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GENERAL DECAY OF SOLUTIONS FOR A VISCOELASTIC EQUATION WITH BALAKRISHNAN-TAYLOR DAMPING

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Abstract. A viscoelastic equation with Balakrishnan-Taylor damping and nonlinear boundary/interior sources is considered in a bounded domain. Under appropriate assumptions on the relaxation function and with certain initial data and by adopting the perturbed energy method, we establish uniform decay rate of the solution energy in terms of the behavior of the relaxation function, which are not necessarily of exponential or polynomial decay.

1. INTRODUCTION

In this paper, we study the following viscoelastic problem with Balakrishnan-Taylor damping and nonlinear boundary/interior sources:

$$\begin{aligned}
&u_{tt}(t) - M(t)\Delta u(t) + \int_0^t g(t-s)\Delta u(s)ds \\
&\quad = |u|^{p-1}u, \text{ in } \Omega\times(0,\infty), \\
&u = 0, \text{ on } \Gamma_0\times(0,\infty), M(t)\frac{\partial u}{\partial\nu} - \int_0^t g(t-s)\frac{\partial}{\partial\nu}u(s)ds + \alpha u_t \\
&\quad = |u|^{k-1}u, \text{ on } \Gamma_1\times(0,\infty) \\
&u(0) = u_0(x),\ u_t(0) = u_1(x),\ x\in\Omega,
\end{aligned} \tag{1.1}$$

where $M(t) = a + b\|\nabla u\|_2^2 + \sigma\int_\Omega \nabla u\cdot\nabla u_t dx$, $a>0$, $b>0$, $\sigma>0$, $\alpha>0$ and Ω is a bounded domain in R^n $(n\geq 1)$ with smooth boundary $\Gamma = \Gamma_0\cup\Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint with $meas(\Gamma_0)>0$, and ν is the unit outward normal to Γ. The relaxation function g is a positive and uniformly decaying function and

$$\begin{aligned}
&1\leq p\leq \frac{n}{n-2},\ n>2 \text{ and } 1\leq p<\infty,\ \text{if } n=2, \\
&1\leq k<\frac{n-1}{n-2},\ n>2 \text{ and } 1\leq k<\infty,\ \text{if } n=2.
\end{aligned} \tag{1.2}$$

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The equations in (1.1) with $M \equiv 1$ form a class of nonlinear viscoelastic equations used to investigate the motion of viscoelastic materials. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. Hence, questions related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years. For example, Cavalcanti et al. [6] considered the following problem:

$$
\begin{aligned}
&u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)ds = 0, \text{ in } \Omega \times (0,\infty), \\
&u = 0, \text{ on } \Gamma_0 \times (0,\infty), \\
&\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(s)ds + h(u_t) = 0, \text{ on } \Gamma_1 \times (0,\infty) \\
&u(0) = u_0(x),\ u_t(0) = u_1(x),\ x \in \Omega.
\end{aligned}
\tag{1.3}
$$

They showed the global existence of solutions and established some uniform decay results under quite restrictive assumptions on both the damping function h and the relaxation function g. Later, Cavalcanti et al. [5] generalized the result without imposing a growth condition on h and under a weaker assumption on g. Recently, Messaoudi and Mustafa [11] exploited some properties of convex functions [1] and the multiplier method to extend these results. They established an explicit and general decay rate result without imposing any restrictive growth assumption on damping term h and greatly weakened the assumption on g.

In the absence of Balakrishnan-Taylor damping $(\sigma = 0)$, equation $(1.1)_1$ is the model to describe the motion of deformable solids as hereditary effect is incorporated, which was first studied by Torrejon and Yong [15]. They proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera [13] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions.

Conversely, in the presence of Balakrishnan-Taylor damping $(\sigma \neq 0)$ and $g = 0$, equation $(1.1)_1$ is used to study the flutter panel equation and to the spillover problem, which was initially proposed by Balakrishnan and Taylor in 1989 [2], and Bass and Zes [3]. The related problems also concerned by You [17], Clark [8], Tatar and Zarai $[14, 18]$ and Mu et al. [12]. Recently, Zarai et al. [19] considered the following flutter equation with memory term:

$$
\begin{aligned}
&u_{tt}(t) - M(t)\Delta u(t) + \int_0^t g(t-s)\Delta u(s)ds = 0, \text{ in } \Omega \times (0,\infty), \\
&u = 0, \text{ on } \Gamma_0 \times (0,\infty), \\
&M(t)\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(s)ds + \alpha u_t = |u|^{k-2}\, u, \text{ on } \Gamma_1 \times (0,\infty) \\
&u(0) = u_0(x),\ u_t(0) = u_1(x),\ x \in \Omega,
\end{aligned}
$$

which arises in a wind tunnel experiment for a panel at supersonic speeds. They proved the global existence of solutions and a general decay result for the energy by using the multiplier technique..

Motivated by previous works, it is interesting to investigate the uniform decay result of solutions to problem (1.1) with two nonlinear source terms (boundary and interior). Indeed, we show in this study that the decay rate of the solution energy is similar to the relaxation function, which is not necessarily decaying in a polynomial or exponential fashion. Our proof technique closely follows the arguments of $[12, 16]$, with some modifications being needed for our problem. The remainder of this paper is organized as follows. In section 2, we give some notations and assumptions and state the local existence result Theorem 2.1. In section 3, we prove our stability result that is given in Theorem 3.7.

2. Preliminaries Results

In this section, we give assumptions and preliminaries that will be needed throughout the paper. First, we introduce the set

$$H^1_{\Gamma_0} = \left\{ u \in H^1(\Omega) : u\,|_{\Gamma_0} = 0 \right\},$$

and endow $H^1_{\Gamma_0}$ with the Hilbert structure induced by $H^1(\Omega)$, we have that $H^1_{\Gamma_0}$ is a Hilbert space. For simplicity, we denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{q,\Gamma_1} = \|\cdot\|_{L^q(\Gamma_1)}$, $1 \le q \le \infty$. According to (1.2), we have the imbedding : $H^1_{\Gamma_0} \hookrightarrow L^{p+1}(\Omega)$. Let $c_* > 0$ be the optimal constant of Sobolev imbedding which satisfies the inequality

$$\|u\|_{p+1} \le c_* \|\nabla u\|_2, \ \forall u \in H^1_{\Gamma_0}, \tag{2.1}$$

and we use the trace-Sobolev imbedding: $H^1_{\Gamma_0} \hookrightarrow L^{k+1}(\Gamma_1)$, $1 \le k < \frac{n}{n-2}$. In this case, the imbedding constant is denoted by B_*, i.e.

$$\|u\|_{k+1,\Gamma_1} \le B_* \|\nabla u\|_2 . \tag{2.2}$$

Next, we state the assumptions for problem (1.1):

(A1) $g : [0,\infty) \to (0,\infty)$ is a bounded C^1 function satisfying

$$g(0) > 0, \ a - \int_0^\infty g(s)ds = l > 0, \tag{2.3}$$

and there exists a non-increasing positive differentiable function ξ such that

$$g'(t) \le -\xi(t)g(t), \tag{2.4}$$

for all $t \ge 0$.

Now, we state the local existence result, which can be obtained by $[4, 7, 9, 16]$.

Theorem 2.1. *Let the initial data* $\{u_0, u_1\} \in H^1_{\Gamma_0} \cap H^2(\Omega) \times H^1_{\Gamma_0}$. *Suppose that the hypotheses (A1) and (1.2) hold. Then there exists a regular solution* u *of the problem* (1.1) *satisfying*

$$u \in L^\infty\left([0,T); H^1_{\Gamma_0} \cap H^2(\Omega)\right),\ u_t \in L^\infty\left([0,T); H^1_{\Gamma_0}\right),\ u_{tt} \in L^\infty\left([0,T); L^2(\Omega)\right)$$

for some $T > 0$.

3. Uniform Decay

In this section, we prove decay rate estimates for problem (1.1). For this purpose, the energy associated with problem (1.1) is given by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(u(t)),\ \text{for } u \in H^1_{\Gamma_0}, \tag{3.1}$$

where

$$\begin{aligned} J(u(t)) &= \frac{1}{2}\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{b}{4}\|\nabla u(t)\|_2^4 \\ &+ \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} - \frac{1}{k+1}\|u(t)\|_{k+1,\Gamma_1}^{k+1}, \end{aligned} \tag{3.2}$$

and

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2\, ds.$$

Adopting the proof of [11], we still have the following result.

Lemma 3.1. *Let* u *be the solution of* (1.1), *then,* $E(t)$ *is a non-increasing function on* $[0,T)$ *and*

$$\begin{aligned} E'(t) &= -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 - \alpha\|u_t\|_{2,\Gamma_1}^2 \\ &+ \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0. \end{aligned} \tag{3.3}$$

Next, we define a functional F introduced by Cavalcanti et al. [4], which helps in establishing desired results. Setting

$$F(x) = \frac{1}{2}x^2 - \frac{B_\Omega^{p+1}}{p+1}x^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}x^{k+1},\ x > 0, \tag{3.4}$$

where

(3.5) $$B_\Omega = \sup_{\substack{u\in H^1_{\Gamma_0},\\ u\neq 0}} \frac{\|u\|_{p+1}}{\sqrt{\left(l+\frac{b}{2}\|\nabla u\|_2^2\right)\|\nabla u\|_2^2}} \text{ and } B_\Gamma = \sup_{\substack{u\in H^1_{\Gamma_0},\\ u\neq 0}} \frac{\|u\|_{k+1,\Gamma_1}}{\sqrt{\left(l+\frac{b}{2}\|\nabla u\|_2^2\right)\|\nabla u\|_2^2}}.$$

Remark 3.2. (i) As in [4], we can verify that the functional F is increasing in $(0,\lambda_0)$, decreasing in (λ_0,∞), and F has a maximum at λ_0 with the maximum value

(3.6) $$\begin{aligned} d &\equiv F(\lambda_0)\\ &= \frac{1}{2}\lambda_0^2 - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1}, \end{aligned}$$

where λ_0 is the first positive zero of the derivative function $F'(x)$.

(ii) From (3.1), (3.2), (1.2), (2.3) and the definition of F, we have

(3.7) $$\begin{aligned} E(t) \geq J(u(t)) &\geq \frac{1}{2}\gamma^2(t) - \frac{B_\Omega^{p+1}}{p+1}\gamma^{p+1}(t) - \frac{B_\Gamma^{k+1}}{k+1}\gamma^{k+1}(t)\\ &= F(\gamma(t)),\ t\geq 0, \end{aligned}$$

where

$$\gamma^2(t) = l\|\nabla u(t)\|_2^2 + \frac{b}{2}\|\nabla u(t)\|_2^4 + (g\circ\nabla u)(t).$$

Now, if one considers $\gamma(t)<\lambda_0$, then, from (3.7), we get

$$E(t)\geq F(\gamma(t)) > \gamma^2(t)\left(\frac{1}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p-1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k-1}\right),\ t\geq 0,$$

which together with the identity

(3.8) $$1 - B_\Omega^{p+1}\lambda_0^{p-1} - B_\Gamma^{k+1}\lambda_0^{k-1} = 0$$

give

(3.9) $$E(t)\geq F(\gamma(t)) \geq c_0\gamma^2(t),$$

where $c_0 = \begin{cases} \frac{p-1}{2(p+1)}, & \text{if } k\geq p,\\ \frac{k-1}{2(k+1)}, & \text{if } p\geq k. \end{cases}$ and the identity (3.8) is derived because λ_0 is the first positive zero of the derivative function $F'(x)$.

Lemma 3.3. *Let $u_0\in H^1_{\Gamma_0}, u_1\in L^2(\Omega)$ and the hypotheses (A1) and (1.2) hold. Assume further that $\gamma(0)=\sqrt{l\|\nabla u_0\|_2^2+\frac{b}{2}\|\nabla u_0\|_2^4}<\lambda_0$ and $E(0)<d$. Then,*

(3.10) $$\gamma(t) = \sqrt{l\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u(t)\|_2^4 + (g\circ\nabla u)(t)} < \lambda_0,$$

for all $t \in [0,T)$.

Proof. Using (3.7) and considering $E(t)$ is a non-increasing function, we obtain

$$F(\gamma(t)) \le E(t) \le E(0) < d,\ t \in [0,T). \tag{3.11}$$

In addition, from Remark 3.2 (i), we see that F is increasing in $(0,\lambda_0)$, decreasing in (λ_0,∞) and $F(\lambda) \to -\infty$ as $\lambda \to \infty$. Thus, as $E(0) < d$, there exist $\lambda_2' < \lambda_0 < \lambda_2$ such that $F(\lambda_2') = F(\lambda_2) = E(0)$. Besides, through the assumption $\gamma(0) < \lambda_0$, we observe that

$$F(\gamma(0)) \le E(0) = F\left(\lambda_2'\right).$$

This implies that $\gamma(0) \le \lambda_2'$. Next, we will prove that

$$\gamma(t) \le \lambda_2',\ t \in [0,T). \tag{3.12}$$

To establish (3.12), we argue by contradiction. Suppose that (3.12) does not hold, then there exists $t^* \in (0,T)$ such that $\gamma(t^*) > \lambda_2'$.

Case 1. If $\lambda_2' < \gamma(t^*) < \lambda_0$, then

$$F(\gamma(t^*)) > F(\lambda_2') = E(0) \ge E(t^*).$$

This contradicts (3.11).

Case 2. If $\gamma(t^*) \ge \lambda_0$, then by continuity of $\gamma(t)$, there exists $0 < t_1 < t^*$ such that

$$\lambda_2' < \gamma(t_1) < \lambda_0,$$

then

$$F(\gamma(t_1)) > F(\lambda_2') = E(0) \ge E(t_1).$$

This is also a contradiction of (3.11). Thus, we have proved (3.10). ■

Theorem 3.4. *Let* $u_0 \in H^1_{\Gamma_0}, u_1 \in L^2(\Omega)$ *and* $(A1)$ *and* (1.2) *hold. Assume further that* $\gamma(0) < \lambda_0$ *and* $E(0) < d$, *then the problem* (1.1) *admits a global solution.*

Proof. It follows from (3.10), (3.9) and (3.7) that

$$\begin{aligned}\frac{1}{2}\|u_t\|_2^2 + c_0\gamma^2(t) \le \frac{1}{2}\|u_t\|_2^2 + F(\gamma(t)) \le \frac{1}{2}\|u_t\|_2^2 + J(u(t))\\ = E(t) \le E(0) < d.\end{aligned} \tag{3.13}$$

Thus, we establish the boundedness of u_t in $L^2(\Omega)$ and the boundedness of u in $H^1_{\Gamma_0}$. Moreover, from (2.1), (2.2) and (3.13), we also obtain

$$\begin{aligned}\|u\|_{p+1}^{p+1}+\|u\|_{k+1,\Gamma_1}^{k+1} &\le c_*^{p+1}\|\nabla u\|_2^{p+1}+B_*^{k+1}\|\nabla u\|_2^{k+1}\\ &\le \frac{1}{l}\left(c_*^{p+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{p-1}{2}}+B_*^{k+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{k-1}{2}}\right)l\|\nabla u\|_2^2\\ &\le L\gamma^2(t),\end{aligned}$$

which implies that the boundedness of u in $L^{p+1}(\Omega)$ and in $L^{k+1}(\Gamma_1)$ with $L=\frac{1}{l}\left(c_*^{p+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{p-1}{2}}+B_*^{k+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{k-1}{2}}\right)$. Hence, it must have $T=\infty$. qed

Now, we shall investigate the asymptotic behavior of the energy function $E(t)$. First, we define some functionals and establish several lemmas. Let

(3.14) $$G(t)=ME(t)+\varepsilon\Phi(t)+\Psi(t),$$

where

(3.15) $$\Phi(t)=\int_\Omega u_t u\,dx+\frac{\sigma}{4}\|\nabla u\|_2^4$$

(3.16) $$\Psi(t)=\int_\Omega u_t\int_0^t g(t-s)\left(u(s)-u(t)\right)ds\,dx,$$

and M, ε are some positive constants to be be specified later.

Lemma 3.5. *There exist two positive constants β_1 and β_2 such that the relation*

(3.17) $$\beta_1 E(t)\le G(t)\le \beta_2 E(t)$$

holds, for $\varepsilon>0$ small enough while $M>0$ is large enough.

Proof. By Hölder's inequality, Young's inequality, (2.1) and (2.3), we deduce that

$$\begin{aligned}|G(t)-ME(t)| &\le \varepsilon|\Phi(t)|+|\Psi(t)|\\ &\le \frac{\varepsilon+1}{2}\|u_t\|^2+\frac{\varepsilon c_*^2}{2}\|\nabla u\|_2^2+\frac{\varepsilon\sigma}{4}\|\nabla u\|_2^4\\ &\quad+\int_\Omega\left(\int_0^t g(t-s)\left(u(s)-u(t)\right)ds\right)^2dx\\ &\le \frac{\varepsilon+1}{2}\|u_t\|^2+\frac{\varepsilon c_*^2}{2}\|\nabla u\|_2^2+\frac{\varepsilon\sigma}{4}\|\nabla u\|_2^4+\frac{c_*^2(a-l)}{2}\left(g\circ\nabla u\right)(t)\\ &\le c_1\left(\frac{1}{2}\|u_t\|_2^2+c_0\left(l\|\nabla u\|_2^2+\left(g\circ\nabla u\right)(t)+\frac{b}{2}\|\nabla u\|_2^4\right)\right),\end{aligned}$$

where $c_1 = \max(\varepsilon+1, \frac{\varepsilon c_*^2}{2c_0 l}, \frac{c_*^2(a-l)}{2c_0}, \frac{\varepsilon\sigma}{2bc_0})$. Employing (3.13) and selecting $\varepsilon > 0$ small enough and M sufficiently large, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq G(t) \leq \beta_2 E(t). \qquad \blacksquare$$

Lemma 3.6. *Let $(A1)$ and (1.2) hold, then, for any $t_0 > 0$, the functional $G(t)$ verifies, along solution of (1.1),*

$$G'(t) \leq -c_6 E(t) + c_7 (g \circ \nabla u)(t) - c_5 E(0) E'(t), \tag{3.18}$$

where c_i, $i = 5, 6, 7$ are some positive constants given in the proof.

Proof. In the following, we estimate the derivative of $G(t)$. From (3.15) and (1.1), we have

$$\begin{aligned} \Phi'(t) = &\|u_t\|_2^2 - \left(a + b\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla u(s)dsdx \\ &- \alpha \int_{\Gamma_1} u_t u d\Gamma + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1} \end{aligned} \tag{3.19}$$

Utilizing Hölder's inequality, Young's inequality, (2.2) and (2.3), the third and fourth terms on the right-hand side of (3.19) can be estimated as follows, for η, $\delta > 0$,

$$\int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla u(s)dsdx \leq (\eta + a - l)\|\nabla u\|_2^2 + \frac{a-l}{4\eta}(g \circ \nabla u)(t), \tag{3.20}$$

and

$$\left|\alpha \int_{\Gamma_1} u_t u d\Gamma\right| \leq \delta\alpha B_*^2 \|\nabla u\|_2^2 + \frac{\alpha}{4\delta}\|u_t\|_{2,\Gamma_1}^2. \tag{3.21}$$

A substitution of (3.20)-(3.21) into (3.19) yields

$$\begin{aligned} \Phi'(t) \leq &\|u_t\|_2^2 - \left(-\eta + l - \delta\alpha B_*^2\right)\|\nabla u\|_2^2 + \frac{a-l}{4\eta}(g \circ \nabla u)(t) + \frac{\alpha}{4\delta}\|u_t\|_{2,\Gamma_1}^2 \\ &+ \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}. \end{aligned}$$

Letting $\eta = \frac{l}{2} > 0$ and $\delta = \frac{l}{8\alpha B_*^2}$ in above inequality, we obtain

$$\begin{aligned} \Phi'(t) \leq &\|u_t\|_2^2 - \frac{l}{4}\|\nabla u\|_2^2 + \frac{a-l}{2l}(g \circ \nabla u)(t) + \frac{2\alpha B_*^2}{l}\|u_t\|_{2,\Gamma_1}^2 \\ &+ \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}. \end{aligned} \tag{3.22}$$

Next, we estimate $\Psi'(t)$. Taking the derivative of $\Psi(t)$ in (3.16) and using (1.1), we obtain

$$
\begin{aligned}
\Psi'(t) = & \int_\Omega \left(a + b \|\nabla u\|_2^2\right) \nabla u(t) \int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds dx \\
& + \int_\Omega \sigma \left(\int_\Omega \nabla u \nabla u_t dx\right) \nabla u(t) \int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds dx \\
& - \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) ds\right) \left(\int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds\right) dx \\
& + \alpha \int_{\Gamma_1} u_t \int_0^t g(t-s) \left(u(t) - u(s)\right) ds d\Gamma \\
& - \int_{\Gamma_1} |u|^{k-1} u \int_0^t g(t-s) \left(u(t) - u(s)\right) ds d\Gamma \\
& - \int_\Omega |u|^{p-1} u \int_0^t g(t-s) \left(u(t) - u(s)\right) ds dx \\
& - \int_\Omega u_t \int_0^t g'(t-s) \left(u(t) - u(s)\right) ds dx - \left(\int_0^t g(s) ds\right) \|u_t\|_2^2 .
\end{aligned} \tag{3.23}
$$

Similar to deriving (3.22), in what follows we will estimate the right-hand side of (3.23). Using Young's inequality, Hölder's inequality, $l \|\nabla u\|_2^2 \leq \frac{E(t)}{c_0} \leq \frac{E(0)}{c_0}$ by (3.13), $E'(t) \leq -\sigma \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2\right)^2$ by (3.3), (2.3) and (2.2), we have, for $\delta > 0$,

$$
\begin{aligned}
& \left| \int_\Omega \left(a + b \|\nabla u\|_2^2\right) \nabla u(t) \int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds dx \right| \\
& \leq \left| \int_\Omega \left(a + \frac{b}{c_0 l} E(0)\right) \nabla u(t) \int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds dx \right| \\
& \leq \delta \|\nabla u\|_2^2 + \frac{a-l}{4\delta} \left(a + \frac{b}{c_0 l} E(0)\right)^2 (g \circ \nabla u)(t),
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
& \left| \int_\Omega \sigma \left(\int_\Omega \nabla u \nabla u_t dx\right) \nabla u(t) \int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds dx \right| \\
& \leq \sigma^2 \left(\int_\Omega \nabla u \nabla u_t dx\right)^2 l \|\nabla u\|_2^2 + \frac{1}{4l} \int_\Omega \left(\int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds\right)^2 dx \\
& \leq \frac{-\sigma}{c_0} E(0) E'(t) + \frac{a-l}{4l} (g \circ \nabla u)(t),
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
& \left| \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) ds\right) \left(\int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s)\right) ds\right) dx \right| \\
& \leq 2\delta (a-l)^2 \|\nabla u\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right) (a-l) (g \circ \nabla u)(t),
\end{aligned} \tag{3.26}
$$

and

$$\left|\alpha \int_{\Gamma_1} u_t \int_0^t g(t-s)\left(u(t)-u(s)\right) ds d\Gamma\right| \tag{3.27}$$
$$\leq \frac{\alpha}{2}\left\|u_t\right\|_{2,\Gamma_1}^2 + \frac{(a-l)\alpha B_*^2}{2}\left(g \circ \nabla u\right)(t)$$

As for the the fifth and sixth terms on the right-hand side of (3.23), using Hölder's inequality, Young's inequality, (2.1)-(2.3) and (3.13), we obtain

$$\begin{aligned} &\int_{\Gamma_1} |u|^{k-1} u \int_0^t g(t-s)\left(u(t)-u(s)\right) ds d\Gamma \\ &\leq \delta \left\|u\right\|_{2k,\Gamma_1}^{2k} + \frac{(a-l)B_*^2}{4\delta}\left(g \circ \nabla u\right)(t) \\ &\leq \delta B_*^{2k}\left(\frac{E(0)}{lc_0}\right)^{k-1}\left\|\nabla u\right\|_2^2 + \frac{(a-l)B_*^2}{4\delta}\left(g \circ \nabla u\right)(t), \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\left|\int_{\Omega} |u|^{p-1} u \int_0^t g(t-s)\left(u(t)-u(s)\right) ds dx\right| \\ &\leq \delta \left\|u\right\|_{2p}^{2p} + \frac{(a-l)c_*^2}{4\delta}\left(g \circ \nabla u\right)(t) \\ &\leq \delta c_*^{2p}\left(\frac{E(0)}{lc_0}\right)^{p-1}\left\|\nabla u\right\|_2^2 + \frac{(a-l)c_*^2}{4\delta}\left(g \circ \nabla u\right)(t). \end{aligned} \tag{3.29}$$

Exploiting Hölder's inequality, Young's inequality and (A1) to estimate the seventh term, we have

$$\left|\int_{\Omega} u_t \int_0^t g'(t-s)\left(u(t)-u(s)\right) ds dx\right| \leq \delta \left\|u_t\right\|_2^2 - \frac{g(0)c_*^2}{4\delta}\left(g' \circ \nabla u\right)(t). \tag{3.30}$$

Then, combining these estimates (3.24)-(3.30), (3.23) becomes

$$\begin{aligned} \Psi'(t) \leq &-\left(\int_0^t g(s)ds - \delta\right)\left\|u_t\right\|_2^2 + c_2\delta\left\|\nabla u\right\|_2^2 + c_3\left(g \circ \nabla u\right)(t) \\ &+\frac{\alpha}{2}\left\|u_t\right\|_{2,\Gamma_1}^2 - \frac{g(0)c_*^2}{4\delta}\left(g' \circ \nabla u\right)(t) - \frac{\sigma}{c_0}E(0)E'(t), \end{aligned} \tag{3.31}$$

where $c_2 = 1 + 2(a-l)^2 + c_*^{2p}\left(\frac{E(0)}{lc_0}\right)^{p-1} + B_*^{2k}\left(\frac{E(0)}{lc_0}\right)^{k-1}$ and $c_3 = (a-l)$ $\left(\frac{1+\left(a+\frac{b}{c_0 l}E(0)\right)^2}{4\delta} + 2\delta + \frac{1}{4l} + \frac{\alpha B_*^2}{2} + \frac{c_*^2+B_*^2}{4\delta}\right)$.

Thus, we conclude from (3.14), (3.3), (3.22), and (3.31) that

$$
\begin{aligned}
& G'(t) \\
= & ME'(t) + \varepsilon\Phi'(t) + \Psi'(t) \\
\leq & -\left(\frac{M}{2} - \frac{g(0)c_*^2}{4\delta}\right)\left(-g' \circ \nabla u\right)(t) - (g_0 - \delta - \varepsilon)\|u_t\|_2^2 + \left(c_2\delta - \frac{\varepsilon l}{4}\right)\|\nabla u\|_2^2 \\
& - \alpha^2\left(M - \frac{2B_*^2}{l} - \frac{1}{2}\right)\|u_t\|_{2,\Gamma_1}^2 + \left(c_3 + \frac{(a-l)\varepsilon}{2l}\right)(g \circ \nabla u)(t) - \frac{\sigma}{c_0}E(0)E'(t) \\
& + \varepsilon\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right),
\end{aligned}
$$

where we have used the fact that for any $t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0, \ \forall\, t \geq t_0,$$

because g is positive and continuous with $g(0) > 0$. At this point, we choose $\varepsilon > 0$ small enough so that Lemma 3.5 holds and $\varepsilon < \frac{g_0}{2}$. Once ε is fixed, we choose δ to satisfy

$$\delta < \min\left\{\frac{\varepsilon l}{8c_2}, \frac{g_0}{4}\right\},$$

and then pick M sufficiently large such that

$$M > \max\left\{\frac{g(0)c_*^2}{2\delta}, \frac{2B_*^2}{l} + \frac{1}{2}\right\}.$$

Hence, for all $t \geq t_0$, we arrive at

$$
\begin{aligned}
G'(t) \leq & -\frac{\varepsilon l}{8}\|\nabla u\|_2^2 - \frac{g_0}{4}\|u_t\|_2^2 + c_4(g \circ \nabla u)(t) - c_5E(0)E'(t) \\
& + \varepsilon\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right),
\end{aligned}
$$

which yields (if needed, one can choose ε sufficiently small)

$$G'(t) \leq -c_6E(t) + c_7(g \circ \nabla u)(t) - c_5E(0)E'(t), \tag{3.32}$$

where c_i, $i = 5, 6, 7$ are all positive constants. This completes the proof. ∎

Theorem 3.7. *Let $(A1)$ and (1.2) hold. Assume that $u_0 \in H^1_{\Gamma_0}, u_1 \in L^2(\Omega)$, $\gamma(0) < \lambda_0$ and $E(0) < d$. Then, there exist two positive constants K and k such that the solution of (1.1) satisfies*

$$E(t) \leq Ke^{-k\int_{t_0}^t \xi(s)ds}, \text{ for } t \geq 0. \tag{3.33}$$

Proof. It follows from (3.32), (2.4) and (3.3) that

$$\begin{aligned}\xi(t)G'(t) &\leq -c_6\xi(t)E(t) + c_7\xi(t)\left(g \circ \nabla u\right)(t) - c_5E(0)\xi(t)E'(t) \\ &\leq -c_6\xi(t)E(t) - c_7\left(g' \circ \nabla u\right)(t) - c_5E(0)\xi(0)E'(t) \\ &\leq -c_6\xi(t)E(t) - c_8E'(t),\end{aligned}$$

where $c_8 = 2c_7 + c_5E(0)\xi(0)$ and $t \geq t_0$. This infers that

$$L'(t) \leq -c_6\xi(t)E(t) \leq -k\xi(t)L(t), \text{ for } t \geq t_0, \tag{3.34}$$

where $L(t) = \xi(t)G(t)+c_8E(t)$ is equivalent to $E(t)$ by Lemma 3.5 and k is a positive constant. An integration of (3.34) leads to

$$L(t) \leq L(t_0)e^{-k\int_{t_0}^t \xi(s)ds}, \text{ for } t \geq t_0.$$

Again, employing $L(t)$ is equivalent to $E(t)$ leads to

$$E(t) \leq Ke^{-k\int_{t_0}^t \xi(s)ds}, \text{ for } t \geq t_0, \tag{3.35}$$

where K is a positive constant. Thus, (3.33) follows from (3.35) and by virture of continuity and boundedness of $E(t)$. This completes the proof. ∎

Remark 3.8. We illustrate the energy decay rate given by Theorem 3.7 through the following examples which are introduced in [10].

(i) If

$$\xi(t) = \alpha, \ \alpha > 0,$$

then (3.35) gives the exponential decay estimate

$$E(t) \leq Ke^{-k\alpha t}.$$

Similarly, if

$$\xi(t) = \alpha\left(1+t\right)^{-1}, \ \alpha > 0,$$

then we obtain the polynomial decay estimate

$$E(t) \leq K\left(1+t\right)^{-\alpha k}.$$

(ii) If

$$g(t) = \alpha e^{-\alpha_1(\ln(1+t))^\nu},$$

with α, α_1, $\nu > 1$, then (2.4) holds for

$$\xi(t) = \frac{\alpha_1\nu\left(\ln\left(1+t\right)\right)^{\nu-1}}{1+t}.$$

Thus (3.35) gives the estimate

$$E(t) \leq K e^{-k\alpha_1 (\ln(1+t))^{\nu}}.$$

(iii) If

$$g(t) = \frac{\alpha}{(2+t)^{\nu} (\ln(2+t))^{\alpha_1}},$$

where $\alpha > 0$ and $\nu > 1$ and $\alpha_1 \in R$ (or $\nu = 1$ and $\alpha_1 > 1$) . Then for

$$\xi(t) = \frac{\nu (\ln(2+t)) + \alpha_1}{(2+t) (\ln(2+t))^{\alpha_1}},$$

we obtain from (3.35) that

$$E(t) \leq \frac{K}{\left[(2+t)^{\nu} (\ln(2+t))^{\alpha_1}\right]^{k}}.$$

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