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y)$.

In this case, we say α is the based mapping of T. If $\alpha(x, y) = 1$ for all $x, y \in X$, then the mapping T is called a *convex contraction mapping*.

Now, we establish new approximate fixed point theorem for partial generalized convex contraction mappings in α-complete metric spaces.

Theorem 3.2. *Let (X, d) be a metric space and $T : X \to X$ be a partial generalized convex contraction with the based mapping $\alpha : X \times X \to [0, \infty)$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.*

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point.

Proof. Starting from $x_0 \in X$ in hypothesis and then $\alpha(x_0, Tx_0) \geq 1$. We will construct the sequence $\{x_n\}$ in X by $x_{n+1} = T^{n+1}x_0$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{\widetilde{n}} = x_{\widetilde{n}+1}$ for some $\widetilde{n} \in \mathbb{N} \cup \{0\}$, then we have nothing to prove. So, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $\mu = d(Tx_0, T^2x_0) + d(x_0, Tx_0)$ and $\omega = a + b$. Now we obtain that $d(Tx_0, T^2x_0) \leq \mu$. Since $\alpha(x_0, Tx_0) \geq 1$, by the partial generalized convex contractive condition, we get

$$\begin{aligned} d(T^2x_0, T^3x_0) &\leq ad(Tx_0, T^2x_0) + bd(x_0, Tx_0) \\ &\leq \omega\mu. \end{aligned}$$

Follows from $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ and T is α-addmissible that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \geq 1$, we have

$$\begin{aligned} d(T^3x_0, T^4x_0) &\leq ad(T^2x_0, T^3x_0) + bd(Tx_0, T^2x_0) \\ &\leq ad(Tx_0, T^2x_0) + bd(x_0, Tx_0) + bd(Tx_0, T^2x_0) \\ &\leq ad(x_0, Tx_0) + ad(Tx_0, T^2x_0) + bd(x_0, Tx_0) + bd(Tx_0, T^2x_0) \\ &\leq \omega\mu. \end{aligned}$$

By continuous this process, it is easy to see that $d(T^m x_0, T^{m+1}x_0) \leq \omega^l \mu$, where $m = 2l$ or $m = 2l + 1$ for all $l \in \mathbb{N}$. This implies that $d(T^m x_0, T^{m+1}x_0) \to 0$ as $m \to \infty$. Therefore, T is an asymptotically regular at a point $x_0 \in X$. By using Lemma 2.5, we conclude that T has the approximate fixed point property.

Next, we show that T has a fixed point provide that T is α-continuous and (X, d) is an α-complete metric space. Firstly, we claim that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N}$ such that $n > m$. For the proof it, we distinguish the following cases.

Case 1. m is odd number such that $m = 2l + 1$, where $l \in \mathbb{N}$. Now we obtain that

$$\begin{aligned}
&d(T^m x_0, T^n x_0)\\
&\leq d(T^m x_0, T^{m+1}x_0) + d(T^{m+1}x_0, T^{m+2}x_0) + \cdots + d(T^{n-1}x_0, T^n x_0)\\
&\leq \omega^l \mu + \omega^{l+1}\mu + \omega^{l+1}\mu + \omega^{l+2}\mu + \omega^{l+2}\mu + \cdots\\
&\leq 2\omega^l \mu + 2\omega^{l+1}\mu + 2\omega^{l+2}\mu + \cdots\\
&\leq \frac{2\omega^l \mu}{1-\omega}.
\end{aligned}$$

Case 2. m is even number such that $m = 2l$, where $l \in \mathbb{N}$. Now we obtain that

$$\begin{aligned}
&d(T^m x_0, T^n x_0)\\
&\leq d(T^m x_0, T^{m+1}x_0) + d(T^{m+1}x_0, T^{m+2}x_0) + \cdots + d(T^{n-1}x_0, T^n x_0)\\
&\leq \omega^l \mu + \omega^l\mu + \omega^{l+1}\mu + \omega^{l+1}\mu + \omega^{l+2}\mu + \cdots\\
&\leq 2\omega^l \mu + 2\omega^{l+1}\mu + 2\omega^{l+2}\mu + \cdots\\
&\leq \frac{2\omega^l \mu}{1-\omega}.
\end{aligned}$$

Therefore, $d(T^m x_0, T^n x_0) \leq \frac{2\omega^l \mu}{1-\omega}$ for $n > m$ and $m = 2l$ or $m = 2l + 1$ for all $l \in \mathbb{N}$. Yields to $\frac{2\omega^l \mu}{1-\omega} \to 0$ as $m \to \infty$ that $\{x_n\}$ is a Cauchy sequence in X. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is a Cauchy sequence in X, by using α-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Follows from T is α-continuous that $Tx_n \to Tx^*$ as $n \to \infty$. Therefore $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*$ and thus T has a fixed point. This completes the proof. ■

Now we give some example to illustrate the usability of Theorem 3.2.

Example 3.3. Let $X = (0, \infty)$ and $d : X \times X \to \mathbb{R}$ defined by $d(x - y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \dfrac{x+3}{4}, & x \in [1, 2],\\ x, & x \in (2, 3),\\ x^2 - 8x + 20, & \text{otherwise} \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} \dfrac{e^x + e^y + 1}{|e^x - e^y| + 1}, & x, y \in [1,2], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, (X, d) is not complete metric space and T is not continuous. Therefore, results of Miandaragh et al. [10] can not be applied to this case.

Next, we show that Theorem 3.2 can be guarantee the existence of fixed point of T. Firstly, we will show that T is a partial generalized convex contraction with $a = \frac{2}{3}$ and $b = \frac{1}{9}$.

For $\alpha(x, y) \geq 1$, we have $x, y \in [1, 2]$ and thus

$$\begin{aligned} d(T^2x, T^2y) &= \left| T\left(\frac{x+3}{4}\right) - T\left(\frac{y+3}{4}\right) \right| \\ &= \left| \frac{x+15}{16} - \frac{y+15}{16} \right| \\ &= \frac{1}{16}|x-y| \\ &\leq \frac{5}{18}|x-y| \\ &= \frac{2}{3}\left| \frac{x+3}{4} - \frac{y+3}{4} \right| + \frac{1}{9}|x-y| \\ &= ad(Tx, Ty) + bd(x, y). \end{aligned}$$

Therefore, T is a partial generalized convex contraction with $a = \frac{2}{3}$ and $b = \frac{1}{9}$.

Moreover, it is easy to see that T is an α-admissible and there exists $x_0 = 1.5 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(1.5, T(1.5)) = \alpha(1.5, 1.125) \geq 1.$$

Also, T is α-continuous mapping. By simple calculation, we see that (X, d) is an α-complete metric space. Indeed, let $\{x_n\}$ be Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then $x_n \in [1, 2]$ for all $n \in \mathbb{N}$. Since $([1, 2], d)$ is complete, we get $x_n \to x^*$ as $n \to \infty$, where $x^* \in [1, 2]$. Thus (X, d) is an α-complete metric space. Therefore, by using Theorem 3.2, we get T has a fixed point in X. In this case, T have many fixed points such as 1, 4 and 5.

We obtain that Theorem 3.2 don't claim the uniqueness of fixed point. To assure the uniqueness of the fixed point, we will add the property (H).

Theorem 3.4. *Adding property* (H) *to the hypotheses of Theorem 3.2, we obtain uniqueness of the fixed point of* T.

Proof. Let x^* and y^* be fixed points of T. By property (H), we can choose $z \in X$ such that $\alpha(x^*, z) \geq 1$ and $\alpha(y^*, z) \geq 1$. Since T is α-admissible, we get $\alpha(x^*, T^m z) \geq 1$ and $\alpha(y^*, T^m z) \geq 1$ for all $m \in \mathbb{N}$. Put $\nu = d(x^*, T^2 z) + d(x^*, Tz)$ and $\omega = a + b$. Since $\alpha(x^*, Tz) \geq 1$, we get

$$\begin{aligned} d(x^*, T^3 z) &= d(T^2 x^*, T^2(Tz)) \\ &\leq ad(x^*, T^2 z) + bd(x^*, Tz) \\ &\leq \omega\nu. \end{aligned}$$

Follows from $\alpha(x^*, T^2 z) \geq 1$ that

$$\begin{aligned} d(x^*, T^4 z) &= d(T^2 x^*, T^2(T^2 z)) \\ &\leq ad(x^*, T^3 z) + bd(x^*, T^2 z) \\ &\leq a^2 d(x^*, T^2 z) + abd(x^*, Tz) + bd(x^*, T^2 z) \\ &\leq ad(x^*, T^2 z) + bd(x^*, T^2 z) + ad(x^*, Tz) + bd(x^*, Tz) \\ &= \omega\nu. \end{aligned}$$

Also, from $\alpha(x^*, T^3 z) \geq 1$, we get

$$\begin{aligned} d(x^*, T^5 z) &= d(T^2 x^*, T^2(T^3 z)) \\ &\leq ad(x^*, T^4 z) + bd(x^*, T^3 z) \\ &\leq a(\omega\nu) + b(\omega\nu) \\ &= (a+b)(\omega\nu) \\ &= \omega(\omega\nu) \\ &= \omega^2\nu. \end{aligned}$$

Similarly, from $\alpha(x^*, T^4 z) \geq 1$, we have

$$\begin{aligned} d(x^*, T^6 z) &= d(T^2 x^*, T^2(T^4 z)) \\ &\leq ad(x^*, T^5 z) + bd(x^*, T^4 z) \\ &\leq a[ad(x^*, T^4 z) + bd(x^*, T^3 z)] + bd(x^*, T^4 z) \\ &= (a^2 + b)d(x^*, T^4 z) + abd(x^*, T^3 z) \\ &\leq (a+b)d(x^*, T^4 z) + (a+b)d(x^*, T^3 z) \\ &\leq (a+b)(\omega\nu) + (a+b)(\omega\nu) \\ &= 2(a+b)(\omega\nu) \\ &= 2\omega(\omega\nu) \\ &= 2\omega^2\nu. \end{aligned}$$

By continuing this process, we get $d(x^*, T^m z) \leq 2\omega^{l-1}\nu$, where $m = 2l-1$ or $m = 2l$ for all $l \in \mathbb{N}$. Since $\omega < 1$, therefore $T^m z \to x^*$ as $m \to \infty$. Similarly, we can prove that $T^m z \to y^*$ as $m \to \infty$. By the uniqueness of limit, we have $x^* = y^*$ and then T has a unique fixed point. This completes the proof. ■

Corollary 3.5. *Let (X, d) be a metric space and $T : X \to X$ be a generalized convex contraction with based mapping $\alpha : X \times X \to [0, \infty)$, that is,*

$$\alpha(x, y)d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y) \tag{3.2}$$

for all $x, y \in X$, where $a, b \in [0, 1)$ with $a + b < 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point, and also T has a unique fixed point whenever X has the property (H).

Proof. We will show that T is partial generalized convex contraction with based mapping α. Suppose that $\alpha(x, y) \geq 1$ and then

$$\begin{aligned} d(T^2x, T^2y) &\leq \alpha(x, y)d(T^2x, T^2y) \\ &\leq ad(Tx, Ty) + bd(x, y). \end{aligned}$$

This implies that T is a partial generalized convex contraction with based mapping α. By Theorem 3.2 and Theorem 3.4, we get the desired result. ■

Remark 3.6. It is easy to obtain that Theorem 3.1 of Miandaragh et al. [10] is a special case of Corollary 3.5.

Corollary 3.7. *Let (X, d) be a metric space and $T : X \to X$ be a mapping such that*

$$[d(T^2x, T^2y) + \tau]^{\alpha(x,y)} \leq ad(Tx, Ty) + bd(x, y) + \tau \tag{3.3}$$

for all $x, y \in X$, where $a, b \in [0, 1)$ with $a + b < 1$ and $\tau \geq 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point, and also T has a unique fixed point whenever X has the property (H).

Proof. We will show that T is partial generalized convex contraction with based mapping α. Suppose that $\alpha(x, y) \geq 1$ and hence

$$\begin{aligned} d(T^2x, T^2y) + \tau &\leq [d(T^2x, T^2y) + \tau]^{\alpha(x,y)} \\ &\leq ad(Tx, Ty) + bd(x, y) + \tau. \end{aligned}$$

This implies that

$$d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y),$$

that is T is a partial generalized convex contraction with based mapping α. By Theorem 3.2 and Theorem 3.4, we get the desired result. ∎

Corollary 3.8. *Let (X, d) be a metric space and $T : X \to X$ be a mapping such that*

$$[\tau - 1 + \alpha(x, y)]^{d(T^2x, T^2y)} \leq \tau^{ad(Tx,Ty)+bd(x,y)} \tag{3.4}$$

for all $x, y \in X$, where $a, b \in [0, 1)$ with $a + b < 1$ and $\tau > 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point, and also T has a unique fixed point whenever X has the property (H).

Proof. We will show that T is partial generalized convex contraction with based mapping α. Suppose that $\alpha(x, y) \geq 1$ and hence

$$\begin{aligned} \tau^{d(T^2x,T^2y)} &\leq [\tau - 1 + \alpha(x, y)]^{d(T^2x,T^2y)} \\ &\leq \tau^{ad(Tx,Ty)+bd(x,y)}. \end{aligned}$$

This implies that

$$d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y),$$

that is T is a partial generalized convex contraction with based mapping α. By Theorem 3.2 and Theorem 3.4, we get the desired result. ∎

Next, we give the concept of partial generalized convex contraction mappings of order 2 and prove approximate fixed point results for such mappings in α-complete metric spaces.

Definition 3.9. Let (X, d) be a metric space. The mapping $T : X \to X$ is called a *partial generalized convex contraction of order 2* if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ satisfies the following condition:

$$\begin{aligned} &\text{for all } x, y \in X \ \ \alpha(x, y) \geq 1 \Longrightarrow \ d(T^2x, T^2y) \\ &\leq a_1d(x, Tx) + a_2d(Tx, T^2x) + b_1d(y, Ty) + b_2d(Ty, T^2y). \end{aligned} \tag{3.5}$$

In this case, we say α is the based mapping of T. If $\alpha(x, y) = 1$ for all $x, y \in X$, then the mapping T is called a *convex contraction mapping of order 2*.

Theorem 3.10. *Let (X, d) be a metric space and $T : X \to X$ be a partial generalized convex contraction mapping of order 2 with based mapping $\alpha : X \times X \to [0, \infty)$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.*

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Moreover, T has a unique fixed point provide that X has the property (H).

Proof. Let $x_0 \in X$ be given point in assumption and so $\alpha(x_0, Tx_0) \geq 1$. Construct the sequence $\{x_n\}$ in X by $x_{n+1} = T^{n+1}x_0$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists $\widetilde{n} \in \mathbb{N} \cup \{0\}$ such that $x_{\widetilde{n}} = x_{\widetilde{n}+1}$, then $x_{\widetilde{n}}$ is a fixed point of T. So we have nothing to prove. Next, we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_0, x_1) \geq 1$ and T is α-admissible, we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Let $s = d(Tx_0, T^2x_0) + d(x_0, Tx_0)$, $\mu = 1 - b_2$ and $\rho = a_1 + a_2 + b_1$. Since $\alpha(x_0, Tx_0) \geq 1$, by the partial generalized convex contractive of order 2 condition, we get

$$
\begin{aligned}
& d(T^2x_0, T^3x_0) \\
& \leq a_1 d(x_0, Tx_0) + a_2 d(Tx_0, T^2x_0) + b_1 d(Tx_0, T^2x_0) + b_2 d(T^3x_0, T^2x_0) \\
& \leq a_1 s + (a_2 + b_1)s + b_2 d(T^3x_0, T^2x_0).
\end{aligned}
$$

This implies that, $d(T^2x_0, T^3x_0) \leq \frac{\rho}{\mu}s$. Also, we have

$$
\begin{aligned}
& d(T^3x_0, T^4x_0) \\
& \leq a_1 d(Tx_0, T^2x_0) + a_2 d(T^2x_0, T^3x_0) + b_1 d(T^2x_0, T^3x_0) + b_2 d(T^3x_0, T^4x_0) \\
& \leq a_1 s + (a_2 + b_1)\frac{a_1 + a_2 + b_1}{1 - b_2}s + b_2 d(T^3x_0, T^4x_0)
\end{aligned}
$$

and hence $d(T^3x_0, T^4x_0) \leq (\frac{\rho}{\mu})s$. Similarly, we get $d(T^4x_0, T^5x_0) \leq (\frac{\rho}{\mu})^2 s$ and $d(T^5x_0, T^6x_0) \leq (\frac{\rho}{\mu})^2 s$. By continuing this process, we get $d(T^m x_0, T^{m+1}x_0) \leq (\frac{\rho}{\mu})^l s$, where $m = 2l$ or $m = 2l + 1$ for $l \in \mathbb{N}$. Thus, $d(T^m x_0, T^{m+1}x_0) \to 0$ as $m \to \infty$. This implies that T is an asymptotically regular at a point $x_0 \in X$. By using Lemma 2.5, T has the approximate fixed point property.

Next, we show that T has a fixed point provide that T is α-continuous and (X, d) is an α-complete metric space. Firstly, we claim that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N}$ such that $n > m$. For the proof it, we distinguish the following cases.

Case 1. m is odd number such that $m = 2l + 1$, where $l \in \mathbb{N}$.
Now we obtain that

$$
\begin{aligned}
& d(T^m x_0, T^n x_0) \\
& \leq d(T^m x_0, T^{m+1}x_0) + d(T^{m+1}x_0, T^{m+2}x_0) + \cdots + d(T^{n-1}x_0, T^n x_0)
\end{aligned}
$$

$$
\begin{aligned}
&\le (\frac{\rho}{\mu})^l s + (\frac{\rho}{\mu})^{l+1} s + (\frac{\rho}{\mu})^{l+1} s + (\frac{\rho}{\mu})^{l+2} s + (\frac{\rho}{\mu})^{l+2} s + \cdots \\
&\le 2(\frac{\rho}{\mu})^l s + 2(\frac{\rho}{\mu})^{l+1} s + 2(\frac{\rho}{\mu})^{l+2} s + \cdots \\
&\le \frac{2(\frac{\rho}{\mu})^l s}{1-\frac{\rho}{\mu}}.
\end{aligned}
$$

Case 2. m is even number such that $m = 2l$, where $l \in \mathbb{N}$.
Now we obtain that

$$
\begin{aligned}
& d(T^m x_0, T^n x_0) \\
&\le d(T^m x_0, T^{m+1} x_0) + d(T^{m+1} x_0, T^{m+2} x_0) + \cdots + d(T^{n-1} x_0, T^n x_0) \\
&\le (\frac{\rho}{\mu})^l s + (\frac{\rho}{\mu})^l s + (\frac{\rho}{\mu})^{l+1} s + (\frac{\rho}{\mu})^{l+1} s + (\frac{\rho}{\mu})^{l+2} s + \cdots \\
&\le 2(\frac{\rho}{\mu})^l s + 2(\frac{\rho}{\mu})^{l+1} s + 2(\frac{\rho}{\mu})^{l+2} s + \cdots \\
&\le \frac{2(\frac{\rho}{\mu})^l s}{1-\frac{\rho}{\mu}}.
\end{aligned}
$$

Therefore, we conclude that $d(T^m x_0, T^n x_0) \le \frac{2(\frac{\rho}{\mu})^l s}{1-\frac{\rho}{\mu}}$ where $n > m$ and $m = 2l$ or $m = 2l+1$ for $l \in \mathbb{N}$. This implies that $\{x_n\}$ is a Cauchy sequence in X. Since $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, by using α-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is α-continuous, we get $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. By the uniqueness of the limit $\{x_n\}$, we obtain that $Tx^* = x^*$ and thus T has a fixed point.

For the uniqueness of fixed point of T, we can using a similar technique in the proof of Theorem 3.2. This completes the proof. ■

By applying Theorem 3.10, we get the following results.

Corollary 3.11. *Let (X, d) be a metric space and $T : X \to X$ be a generalized convex contraction mapping of order 2 with based mapping $\alpha : X \times X \to [0, \infty)$, that is,*

(3.6) $\alpha(x, y)d(T^2x, T^2y) \le a_1 d(x, Tx) + a_2 d(Tx, T^2x) + b_1 d(y, Ty) + b_2 d(Ty, T^2y)$

for all $x, y \in X$, where $a, b \in [0, 1)$ with $a + b < 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Then T has the approximate fixed point property.

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point, and also T has a unique fixed point whenever X has the property (H).

Remark 3.12. It is easy to see that Corollary 3.11 is a generalization of Theorem 3.2 of Miandaragh et al. [10].

Corollary 3.13. *Let (X, d) be a metric space and $T : X \to X$ be a mapping such that*

$$\begin{aligned} &[d(T^2x, T^2y) + \tau]^{\alpha(x,y)} \\ &\leq a_1 d(x, Tx) + a_2 d(Tx, T^2x) + b_1 d(y, Ty) + b_2 d(Ty, T^2y) + \tau \end{aligned} \tag{3.7}$$

for all $x, y \in X$, where $a, b \in [0, 1)$ with $a + b < 1$ and $\tau \geq 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point, and also T has a unique fixed point whenever X has the property (H).

Corollary 3.14. *Let (X, d) be a metric space and $T : X \to X$ be a mapping such that*

$$\begin{aligned} &[\tau - 1 + \alpha(x, y)]^{d(T^2x, T^2y)} \\ &\leq \tau^{a_1 d(x,Tx) + a_2 d(Tx,T^2x) + b_1 d(y,Ty) + b_2 d(Ty,T^2y)} \end{aligned} \tag{3.8}$$

for all $x, y \in X$, where $a, b \in [0, 1)$ with $a + b < 1$ and $\tau > 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point, and also T has a unique fixed point whenever X has the property (H).

Remark 3.15. Theorems 3.2, 3.4 and 3.10 generalize many results in the following sense:

1. The condition (3.1) is weaker than some kinds of the contractive conditions and convex contractive conditions such as generalized convex contractive condition [10], Banach's contractive condition [1] etc.;
2. The condition (3.5) is weaker than some kinds of the contractive conditions and convex contractive conditions such as generalized convex contractive of order 2 condition [10], Kannan's contractive condition [8] etc.;
3. For the existence of fixed point, we merely require that α-continuity of T and α-completeness of X, whereas other result demands stronger than this condition.

Consequently, Theorems 3.2, 3.4 and 3.10 extend and improve main results of Miandaragh et al. [10], Banach contraction theorem [1], Kannan contraction theorem [8] and several results in literature.

4. Approximate Fixed Point Theorems in Metric Spaces Endowed with an Arbitrary Binary Relations

In this section, we present approximate fixed point theorems on metric spaces endowed with an arbitrary binary relations. The following notions and definitions are needed.

Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}.$$

Clearly,

$$x, y \in X, \quad x\mathcal{S}y \quad \Longleftrightarrow \quad x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.$$

It is easy to see that $\mathcal{S}$ is the symmetric relation attached to $\mathcal{R}$.

Definition 4.1. Let T be a self mapping on a nonempty set X and $\mathcal{R}$ be a binary relation over X. We say that T is *comparative mapping* if

$$x, y \in X, \quad x\mathcal{S}y \Longrightarrow (Tx)\mathcal{S}(Ty).$$

Definition 4.2. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The metric space X is said to be *$\mathcal{S}$-complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$, converges in X.

Definition 4.3. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. We say that $T : X \to X$ is an *$\mathcal{S}$-continuous mapping* on (X, d) if for each sequence $\{x_n\}$ in X, we have

$$x_n \to x \text{ as } n \to \infty \text{ for some } x \in X \text{ and } x_n\mathcal{S}x_{n+1} \text{ for all}$$
$$n \in \mathbb{N} \quad \Longrightarrow \quad Tx_n \to Tx \text{ as } n \to \infty.$$

Definition 4.4. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The mapping $T : X \to X$ is called a *partial generalized convex contraction mapping with respect to $\mathcal{S}$* if there exist $a, b \in [0, \infty)$, with $a + b < 1$, satisfies the following condition:

$$\text{for } x, y \in X, \quad x\mathcal{S}y \quad \Longrightarrow \quad d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y).$$

Definition 4.5. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The mapping $T : X \to X$ is called a *partial generalized convex contraction mapping of order 2 with respect to $\mathcal{S}$* if there exist $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1+a_2+b_1+b_2 < 1$, satisfies the following condition:

$$\text{for } x, y \in X, \quad x\mathcal{S}y \quad \Longrightarrow \quad d(T^2x, T^2y)$$
$$\leq a_1d(x, Tx) + a_2d(Tx, T^2x) + b_1d(y, Ty) + b_2d(Ty, T^2y).$$

Definition 4.6. Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. We say that X has the *property (H_S)* if for each $x, y \in X$, there exists $z \in X$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$.

Theorem 4.7. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$ be a partial generalized convex contraction mapping with respect to $\mathcal{S}$. Assume that T is comparative mapping and there exists $x_0 \in X$ such that $(x_0)\mathcal{S}(Tx_0)$. Then T has the approximate fixed point property.*

In addition, if T is $\mathcal{S}$-continuous and (X, d) is an $\mathcal{S}$-complete metric space, then T has a fixed point and T has a unique fixed point whenever X has the property (H_S).

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in x\mathcal{S}y, \\ 0, & \text{otherwise.} \end{cases}$$

From that there exists $x_0 \in X$ such that $(x_0)\mathcal{S}(Tx_0)$, we get $\alpha(x_0, Tx_0) = 1$. It follows from T is comparative mapping that T is α-admissible mapping. Since T is a partial generalized convex contraction mapping with respect to $\mathcal{S}$, we have, for all $x, y \in X$,

$$x\mathcal{S}y \quad \Longrightarrow \quad d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y)$$

and then

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y).$$

This implies that T is a partial generalized convex contraction with based mapping α. Now all the hypotheses of Theorem 3.2 are satisfied. So, T has approximate fixed point. Furthermore, $\mathcal{S}$-continuity of T and the $\mathcal{S}$-completeness of X yield the existence of fixed point of T. Finally, the uniqueness of fixed of T follows from Theorem 3.4. This completes the proof. ■

Theorem 4.8. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$ be a partial generalized convex contraction mapping of order 2 with respect to $\mathcal{S}$. Assume that T is comparative mapping and there exists $x_0 \in X$ such that $(x_0)\mathcal{S}(Tx_0)$. Then T has the approximate fixed point property.*

In addition, if T is $\mathcal{S}$-continuous and (X, d) is an $\mathcal{S}$-complete metric space, then T has a fixed point and T has a unique fixed point whenever X has the property (H_S).

Proof. This proof is similar to Theorem 4.7. ■

5. Approximate Fixed Point Analysis with Graph

Throughout this section, let (X, d) be a metric space. A set $\{(x, x) : x \in X\}$ is called a diagonal of the Cartesian product $X \times X$ and is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

In this section, we give the existence of approximate fixed point theorems on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Definition 5.1. Let X be a nonempty set endowed with a graph G. We say that $T : X \to X$ *preserve edge* if

$$\text{for } x, y \in X, \quad (x, y) \in E(G) \quad \Longrightarrow \quad (Tx, Ty) \in E(G).$$

Definition 5.2. Let (X, d) be a metric space endowed with a graph G. The metric space X is said to be $E(G)$-*complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in X.

Definition 5.3. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping. We say that T is an $E(G)$-*continuous mapping* on (X, d) if for each sequence $\{x_n\}$ in X with

$$x_n \to x \text{ as } n \to \infty \text{ for some } x \in X \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \quad \Longrightarrow \quad Tx_n \to Tx \text{ as } n \to \infty.$$

Definition 5.4. Let (X, d) be a metric space endowed with a graph G. The mapping $T : X \to X$ is called a *partial generalized convex contraction mapping with respect to* $E(G)$ if there exist $a, b \in [0, \infty)$, with $a + b < 1$, satisfies the following condition:

$$\text{for } x, y \in X, \quad (x, y) \in E(G) \quad \Longrightarrow \quad d(T^2x, T^2y) \le ad(Tx, Ty) + bd(x, y).$$

Definition 5.5. Let (X, d) be a metric space endowed with a graph G. The mapping $T : X \to X$ is called a *partial generalized convex contraction mapping of order 2 with with respect to* $E(G)$ if there exist $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, satisfies the following condition:

$$\text{for } x, y \in X, \quad (x, y) \in E(G) \quad \Longrightarrow \quad d(T^2x, T^2y)$$
$$\le a_1 d(x, Tx) + a_2 d(Tx, T^2x) + b_1 d(y, Ty) + b_2 d(Ty, T^2y).$$

Example 5.6. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. It is easy to see that T is trivially partial generalized convex contraction mapping and trivially partial generalized convex contraction mapping of order 2 with respect to graph G, where $G = (V(G), E(G)) := (X, \Delta)$.

Definition 5.7. Let X be a nonempty set endowed with a graph G. We say that X has the *property* (H_E) if for each $x, y \in X$, there exists $z \in X$ such that $(x, z) \in E(G)$ and $(y, z) \in E(G)$.

Theorem 5.8. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a partial generalized convex contraction mapping with respect to $E(G)$. Assume that T preserve edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Then T has the approximate fixed point property.*

In addition, if T is $E(G)$-continuous and (X, d) is an $E(G)$-complete metric space, then T has a fixed point and T has a unique fixed point whenever X has the property (H_E).

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

From that there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, we get $\alpha(x_0, Tx_0) = 1$. It follows from T preserve edge that T is α-admissible mapping. Since T is a partial generalized convex contraction mapping with respect to $E(G)$, we have, for all $x, y \in X$,

$$(x, y) \in E(G) \quad \Longrightarrow \quad d(T^2x, T^2y) \le ad(Tx, Ty) + bd(x, y)$$

and then

$$\alpha(x, y) \ge 1 \quad \Longrightarrow \quad d(T^2x, T^2y) \le ad(Tx, Ty) + bd(x, y).$$

This implies that T is a partial generalized convex contraction with based mapping α. Now all the hypotheses of Theorem 3.2 are satisfied. So, T has approximate fixed point. Furthermore, $E(G)$-continuity of T and the $E(G)$-completeness of X yield the existence of fixed point of T. Finally, the uniqueness of fixed of T follows from Theorem 3.4. This completes the proof. ■

Theorem 5.9. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a partial generalized convex contraction mapping of order 2 with respect to $E(G)$. Assume that T preserve edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Then T has the approximate fixed point property.*

In addition, if T is $E(G)$-continuous and (X, d) is an $E(G)$-complete metric space, then T has a fixed point and T has a unique fixed point whenever X has the property (H_E).

Proof. This proof is similar to Theorem 5.8. ■

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