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COMMUTATORS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS ON NON-HOMOGENEOUS METRIC MEASURE SPACES

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Abstract. Let (X, d, μ) be a metric measure space satisfying both the geometrically doubling and the upper doubling measure conditions, which is called non-homogeneous metric measure space. In this paper, via a sharp maximal operator, the boundedness of commutators generated by multilinear singular integral with $RBMO(\mu)$ function on non-homogeneous metric measure spaces in m-multiple Lebesgue spaces is obtained.

1. Introduction

It is well known that the standard singular integral theory is constructed with the assumption of spaces satisfying the doubling measure condition. We recall that μ is said to satisfy the doubling measure condition if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in \mathrm{supp}\mu$ and $r > 0$. A metric measure space (X, d, μ) equipped with a non-negative doubling measure μ is called a space of homogeneous type. In case of non-doubling measures, a non-negative measure μ only need to satisfy the polynomial growth condition, i.e., for all $x \in \mathbb{R}^n$ and $r > 0$, there exist a constant $C_0 > 0$ and $k \in (0, n]$ such that,

$$\mu(B(x, r)) \le C_0 r^k, \tag{1.1}$$

where $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. This breakthrough brings rapid development in harmonic analysis (see [2,5,8-9,17-23]). And the analysis on non-doubling measures has important applications in solving the long-standing open Painlevé's problem (see [18]).

However, as stated by Hytönen in [11], the measure satisfying (1.1) does not include the doubling measure as special cases. To solve this problem, a kind of metric

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measure space (X, d, μ) satisfying geometrically doubling and the upper doubling measure condition (see Definition 1.1 and 1.2) was introduced by Hytönen in [11]. The space is also called non-homogeneous metric measure space. The highlight of this kind of space is that it includes both the homogeneous space and metric space with polynomial growth measures as special cases. From then on, some results paralled to homogeneous spaces and non-doubling measures space are obtained (see [1,5,11-16] and the references therein). Hytönen et al. in [14] and Bui and Duong in [1] independently introduced the atomic Hardy space $H^1(\mu)$ and proved that the dual space of $H^1(\mu)$ is $RBMO(\mu)$. In [1], the authors also proved that Calderón-Zygmund operator and commutators generated by Calderón-Zygmund operator with $RBMO$ function are bounded in $L^p(\mu)$ for $1 < p < \infty$. Recently, some equivalent characterizations are established by Liu et al. in [16] for the boundedness of Calderón-Zygmund operators on $L^p(\mu)$ for $1 < p < \infty$. In [4], Fu et al. established boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces on non-homogeneous spaces.

On the other hand, the theory on multilinear singular integral operators has been considered by many researchers. In [3], Coifman and Meyers firstly established the theory of bilinear Calderón-Zygmund operators. Later, Grafakos and Torres [6-7] established the boundedness of multilinear singular integral on the product Lebesgue spaces and Hardy spaces. The properties of multilinear singular integral and commutators on non-doubling measures spaces $(\mathbb{R}^n, \mu)$ were established by Xu in [21-22]. Weighted norm inequalities for multilinear Calderón-Zygmund operators on non-homogeneous metric measure spaces were also constructed in [10].

In the setting of non-homogeneous metric measure spaces, it is natural to ask whether commutators of multilinear singular integral operators is also bounded in m-multiple Lebesgue spaces. This paper will give an affirmative answer to this question. In this paper, commutators generated by multilinear singular integral with $RBMO(\mu)$ function on non-homogeneous metric spaces is introduced firstly. And we will prove that it is bounded in m-multiple Lebesgue spaces on non-homogeneous metric spaces, provided that multilinear singular integrals is bounded from m-multiple $L^1(\mu) \times \ldots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, where $L^p(\mu)$ and $L^{p,\infty}(\mu)$ denote the Lebesgue space and weak Lebesgue space respectively. This result in this paper includes the corresponding results on both the homogeneous spaces and $(\mathbb{R}^n, \mu)$ with non-doubling measures space. A variant of sharp maximal operator $M^\sharp$, Kolmogorov's theorem and some good properties of the dominating function λ (see Definition 1.2) are the main tools for proving the results of this paper.

Before stating the main results of this paper, we firstly recall some notations and definitions.

Definition 1.1. ([11]). A metric space (X, d) is called geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset X$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Definition 1.2. ([11]). A metric measure space (X, d, μ) is said to be upper doubling if μ is a Borel measure on X and there exists a dominating function $\lambda : X \times (0, +\infty) \to (0, +\infty)$ and a constant $C_\lambda > 0$, such that for each $x \in X, r \longmapsto \lambda(x, r)$ is non-decreasing, and for all $x \in X, r > 0$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2) \tag{1.2}$$

Remark 1.3. (i) A space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x, r) = \mu(B(x, r))$. On the other hand, a metric space (X, d, μ) satisfying the polynomial growth condition (1.1) (in particular, $(X, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$ with μ satisfying (1.1) for some $k \in (0, n]$)) is also an upper doubling measure space if we take $\lambda(x, r) = Cr^k$.

(ii) Let(X, d, μ) be an upper doubling space and λ be a dominating function on $X \times (0, +\infty)$ as in Definition 1.2. In [14], it was shown that there exists another dominating function $\widetilde{\lambda}$ such that for all $x, y \in X$ with $d(x, y) \leq r$,

$$\widetilde{\lambda}(x, r) \leq \widetilde{C}\widetilde{\lambda}(y, r). \tag{1.3}$$

Thus, we suppose that λ always satisfies (1.3) in this paper.

Definition 1.4. ([14]). Let $\alpha, \beta \in (1, +\infty)$. A ball $B \subset X$ is called (α, β)-doubling if $\mu(\alpha B) \leq \beta\mu(B)$.

As pointed in Lemma 2.3 of [1], there exist plenties of doubling balls with small radii and with large radii. In the rest of this paper, unless α and β are specified otherwise, by an (α, β) doubling ball we mean a $(6, \beta_0)$-doubling with a fixed number $\beta_0 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n = \log_2 N_0$ is viewed as a geometric dimension of the space.

Definition 1.5. ([11]). For any two balls $B \subset Q$, define

$$K_{B,Q} = 1 + \int_{2Q\setminus B} \frac{d\mu(x)}{\lambda(c_B, d(x, c_B))}, \tag{1.4}$$

where c_B is the center of the ball B.

And, for two balls $B \subset Q$, one defines the coefficient $K'_{B,Q}$ as follows. Let $N_{B,Q}$ be the smallest integer satisfying $6^{N_{B,Q}} r_B \geq r_Q$, then we set

$$K'_{B,Q} = 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(c_B, 6^k r_B)}. \tag{1.5}$$

Remark 1.6. In the case that $\lambda(x, ar) = a^t\lambda(x, r)$ for $0 < t < \infty$, $x \in X$, and $a, r > 0$, one knows that $K_{B,Q} \approx K'_{B,Q}$ (see [1,4]). However, in general, we only have $K_{B,Q} \leq CK'_{B,Q}$. In this paper, we always suppose that $\lambda(x, ar) = a^t\lambda(x, r)$ for $0 < t < \infty$, $x \in X$, and $a, r > 0$. So we don't differentiate $K_{B,Q}$ with $K'_{B,Q}$ and always write $K_{B,Q}$ for simplicity in this paper.

Definition 1.7. A kernel $K(\cdot, \cdots, \cdot) \in L^1_{loc}((X)^{m+1}\backslash\{(x, y_1 \cdots, y_j, \cdots, y_m) : x = y_1 = \cdots = y_j = \cdots = y_m\})$ is called an m-linear Calderón-Zygmund kernel if it satisfies:

(i)

$$|K(x, y_1, \cdots, y_j, \cdots, y_m)| \leq C\left[\sum_{j=1}^{m} \lambda(x, d(x, y_j))\right]^{-m} \tag{1.6}$$

for all $(x, y_1 \cdots, y_j, \cdots, y_m) \in (X)^{m+1}$ with $x \neq y_j$ for some j.

(ii) There exists $0 < \delta \leq 1$ such that

$$\begin{aligned} &|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x', y_1, \cdots, y_j, \cdots, y_m)| \\ \leq &\frac{Cd(x, x')^\delta}{\left[\sum_{j=1}^{m} d(x, y_j)\right]^\delta \left[\sum_{j=1}^{m} \lambda(x, d(x, y_j))\right]^m}, \end{aligned} \tag{1.7}$$

provided that $Cd(x, x') \leq \max_{1\leq j\leq m} d(x, y_j)$ and for each j,

$$\begin{aligned} &|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x, y_1, \cdots, y'_j, \cdots, y_m)| \\ \leq &\frac{Cd(y_j, y'_j)^\delta}{\left[\sum_{j=1}^{m} d(x, y_j)\right]^\delta \left[\sum_{j=1}^{m} \lambda(x, d(x, y_j))\right]^m}, \end{aligned} \tag{1.8}$$

provided that $Cd(y_j, y'_j) \leq \max_{1\leq j\leq m} d(x, y_j)$.

A multilinear operator T is called a multilinear Calderón-Zygmund singular integral operator with the above kernel K satisfying (1.6), (1.7) and (1.8), if for $f_1, \cdots, f_m$ are L^∞ functions with compact support and $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$,

$$T(f_1, \cdots f_m)(x) = \int_{X^m} K(x, y_1, \cdots y_m) f_1(y_1) \cdots f_m(y_m) d\mu(y_1) \cdots d\mu(y_m). \tag{1.9}$$

Remark 1.8. Because $\max_{1\leq j\leq m} d(x, y_j) \leq \sum_{j=1}^{m} d(x, y_j) \leq m \max_{1\leq j\leq m} d(x, y_j)$, (ii) in Definition 1.7 is equivalent to (ii') in the following statement.

(ii') There exists $0 < \delta \leq 1$ such that

$$|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x', y_1, \cdots, y_j, \cdots, y_m)| \le \frac{Cd(x, x')^\delta}{\left[\max\limits_{1\le j\le m} d(x, y_j)\right]^\delta \left[\sum\limits_{j=1}^m \lambda(x, d(x, y_j))\right]^m}, \tag{1.10}$$

provided that $Cd(x, x') \le \max\limits_{1\le j\le m} d(x, y_j)$ and for each j,

$$|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x, y_1, \cdots, y_j', \cdots, y_m)| \le \frac{Cd(y_j, y_j')^\delta}{\left[\max\limits_{1\le j\le m} d(x, y_j)\right]^\delta \left[\sum\limits_{j=1}^m \lambda(x, d(x, y_j))\right]^m}, \tag{1.11}$$

provided that $Cd(y_j, y_j') \le \max\limits_{1\le j\le m} d(x, y_j)$.

Definition 1.9. ([1]). Let $\rho > 1$ be some fixed constant. A function $b \in L^1_{loc}(\mu)$ is said to belong to $RBMO(\mu)$ if there exists a constant $C > 0$ such that for any ball B

$$\frac{1}{\mu(\rho B)} \int_B |b(x) - m_{\widetilde{B}}(b)| d\mu(x) \le C, \tag{1.12}$$

and for any two doubling balls $B \subset Q$,

$$|m_B(b) - m_Q(b)| \le CK_{B,Q}, \tag{1.13}$$

where $\widetilde{B}$ is the smallest (α, β)-doubling ball of the form $6^k B$ with $k \in \mathbb{N}\bigcup\{0\}$, and $m_{\widetilde{B}}(b)$ is the mean value of b on $\widetilde{B}$, namely,

$$m_{\widetilde{B}}(b) = \frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}} b(x) d\mu(x).$$

The minimal constant C appearing in (1.12) and (1.13) is defined to be the $RBMO(\mu)$ norm of b and denoted by $||b||_*$.

For $1 \le i \le k$, we denote by C_i^k the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(i)\}$ of $\{1, 2, \cdots, k\}$ with i different elements. For any $\sigma \in C_i^k$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \cdots, k\}\backslash\sigma$. Moreover, for $b_i \in RBMO(\mu), i = 1, \cdots, k$, let $\vec{b} = (b_1, b_2, \cdots, b_k)$ be a finite family of locally integrable function. For all $1 \le i \le k$ and $\sigma = \{\sigma(1), \cdots, \sigma(i)\} \in C_i^k$, we set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(i)})$ and the product $b_\sigma(x) = b_{\sigma(1)}(x) \cdots b_{\sigma(i)}(x)$. Also, we denote $\vec{f} = (f_1, \cdots, f_k)$, $\vec{f}_\sigma = (f_{\sigma(1)}, \cdots, f_{\sigma(i)})$ and $\vec{b}_{\sigma'} \vec{f}_{\sigma'} = (b_{\sigma'(i+1)} f_{\sigma'(i+1)}, \cdots, b_{\sigma'(k)} f_{\sigma'(k)})$.

Definition 1.10. A kind of commutators generated by multilinear singular integral operator T with $b_i \in RBMO(\mu), i = 1, \cdots, k$ is defined as follows:

$$[\vec{b}, T](\vec{f})(x) = \sum_{i=0}^{k} \sum_{\sigma \in C_i^k} (-1)^{k-i} b_\sigma(x) T(\vec{f_\sigma}, \vec{b}_{\sigma'} \vec{f}_{\sigma'})(x). \tag{1.14}$$

In particular, when $k = 2$, we can obtain

$$\begin{aligned} [b_1, b_2, T](f_1, f_2)(x) &= b_1(x)b_2(x)T(f_1, f_2)(x) - b_1(x)T(f_1, b_2 f_2)(x) \\ &\quad - b_2(x)T(b_1 f_1, f_2)(x) + T(b_1 f_1, b_2 f_2)(x). \end{aligned} \tag{1.15}$$

Also, we define $[b_1, T]$ and $[b_2, T]$ as follows respectively.

$$[b_1, T](f_1, f_2)(x) = b_1(x)T(f_1, f_2)(x) - T(b_1 f_1, f_2)(x), \tag{1.16}$$

$$[b_2, T](f_1, f_2)(x) = b_2(x)T(f_1, f_2)(x) - T(f_1, b_2 f_2)(x). \tag{1.17}$$

For the sake of simplicity and without loss of generality, we only consider the case of $k = 2$ in this paper. Let us state the main result as follows.

Theorem 1.11. *Suppose that μ is a Radon measure with $||\mu|| = \infty$. $[b_1, b_2, T]$ is defined by (1.15). Let $1 < p_1, p_2 < +\infty$, $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, $b_1 \in RBMO(\mu)$ and $b_2 \in RBMO(\mu)$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$||[b_1, b_2, T](f_1, f_2)||_{L^q(\mu)} \le C||f_1||_{L^{p_1}(\mu)}||f_2||_{L^{p_2}(\mu)}, \tag{1.18}$$

where $\dfrac{1}{q} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$.

Throughout this paper, C always denotes a positive constant independent of the main parameters involved, but it may be different from line to line. And p' is the conjugate index of p, namely, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

2. Proof of Main Results

To prove the main theorem, we firstly give some notations and lemmas.

Let $f \in L^1_{loc}(\mu)$, the sharp maximal operator is defined by

$$\begin{aligned} &M^\sharp f(x) \\ &= \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_{\widetilde{B}}(f)| d\mu(y) + \sup_{(B,Q) \in \Delta_x} \frac{|m_B(f) - m_Q(f)|}{K_{B,Q}}, \end{aligned} \tag{2.1}$$

where $\Delta_x := \{(B,Q) : x \in B \subset Q$ and $B,\ Q$ are doubling balls$\}$ and the non-centered doubling maximal operator is denoted by

$$Nf(x) = \sup_{\substack{B \ni x, \\ B \text{ doubling}}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

For any $0 < \delta < 1$, we also define that

$$M_\delta^\sharp f(x) = \{M^\sharp(|f|^\delta)(x)\}^{1/\delta} \tag{2.2}$$

and

$$N_\delta f(x) = \{N(|f|^\delta)(x)\}^{1/\delta}. \tag{2.3}$$

We can obtain that for any $f \in L^1_{loc}(\mu)$,

$$|f(x)| \leq N_\delta f(x) \tag{2.4}$$

for $\mu - a.e.\ x \in X$. Let us give an explanation for inequality (2.4). By the Lebesgue differential theorem, we obtain that $|f(x)| \leq Nf(x)$. Hence

$$|f(x)| = [|f(x)|^\delta]^{1/\delta} \leq \{N(|f|^\delta)(x)\}^{1/\delta} = N_\delta f(x).$$

Let $\rho > 1$, $p \in (1, \infty)$ and $r \in (1, p)$, the non-centered maximal operator $M_{r,(\rho)} f$ is defined by

$$M_{r,(\rho)} f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(\rho B)} \int_B |f(y)|^r d\mu(y) \right\}^{1/r}. \tag{2.5}$$

When $r = 1$, we simply write $M_{1,(\rho)} f(x)$ as $M_{(\rho)} f$. If $\rho \geq 5$, then the operator $M_{(\rho)} f$ is bounded on $L^p(\mu)$ for $p > 1$ and $M_{r,(\rho)}$ is bounded on $L^p(\mu)$ for $p > r$ (see [1]).

From Theorem 4.2 in [1], it is easy to obtain that

Lemma 2.1. *Let $f \in L^1_{loc}(\mu)$ with $\int_X f(x) d\mu(x) = 0$ if $||\mu|| < \infty$. For $1 < p < \infty$ and $0 < \delta < 1$, if $\inf(1, N_\delta f) \in L^p(\mu)$, then there exists a constant $C > 0$ such that*

$$||N_\delta(f)||_{L^p(\mu)} \leq C ||M_\delta^\sharp(f)||_{L^p(\mu)}. \tag{2.6}$$

Lemma 2.2. ([4,19]). *Let $1 \leq p < \infty$ and $1 < \rho < \infty$. Then $b \in RBMO(\mu)$ if and only if for any ball $B \subset X$,*

$$\left\{ \frac{1}{\mu(\rho B)} \int_B |b_B - m_{\widetilde{B}}(b)|^p d\mu(x) \right\}^{1/p} \leq C ||b||_*, \tag{2.7}$$

and for any two doubling balls $B \subset Q$,

$$|m_B(b) - m_Q(b)| \leq C K_{B,Q} ||b||_*. \tag{2.8}$$

Lemma 2.3. ([4]).

$$|m_{\widetilde{6^j\frac{6}{5}B}}(b) - m_{\widetilde{B}}(b)| \leq Cj||b||_*. \tag{2.9}$$

Lemma 2.4. ([10]). *Suppose that μ is a Radon measure with $||\mu|| = \infty$. Let T be defined by (1.9) with $m = 2$. Let $1 < p_1, p_2 < +\infty$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$||T(f_1, f_2)||_{L^q(\mu)} \leq C||f_1||_{L^{p_1}(\mu)}||f_2||_{L^{p_2}(\mu)}, \tag{2.10}$$

where $\dfrac{1}{q} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$.

Lemma 2.5. *Suppose that $[b_1, b_2, T]$ is defined by (1.15), $0 < \delta < 1/2$, $1 < p_1, p_2, q < \infty$, $1 < r < q$ and $b_1, b_2 \in RBMO(\mu)$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that for any $x \in X$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$,*

$$\begin{aligned} &M_\delta^\sharp[b_1, b_2, T](f_1, f_2)(x) \leq C||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) \\ &+ C||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) + C||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) \\ &+ C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x), \end{aligned} \tag{2.11}$$

$$\begin{aligned} M_\delta^\sharp[b_1, T](f_1, f_2)(x) &\leq C||b_1||_* M_{r,(6)}(T(f_1, f_2))(x) \\ &+ C||b_1||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} M_\delta^\sharp[b_2, T](f_1, f_2)(x) &\leq C||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) \\ &+ C||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x). \end{aligned} \tag{2.13}$$

Proof. Because $L^\infty(\mu)$ with compact support is dense in $L^p(\mu)$ for $1 < p < \infty$, we only consider the situation of $f_1, f_2 \in L^\infty(\mu)$ with compact support. Also, by Corollary 3.11 in [4], without loss of generality, we can assume that $b_1, b_2 \in L^\infty(\mu)$.

As in the proof of Theorem 9.1 in [19], to obtain (2.11), it suffices to show that

$$\begin{aligned} &\left(\frac{1}{\mu(6B)} \int_B \Big| |[b_1, b_2, T](f_1, f_2)(z)|^\delta - |h_B|^\delta \Big| d\mu(z)\right)^{1/\delta} \\ &\leq C||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + C||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\ &\quad + C||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) + C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x), \end{aligned} \tag{2.14}$$

holds for any x and ball B with $x \in B$, and

$$
\begin{aligned}
&|h_B - h_Q| \\
&\le CK_{B,Q}^2 \Big[||b_1||_* ||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + ||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\
&\quad + ||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) + ||b_1||_* ||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x) \Big].
\end{aligned} \tag{2.15}
$$

for any balls $B \subset Q$ with $x \in B$, where B is an arbitrary ball and Q is a doubling ball. For any ball B, we denote

$$
h_B := m_B(T((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{X \setminus \frac{6}{5}B}, (b_2 - m_{\widetilde{B}}(b_2)) f_2 \chi_{X \setminus \frac{6}{5}B})),
$$

and

$$
h_Q := m_Q(T((b_1 - m_Q(b_1)) f_1 \chi_{X \setminus \frac{6}{5}Q}, (b_2 - m_Q(b_2)) f_2 \chi_{X \setminus \frac{6}{5}Q})).
$$

Write

$$
[b_1, b_2, T] = T((b_1 - b_1(z)) f_1, (b_2 - b_2(z)) f_2),
$$

and

$$
\begin{aligned}
&T((b_1 - m_{\widetilde{B}}(b_1)) f_1, (b_2 - m_{\widetilde{B}}(b_2)) f_2) \\
&= T((b_1 - b_1(z) + b_1(z) - m_{\widetilde{B}}(b_1)) f_1, (b_2 - b_2(z) + b_2(z) - m_{\widetilde{B}}(b_2)) f_2) \\
&= (b_1(z) - m_{\widetilde{B}}(b_1))(b_2(z) - m_{\widetilde{B}}(b_2)) T(f_1, f_2) \\
&\quad - (b_1(z) - m_{\widetilde{B}}(b_1)) T(f_1, (b_2 - b_2(z)) f_2) \\
&\quad - (b_2(z) - m_{\widetilde{B}}(b_2)) T((b_1 - b_1(z)) f_1, f_2) \\
&\quad + T((b_1 - b_1(z)) f_1, (b_2 - b_2(z)) f_2).
\end{aligned} \tag{2.16}
$$

Then

$$
\begin{aligned}
&\left(\frac{1}{\mu(6B)} \int_B \Big| |[b_1, b_2, T](f_1, f_2)(z)|^\delta - |h_B|^\delta \Big| d\mu(z) \right)^{1/\delta} \\
&\le C \left(\frac{1}{\mu(6B)} \int_B |[b_1, b_2, T](f_1, f_2)(z) - h_B|^\delta d\mu(z) \right)^{1/\delta} \\
&\le C \left(\frac{1}{\mu(6B)} \int_B |(b_1(z) - m_{\widetilde{B}}(b_1))(b_2(z) - m_{\widetilde{B}}(b_2)) T(f_1, f_2)(z)|^\delta d\mu(z) \right)^{1/\delta} \\
&\quad + C \left(\frac{1}{\mu(6B)} \int_B |(b_1(z) - m_{\widetilde{B}}(b_1)) T(f_1, (b_2 - b_2(z)) f_2)(z)|^\delta d\mu(z) \right)^{1/\delta} \\
&\quad + C \left(\frac{1}{\mu(6B)} \int_B |(b_2(z) - m_{\widetilde{B}}(b_2)) T((b_1 - b_1(z)) f_1, f_2)(z)|^\delta d\mu(z) \right)^{1/\delta} \\
&\quad + C \left(\frac{1}{\mu(6B)} \int_B |T((b_1 - m_{\widetilde{B}}(b_1)) f_1, (b_2 - m_{\widetilde{B}}(b_2)) f_2)(z) - h_B|^\delta d\mu(z) \right)^{1/\delta} \\
&=: E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{2.17}
$$

We firstly estimate E_1. Let $r_1, r_2 > 1$ such that $\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{\delta}$. By Hölder's inequality and Lemma 2.2, we obtain

$$\begin{aligned} E_1 &\le C\bigg(\frac{1}{\mu(6B)}\int_B |b_1(z) - m_{\widetilde{B}}(b_1)|^{r_1} d\mu(z)\bigg)^{1/r_1} \\ &\quad \times \bigg(\frac{1}{\mu(6B)}\int_B |b_2(z) - m_{\widetilde{B}}(b_2)|^{r_2} d\mu(z)\bigg)^{1/r_2} \\ &\quad \times \bigg(\frac{1}{\mu(6B)}\int_B |T(f_1, f_2)|^{r} d\mu(z)\bigg)^{1/r} \\ &\le C||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x). \end{aligned} \tag{2.18}$$

For E_2, let $s > 1$ such that $\frac{1}{s} + \frac{1}{r} = \frac{1}{\delta}$, by Hölder's inequality and Lemma 2.2, we deduce

$$\begin{aligned} E_2 &\le C\bigg(\frac{1}{\mu(6B)}\int_B |b_1(z) - m_{\widetilde{B}} b_1|^{s} d\mu(z)\bigg)^{1/s} \\ &\quad \times \bigg(\frac{1}{\mu(6B)}\int_B |[b_2, T](f_1, f_2)|^{r} d\mu(z)\bigg)^{1/r} \\ &\le C||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x). \end{aligned} \tag{2.19}$$

Similar to estimate E_2, we immediately get

$$E_3 \le C||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x). \tag{2.20}$$

Let us turn to estimate E_4. Denote $f_j^1 = f_j \chi_{\frac{6}{5}B}$ and $f_j^2 = f_j - f_j^1$ for $j = 1, 2$. Then

$$\begin{aligned} E_4 &\le C\bigg(\frac{1}{\mu(6B)}\int_B |T((b_1 - m_{\widetilde{B}} b_1) f_1^1(z), (b_2 - m_{\widetilde{B}} b_2) f_2^1)(z)|^{\delta} d\mu(z)\bigg)^{1/\delta} \\ &\quad + C\bigg(\frac{1}{\mu(6B)}\int_B |T((b_1 - m_{\widetilde{B}} b_1) f_1^1(z), (b_2 - m_{\widetilde{B}} b_2) f_2^2)(z)|^{\delta} d\mu(z)\bigg)^{1/\delta} \\ &\quad + C\bigg(\frac{1}{\mu(6B)}\int_B |T((b_1 - m_{\widetilde{B}} b_1) f_1^2(z), (b_2 - m_{\widetilde{B}} b_2) f_2^1)(z)|^{\delta} d\mu(z)\bigg)^{1/\delta} \\ &\quad + C\bigg(\frac{1}{\mu(6B)}\int_B |T((b_1 - m_{\widetilde{B}} b_1) f_1^2(z), (b_2 - m_{\widetilde{B}} b_2) f_2^2)(z) - h_B|^{\delta} d\mu(z)\bigg)^{1/\delta} \\ &=: E_{41} + E_{42} + E_{43} + E_{44}. \end{aligned} \tag{2.21}$$

To estimate E_{41}, we need the classical Kolmogorov's theorem: Let (X, μ) be a probability measure space and let $0 < p < q < \infty$, then there exists a constant $C > 0$, such that $||f||_{L^p(\mu)} \leq C||f||_{L^{q,\infty}(\mu)}$ for any measurable function f. Let $p = \delta$ and $q = 1/2$ such that $0 < \delta < 1/2$. Using Kolmogorov's theorem, the boundedness of T, Lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned}
E_{41} \leq & C||T((b_1 - m_{\widetilde{B}}b_1)f_1^1, (b_2 - m_{\widetilde{B}}b_2)f_2^1)||_{L^{1/2,\infty}(\frac{6}{5}B, \frac{d\mu(z)}{\mu(6B)})} \\
\leq & C\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |(b_1 - m_{\widetilde{B}}b_1)f_1(z)|d\mu(z) \\
& \times \frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |(b_2 - m_{\widetilde{B}}b_2)f_2(z)|d\mu(z) \\
\leq & C\bigg(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |(b_1 - m_{\widetilde{B}}b_1|^{p_1'}d\mu(z)\bigg)^{1/p_1'} \\
& \times \bigg(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |f_1(z)|^{p_1}d\mu(z)\bigg)^{1/p_1} \\
& \times \bigg(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |(b_2 - m_{\widetilde{B}}b_2)|^{p_2'}d\mu(z)\bigg)^{1/p_2'} \\
& \times \bigg(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |f_2(z)|^{p_2}d\mu(z)\bigg)^{1/p_2} \\
\leq & C||b_1||_*||b_2||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).
\end{aligned} \tag{2.22}$$

To compute E_{42}, using (i) of Definition 1.7, Lemma 2.1, Lemma 2.2, Hölder's inequality and the properties of λ, we know

$$\begin{aligned}
E_{42} \leq & C\frac{1}{\mu(6B)}\int_X\int_X\int_{X\setminus\frac{6}{5}B} \frac{|b_1(y_1) - m_{\widetilde{B}}b_1||f_1^1(y_1)|}{[\lambda(z, d(z, y_1)) + \lambda(z, d(z, y_2))]^2} \\
& \times |b_2(y_2) - m_{\widetilde{B}}b_2||f_2^2(y_2)|d\mu(y_1)d\mu(y_2)d\mu(z) \\
\leq & C\frac{1}{\mu(6B)}\int_B\int_{\frac{6}{5}B} |b_1(y_1) - m_{\widetilde{B}}b_1||f_1(y_1)|d\mu(y_1) \\
& \times \int_{X\setminus\frac{6}{5}B} \frac{|b_2(y_2) - m_{\widetilde{B}}b_2||f_2(y_2)|d\mu(y_2)}{[\lambda(z, d(z, y_2))]^2}d\mu(z) \\
\leq & C\bigg(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |b_1(y_1) - m_{\widetilde{B}}b_1|^{p_1'}d\mu(y_1)\bigg)^{1/p_1'} \\
& \times \bigg(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B} |f_1(y_1)|^{p_1}d\mu(y_1)\bigg)^{1/p_1} \\
& \times \mu(B)\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}B} \frac{|b_2(y_2) - m_{\widetilde{B}}b_2||f_2(y_2)|}{[\lambda(z, 6^{k-1}\frac{6}{5}r_B)]^2}d\mu(y_2)
\end{aligned} \tag{2.23}$$

$$\leq C||b_1||_*M_{p_1,(5)}f_1(x)\sum_{k=1}^{\infty}6^{-km}\frac{\mu(B)}{\mu(\frac{6}{5}B)}\frac{\mu(\frac{6}{5}B)}{\lambda(z,\frac{6}{5}r_B)}$$
$$\times\frac{1}{\lambda(z,6^{k-1}\frac{6}{5}r_B)}\int_{6^k\frac{6}{5}B}|b_2(y_2)-m_{\widetilde{B}}b_2||f_2(y_2)|d\mu(y_2)$$
$$\leq C||b_1||_*M_{p_1,(5)}f_1(x)\sum_{k=1}^{\infty}6^{-km}\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}$$
$$\times\int_{6^k\frac{6}{5}B}|b_2(y_2)-m_{\widetilde{6^k\frac{6}{5}B}}(b_2)+m_{\widetilde{6^k\frac{6}{5}B}}(b_2)-m_{\widetilde{B}}b_2||f_2(y_2)|d\mu(y_2)$$
$$\leq C||b_1||_*M_{p_1,(5)}f_1(x)\sum_{k=1}^{\infty}6^{-km}\Bigg[\bigg(\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}$$
$$\times\int_{6^{k+1}\frac{6}{5}B}|b_2(y_2)-m_{\widetilde{6^k\frac{6}{5}B}}(b_2)|^{p_2'}d\mu(y_2)\bigg)^{1/p_2'}$$
$$\times\bigg(\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|f_2(y_2)|^{p_2}d\mu(y_2)\bigg)^{1/p_2}$$
$$+Ck||b_2||_*\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|f_2(y_2)|d\mu(y_2)\Bigg]$$
$$\leq C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).$$

Similarly, we get

$$E_{43}\leq C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.24}$$

Let us turn to estimate E_{44}. For $z_0\in B$, by (ii) of Definition 1.7, Lemma 2.1, Lemma 2.2, Hölder's inequality and the properties of λ, we obtain

$$\begin{aligned}
&|T((b_1-m_{\widetilde{B}}b_1)f_1^2,(b_2-m_{\widetilde{B}}b_2)f_2^2)(z)\\
&\quad-T((b_1-m_{\widetilde{B}}b_1)f_1^2,(b_2-m_{\widetilde{B}}b_2)f_2^2)(z_0)|\\
&\leq C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|\\
&\quad\times|\prod_{i=1}^{2}(b_i(y_i)-m_{\widetilde{B}}b_i)f_i(y_i)|d\mu(y_1)d\mu(y_2)\\
&\leq C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B}\frac{d(z,z_0)^{\delta}|\prod_{i=1}^{2}(b_i(y_i)-m_{\widetilde{B}}b_i)f_i(y_i)|d\mu(y_1)d\mu(y_2)}{(d(z,y_1)+d(z,y_2))^{\delta}[\sum_{j=1}^{2}\lambda(z,d(z,y_j))]^2}
\end{aligned}\tag{2.25}$$

$$
\begin{aligned}
&\le C\prod_{i=1}^{2}\int_{X\setminus\frac{6}{5}B}\frac{d(z,z_0)^{\delta_i}|b_i(y_i)-m_{\widetilde{B}}b_i||f_i(y_i)|d\mu(y_i)}{d(z,y_i)^{\delta_i}\lambda(z,d(z,y_i))}\\
&\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}B}6^{-k\delta_i}\frac{\mu(5\times 6^k\frac{6}{5}B)}{\lambda(z,5\times 6^k\frac{6}{5}r_B)}\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}|b_i(y_i)-m_{\widetilde{B}}b_i||f_i(y_i)|d\mu(y_i)\\
&\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_i}\left(\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|b_i(y_i)-m_{\widetilde{B}}b_i|^{p_i'}d\mu(y_i)\right)^{1/p_i'}\\
&\quad\times\left(\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{1/p_i}\\
&\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_i}M_{p_i,(6)}f_i(x)\left(\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|b_i(y_i)-m_{\widetilde{6^k\frac{6}{5}B}}(b_i)\right.\\
&\quad\left.+m_{\widetilde{6^k\frac{6}{5}B}}(b_i)-m_{\widetilde{B}}b_i|^{p_i'}d\mu(y_i)\right)^{1/p_i'}\\
&\le C\prod_{i=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_i}k||b_i||_*M_{p_i,(5)}f_i(x)\\
&\le C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).
\end{aligned}
$$

where $\delta_1,\delta_2>0$ and $\delta_1+\delta_2=\delta$.

Taking the mean over $z_0\in B$, we deduce

$$E_{44}\le C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.26}$$

So (2.14) can be obtain from (2.17) to (2.26).

Next we prove (2.15). Consider two balls $B\subset Q$ with $x\in B$, where B is an arbitrary ball and Q is a doubling ball. For any ball B, let $N=N_{B,Q}+1$, then we obtain

$$
\begin{aligned}
&\Big||m_BT((b_1-m_{\widetilde{B}}b_1)f_1^2,(b_2-m_{\widetilde{B}}b_2)f_2^2)|\\
&\quad-|m_QT((b_1-m_Qb_1)f_1^2,(b_2-m_Qb_2)f_2^2)|\Big|\\
&\le|m_BT((b_1-m_{\widetilde{B}}b_1)f_1\chi_{X\setminus 6^NB},(b_2-m_{\widetilde{B}}b_2)f_2\chi_{X\setminus 6^NB})\\
&\quad-m_QT((b_1-m_{\widetilde{B}}b_1)f_1\chi_{X\setminus 6^NB},(b_2-m_{\widetilde{B}}b_2)f_2\chi_{X\setminus 6^NB})|\\
&\quad+|m_QT((b_1-m_Qb_1)f_1\chi_{X\setminus 6^NB},(b_2-m_Qb_2)f_2\chi_{X\setminus 6^NB})\\
&\quad-m_QT((b_1-m_{\widetilde{B}}b_1)f_1\chi_{X\setminus 6^NB},(b_2-m_{\widetilde{B}}b_2)f_2\chi_{X\setminus 6^NB})|\\
&\quad+|m_BT((b_1-m_{\widetilde{B}}b_1)f_1\chi_{6^NB\setminus\frac{6}{5}B},(b_2-m_{\widetilde{B}}b_2)f_2\chi_{X\setminus\frac{6}{5}B})|\\
&\quad+|m_BT((b_1-m_{\widetilde{B}}b_1)f_1\chi_{X\setminus\frac{6}{5}B},(b_2-m_{\widetilde{B}}b_2)f_2\chi_{6^NB\setminus\frac{6}{5}B})|
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
&+|m_Q T((b_1-m_Qb_1)f_1\chi_{6^NB\setminus\frac{6}{5}Q},(b_2-m_Qb_2)f_2\chi_{X\setminus 6^NB})|\\
&+|m_Q T((b_1-m_Qb_1)f_1\chi_{X\setminus\frac{6}{5}Q},(b_2-m_Qb_2)f_2\chi_{6^NB\setminus\frac{6}{5}Q})|\\
=:&F_1+F_2+F_3+F_4+F_5+F_6.
\end{aligned}
$$

Using the method to estimate E_{44}, we get

$$
F_1\le CK^2_{B,Q}||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.28}
$$

Let us estimate F_2. At first, we compute

$$
\begin{aligned}
&T((b_1-m_Qb_1)f_1\chi_{X\setminus 6^NB},(b_2-m_Qb_2)f_2\chi_{X\setminus 6^NB})(z)\\
&\quad-T((b_1-m_{\widetilde{B}}b_1)f_1\chi_{X\setminus 6^NB},(b_2-m_{\widetilde{B}}b_2)f_2\chi_{X\setminus 6^NB})(z)\\
=&(m_Qb_2-m_{\widetilde{B}}b_2)T((b_1-m_Qb_1)f_1\chi_{X\setminus 6^NB},f_2\chi_{X\setminus 6^NB})(z)\\
&+(m_Qb_1-m_{\widetilde{B}}b_1)T(f_1\chi_{X\setminus 6^NB},(b_2-m_Qb_2)f_2\chi_{X\setminus 6^NB})(z)\\
&+(m_Qb_1-m_{\widetilde{B}}b_1)(m_Qb_2-m_{\widetilde{B}}b_2)T(f_1\chi_{X\setminus 6^NB},f_2\chi_{X\setminus 6^NB})(z).
\end{aligned}\tag{2.29}
$$

Hence

(2.30)

$$
\begin{aligned}
F_2\le&\Big|(m_Qb_2-m_{\widetilde{B}}b_2)\frac{1}{\mu(Q)}\int_Q T((b_1-m_Qb_1)f_1\chi_{X\setminus 6^NB},f_2\chi_{X\setminus 6^NB})(z)d\mu(z)\Big|\\
&+\Big|(m_Qb_1-m_{\widetilde{B}}b_1)\frac{1}{\mu(Q)}\int_Q T((f_1\chi_{X\setminus 6^NB},(b_2-m_Qb_2)f_2\chi_{X\setminus 6^NB})(z)d\mu(z)\Big|\\
&+\Big|(m_Qb_1-m_{\widetilde{B}}b_1)(m_Qb_2-m_{\widetilde{B}}b_2)\frac{1}{\mu(Q)}\int_Q T(f_1\chi_{X\setminus 6^NB},f_2\chi_{X\setminus 6^NB})(z)d\mu(z)\Big|\\
=:&F_{21}+F_{22}+F_{23}.
\end{aligned}
$$

To estimate F_{21}, for $\frac{6}{5}Q\subset 6^NB$, we write

$$
\begin{aligned}
&T((b_1-m_Qb_1)f_1\chi_{X\setminus 6^NB},f_2\chi_{X\setminus 6^NB})(z)\\
=&T((b_1-m_Qb_1)f_1,f_2)(z)-T((b_1-m_Qb_1)f_1\chi_{6^NB}\chi_{\frac{6}{5}Q},f_2\chi_{\frac{6}{5}Q})(z))\\
&-T((b_1-m_Qb_1)f_1\chi_{X\setminus\frac{6}{5}Q},f_2\chi_{6^NB})(z)\\
&-T((b_1-m_Qb_1)f_1\chi_{6^NB},f_2\chi_{X\setminus\frac{6}{5}Q})(z)\\
&+T((b_1-m_Qb_1)f_1\chi_{6^NB\setminus\frac{6}{5}Q},f_2\chi_{6^NB\setminus\frac{6}{5}Q})(z)\\
=:&H_1(z)+H_2(z)+H_3(z)+H_4(z)+H_5(z).
\end{aligned}\tag{2.31}
$$

Let us estimate $H_1(z)$ firstly. Since

$$
\frac{1}{\mu(Q)}\int_Q|T(b_1-b_1(z)f_1,f_2)(z)|d\mu(z)\le CM_{r,(6)}([b_1,T]f_1,f_2)(x)
$$

and by Hölder's inequality, we have

$$\frac{1}{\mu(Q)}\int_Q |(b_1(z)-m_Q(b_1))T(f_1,f_2)(z)|d\mu(z) \le C||b_1||_* M_{r,(6)}(T(f_1,f_2))(x),$$

then we obtain

$$\begin{aligned} |m_Q(H_1)| &\le |m_Q(T(b_1-b_1(z)f_1,f_2))| + |m_Q((b_1(z)-m_Q(b_1))T(f_1,f_2))| \\ &\le CM_{r,(6)}([b_1,T]f_1,f_2)(x) + C||b_1||_* M_{r,(6)}(T(f_1,f_2))(x). \end{aligned} \tag{2.32}$$

For $H_2(z)$, let $r > 1$ and $1 < s_1 < p_1$ such that $\frac{1}{r} = \frac{1}{s_1} + \frac{1}{p_2}$. Denote $\frac{1}{s_2} = \frac{1}{s_1} - \frac{1}{p_1}$, using the fact of Q is a doubling ball, Kolmogorov's inequality, Hölder's inequality and Lemma 2.4, we have

(2.33)

$$\begin{aligned} |m_Q(H_2)| \le& C||H_2||_{L^{r,\infty}(Q,\frac{d\mu(z)}{\mu(Q)})} \\ \le& C\left(\frac{1}{\mu(Q)}\int_{\frac{6}{5}Q} |(b_1-m_Qb_1)f_1|^{s_1}d\mu(z)\right)^{1/s_1}\left(\frac{1}{\mu(Q)}\int_{\frac{6}{5}Q}|f_2|^{p_2}d\mu(z)\right)^{1/p_2} \\ \le& C\left(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|b_1-m_Qb_1|^{s_2}d\mu(z)\right)^{1/s_2}\left(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|f_1|^{p_1}d\mu(z)\right)^{1/p_1} \\ &\times\left(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|f_2|^{p_2}d\mu(z)\right)^{1/p_2} \\ \le& C||b_1||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \end{aligned}$$

For H_3, since $z \in Q$, by (i) of Definition 1.7, Lemma 2.1, Lemma 2.2, Hölder's inequality and Q is a doubling ball, we deduce

$$\begin{aligned} |H_3(z)| &\le C\int_{6^NB}\int_{X\setminus\frac{6}{5}Q}\frac{|b_1(y_1)-m_Qb_1||f_1(y_1)||f_2(y_2)|d\mu(y_1)d\mu(y_2)}{[\sum_{j=1}^2\lambda(z,d(z,y_j))]^2} \\ &\le C\int_{6^NB}|f_2(y_2)|d\mu(y_2)\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}Q}\frac{|b_1(y_1)-m_Qb_1||f_1(y_1)|}{(\lambda(z,6^{k-1}\frac{6}{5}r_Q))^2}d\mu(y_1) \\ &\le C\int_{6^NB}|f_2(y_2)|d\mu(y_2)\sum_{k=1}^{\infty}6^{-km} \\ &\quad\times\int_{6^k\frac{6}{5}Q}\frac{1}{\lambda(z,\frac{6}{5}r_Q)}\frac{|b_1(y_1)-m_Qb_1||f_1(y_1)|d\mu(y_1)}{\lambda(z,6^{k-1}\frac{6}{5}r_Q)} \\ &\le C\frac{1}{\lambda(z,6r_Q)}\int_{6^NB}|f_2(y_2)|d\mu(y_2)\sum_{k=1}^{\infty}6^{-km}\frac{1}{\lambda(z,5\times 6^k\frac{6}{5}r_Q)} \end{aligned} \tag{2.34}$$

$$
\begin{aligned}
&\times \left[\int_{6^k\frac{6}{5}Q} |b_1(y_1) - m_{\widetilde{6^k\frac{6}{5}Q}}(b_1)||f_1(y_1)|d\mu(y_1)\right.\\
&\left.+ \int_{6^k\frac{6}{5}Q} |m_{\widetilde{6^k\frac{6}{5}Q}}(b_1) - m_Q b_1||f_1(y_1)|d\mu(y_1)\right]\\
\le\; & C\frac{1}{\lambda(z,6r_Q)}\int_{6^N B}|f_2(y_2)|d\mu(y_2)\sum_{k=1}^{\infty}6^{-km}\\
&\times\left[\left(\frac{1}{\lambda(z,5\times 6^k\frac{6}{5}r_Q)}\int_{6^k\frac{6}{5}Q}|b_1(y_1)-m_{\widetilde{6^k\frac{6}{5}Q}}(b_1)|^{p_1'}d\mu(y_1)\right)^{1/p_1'}\right.\\
&\times(\frac{1}{\lambda(z,5\times 6^k\frac{6}{5}r_Q)}\int_{6^k\frac{6}{5}Q}|f_1(y_1)|^{p_1}d\mu(y_1))^{1/p_1}\\
&\left.+ k||b_1||_*\frac{1}{\lambda(z,5\times 6^k\frac{6}{5}r_Q)}\int_{6^k\frac{6}{5}Q}|f_1(y_1)|d\mu(y_1)\right]\\
\le\; & C\frac{1}{\lambda(z,6r_Q)}\int_{6^N B}|f_2(y_2)|d\mu(y_2)||b_1||_* M_{p_1,(5)}f_1(x)\\
\le\; & C||b_1||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).
\end{aligned}
$$

Then it yields

$$
|m_Q(H_3)| \le C||b_1||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.35}
$$

In the similar way to estimate $m_Q(H_2)$, we also obtain

$$
|m_Q(H_4)| + |m_Q(H_5)| \le C||b_1||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.36}
$$

From (2.8) in Lemma 2.2, we deduce

$$
\begin{aligned}
F_{21} \le & CK_{B,Q}\{||b_1||_*||b_2||_* M_{r,(6)}(T(f_1,f_2))(x)\\
&+ ||b_1||_* M_{r,(6)}([b_2,T](f_1,f_2))(x)\\
&+ ||b_2||_* M_{r,(6)}([b_1,T](f_1,f_2))(x)\\
&+ ||b_1||_*||b_2||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)\}.
\end{aligned} \tag{2.37}
$$

F_{22} and F_{23} also have similar estimate of F_{21}, therefore,

$$
\begin{aligned}
F_2 \le & CK_{B,Q}\Big\{||b_1||_*||b_2||_* M_{r,(6)}(T(f_1,f_2))(x)\\
&+ ||b_1||_* M_{r,(6)}([b_2,T](f_1,f_2))(x)\\
&+ ||b_2||_* M_{r,(6)}([b_1,T](f_1,f_2))(x)\\
&+ ||b_1||_*||b_2||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)\Big\}.
\end{aligned} \tag{2.38}
$$

From F_3 to F_6, using the similar method to estimate F_1, we conclude

$$F_3 + F_4 + F_5 + F_6 \le C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x). \tag{2.39}$$

Thus (2.15) holds from (2.27) to (2.39).

Next, let us show how to obtain (2.11) from (2.14) and (2.15). From (2.14), if B is doubling ball and $x \in B$, it follows that

$$\begin{aligned}
&|m_B(|[b_1, b_2, T](f_1, f_2)|^\delta) - |h_B|^\delta|^{1/\delta} \\
\le&\left(\frac{1}{\mu(B)}\int_B |[b_1, b_2, T](f_1, f_2)(z) - h_B|^\delta d\mu(z)\right)^{1/\delta} \\
\le&C||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + C||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\
&+ C||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) \\
&+ C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x).
\end{aligned} \tag{2.40}$$

Also, for any ball $B \ni x$ (B may be non-doubling), $K_{B,\widetilde{B}} \le C$. With the help of (2.14), (2.15) and (2.40), then

$$\begin{aligned}
&\left(\frac{1}{\mu(6B)}\int_B \Big||[b_1, b_2, T](f_1, f_2)(z)|^\delta - m_{\widetilde{B}}(|[b_1, b_2, T](f_1, f_2)|^\delta)\Big| d\mu(z)\right)^{1/\delta} \\
\le&C\left(\frac{1}{\mu(6B)}\int_B ||[b_1, b_2, T](f_1, f_2)(z)|^\delta - |h_B|^\delta| d\mu(z)\right)^{1/\delta} \\
&+C\left(\frac{1}{\mu(6B)}\int_B ||h_{\widetilde{B}}|^\delta - m_{\widetilde{B}}(|T(f_1, f_2)|^\delta)| d\mu(z)\right)^{1/\delta} \\
&+C\left(\frac{1}{\mu(6B)}\int_B ||h_B|^\delta - |h_{\widetilde{B}}|^\delta| d\mu(z)\right)^{1/\delta} \\
\le&C\Big[||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + ||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\
&+ ||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) + ||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x)\Big].
\end{aligned} \tag{2.41}$$

On the other hand, for all doubling balls $B \subset Q$ with $x \in B$ such that $K_{B,Q} \le P_0$, where P_0 is the constant in Lemma 9.3 in [19], by (2.15), we get

$$\begin{aligned}
|h_B - h_Q| \le CP_0^2\Big[&||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) \\
&+ ||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\
&+ ||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) \\
&+ ||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x)\Big].
\end{aligned} \tag{2.42}$$

Therefore, by Lemma 2.7 in [14], it follows that

$$\begin{aligned}|h_B - h_Q| \leq CK_{B,Q}\Big[&||b_1||_*||b_2||_*M_{r,(6)}(T(f_1, f_2))(x)\\ &+ ||b_1||_*M_{r,(6)}([b_2, T](f_1, f_2))(x)\\ &+ ||b_2||_*M_{r,(6)}([b_1, T](f_1, f_2))(x)\\ &+ ||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)\Big].\end{aligned} \tag{2.43}$$

For all doubling balls $B \subset Q$ with $x \in B$ and using (2.40) again, we obtain

$$\begin{aligned}&|m_B(|[b_1, b_2, T](f_1, f_2)|^\delta) - m_Q(|[b_1, b_2, T](f_1, f_2)|^\delta)|^{1/\delta}\\ \leq& C|m_B(|[b_1, b_2, T](f_1, f_2)|^\delta) - |h_B|^\delta|^{1/\delta}\\ &+ C||h_Q|^\delta - m_Q(|[b_1, b_2, T](f_1, f_2)|^\delta)|^{1/\delta} + C||h_B|^\delta - |h_Q|^\delta|^{1/\delta}\\ \leq& C|m_B|[b_1, b_2, T](f_1, f_2)|^\delta - |h_B|^\delta|^{1/\delta}\\ &+ C||h_Q|^\delta - m_Q(|[b_1, b_2, T](f_1, f_2)|^\delta)|^{1/\delta} + C|h_B - h_Q|\\ \leq& CK_{B,Q}\Big[||b_1||_*||b_2||_*M_{r,(6)}(T(f_1, f_2))(x)\\ &+ ||b_1||_*M_{r,(6)}([b_2, T](f_1, f_2))(x)\\ &+ ||b_2||_*M_{r,(6)}([b_1, T](f_1, f_2))(x)\\ &+ ||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)\Big].\end{aligned} \tag{2.44}$$

Thus we obtain (2.11) from (2.41) and (2.44).

With the same method to prove (2.11), we also obtain that (2.12) and (2.13) also hold. Here we omit the details. Thus Lemma 2.5 is proved. ■

Proof of Theorem 1.11. Let $0 < \delta < 1/2$, $1 < p_1, p_2, q < \infty$, $\dfrac{1}{q} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$, $1 < r < q$, $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, $b_1 \in RBMO(\mu)$ and $b_2 \in RBMO(\mu)$. By $|f(x)| \leq N_\delta f(x)$, Lemma 2.1, Lemma 2.4, Lemma 2.5, Hörder's inequality and the boundedness of $M_{(\rho)}$ and $M_{r,(\rho)}$ for $\rho \geq 5$ and $q > r$, we obtain

$$\begin{aligned}&||[b_1, b_2, T](f_1, f_2)||_{L^q(\mu)} \leq ||N_\delta([b_1, b_2, T](f_1, f_2))||_{L^q(\mu)}\\ \leq& C||M^\sharp_\delta([b_1, b_2, T](f_1, f_2))||_{L^q(\mu)}\\ \leq& C||b_1||_*||b_2||_*||M_{r,(6)}(T(f_1, f_2))||_{L^q(\mu)}\\ &+ C||b_1||_*||M_{r,(6)}([b_2, T](f_1, f_2))||_{L^q(\mu)}\\ &+ C||b_2||_*||M_{r,(6)}([b_1, T](f_1, f_2))||_{L^q(\mu)}\\ &+ C||b_1||_*||b_2||_*||M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)||_{L^q(\mu)}\\ &C||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)}\end{aligned} \tag{2.45}$$

$$
\begin{aligned}
&\leq + C||b_1||_*||([b_2, T](f_1, f_2))||_{L^q(\mu)} \\
&\quad + C||b_2||_*||([b_1, T](f_1, f_2))||_{L^q(\mu)} \\
&\leq C||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)} \\
&\quad + C||b_1||_*||M^\sharp_\delta([b_2, T](f_1, f_2))||_{L^q(\mu)} \\
&\quad + C||b_2||_*||M^\sharp_\delta([b_1, T](f_1, f_2))||_{L^q(\mu)} \\
&\leq ||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)} \\
&\quad + C||b_1||_*||b_2||_*||M_{r,(6)}(T(f_1, f_2))(x)||_{L^q(\mu)} \\
&\quad + C||b_1||_*||b_2||_*||M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)||_{L^q(\mu)} \\
&\leq C||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)}.
\end{aligned}
$$

Thus the proof of Theorem 1.11 is completed. ■

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