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UPPER BOUNDS FOR THE FIRST EIGENVALUE OF THE LAPLACE OPERATOR ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. Let M be a complete Riemannian manifold with infinite volume and Ω be a compact subdomain in M. In this paper we obtain two upper bound estimates for the first eigenvalue of the Laplacian on the punctured manifold $M \setminus \Omega$ subject to volume growth and lower bound of Ricci curvature, respectively. The proof hinges on asymptotic behavior of solutions of second order differential equations, the max-min principle and Bishop volume comparison theorem.

1. INTRODUCTION

Let M be an n-dimensional complete Riemannian manifold and Δ be the Laplace operator on M. The first Dirichlet eigenvalue $\lambda_1(D)$ for a compact normal subdomain D in M is the smallest λ that satisfies

$$\Delta u + \lambda u = 0$$

for some nontrivial function u on M with $u|_{\partial D} = 0$. The first Dirichlet eigenvalue is characterized by minimizing the Rayleigh quotient

$$\lambda_1(D) = \inf \frac{\int_D |\nabla u|^2}{\int_D u^2},$$

where the infimum is taken over all nontrivial $u \in H_0^1(D)$. For a compact subset Ω of M, the first eigenvalue of $M \setminus \Omega$ is defined by

$$\lambda_1(M \setminus \Omega) = \inf\{\lambda_1(D) : D \subset M \setminus \Omega \ \text{ is a compact doamin}\}.$$

It is clear that $\lambda_1(M) \leq \lambda_1(M \setminus \Omega)$ for all compact subset Ω. In general, $\lambda_1(M) \neq \lambda_1(M \setminus \Omega)$ (see [3]).

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It is well known that $\lambda_1(R^n) = \lambda_1(R^n \setminus \Omega) = 0$. Cheng and Yau showed that $\lambda_1(M) = 0$ if M has polynomial volume growth (see [2]). Do Carmo and Zhou showed that $\lambda_1(M \setminus \Omega) = 0$ if M has polynomial volume growth, and $\lambda_1(M \setminus \Omega) \leq \frac{a^2}{4}$ if M has exponential volume growth, $\mathrm{Vol}(B(r)) \leq C \exp(a\, r)$ for some positive constants C and a (see [3]). The purpose of the present paper is to study upper bounds for $\lambda_1(M \setminus \Omega)$ under certain circumstances.

We shall now summarize our main results. In Section 2, we apply the technique of [3] to prove that if the second order differential equation $(vx')' + \lambda v x = 0$ is nonoscillatory, where v is a positive continuous function on $[t_0, \infty)$, and λ is a nonnegative constant, then

$$\sqrt{\lambda} \leq \frac{1}{2} \liminf_{t\to\infty} \frac{\log V(t)}{t},$$

where $V(t) = \int_{t_0}^{t} v(s)\, ds$. When $v(t) = \exp(a\, t)$, for some nonnegative a, the estimate is sharp.

In Section 3, we use the preceding result to estimate an upper bound of $\lambda_1(M \setminus \Omega)$ subject to volume growth. We prove that

$$\lambda_1(M \setminus \Omega) \leq \frac{1}{4} (\liminf_{r\to\infty} \frac{\log V(r)}{r})^2,$$

where $V(r)$ is the volume of the geodesic ball $B(r)$ with radius r. In case M has polynomial or exponential volume growth, this was a result of Do Carmo and Zhou [3].

In the last section, we start with a discussion of an asymptotic behavior for solutions of the second order differential equation of the form $y'' - (n-1)\, k^2\, y = 0$, where k is a positive continuous function. This discussion is then applied to estimate an upper bound of $\lambda_1(M \setminus \Omega)$ subject to lower bound of Ricci curvature. We prove that if the Ricci curvature is bounded below by $-\,(n-1)\ k(r)^2$, where k is a nonnegative continuous function, then

$$\lambda_1(M \setminus \Omega) \leq \frac{(n-1)^2}{4} [\limsup_{r\to\infty} k(r)]^2.$$

According to Brooks [1], if M_c is an n-dimensional simply connected complete Riemannian manifold with constant negative sectional curvature $-\,c^2$, then $\lambda_1(M_c) = \frac{(n-1)^2}{4}\ c^2$. Our estimate implies that $\lambda_1(M_c \setminus \Omega) = \frac{(n-1)^2}{4}\ c^2$.

2. Oscillation Theorem

In this section we confine our attention to an oscillation criteria in integral form of the second order ordinary differential equation

$$(v(t)x'(t))' + \lambda v(t)x(t) = 0,\ t \geq t_0, \tag{2.1}$$

where $v(t)$ is a positive continuous function on $[t_0, \infty)$, and λ is a nonnegative constant. The equation (2.1) is oscillatory if all solutions of (2.1) have arbitrary large zeros on $[t_0, \infty)$; otherwise, the equation (2.1) is nonoscillatory. We improve a result of M. P. Do Carmo and D. Zhou [3] as follows:

Theorem 2.1. *Let v be a positive continuous function on $[t_0, \infty)$, $\displaystyle\int_{t_0}^{\infty} v(t)\,dt = \infty$, and λ be a nonnegative constant. If the differential equation $(v(t)x'(t))' + \lambda v(t)x(t) = 0,\ t \geq t_0$, is nonoscillatory, then $\sqrt{\lambda} \leq \dfrac{1}{2} \liminf\limits_{t\to\infty} \dfrac{\log V(t)}{t}$, where $V(t) = \int_{t_0}^{t} v(s)\,ds$.*

Proof. There is no loss of generality in assuming that $\liminf\limits_{t\to\infty} \dfrac{\log V(t)}{t} = a$, for some $a \geq 0$. Assume for the sake of contradiction that $\sqrt{\lambda} > \dfrac{a}{2}$. By the nonoscillatory assumption on the differential equation

$$((v(t)x'(t))' + \lambda v(t)x(t) = 0,\ t \geq t_0,$$

there exists a solution $x(t)$ of (2.1) and a positive constant $T > t_0$ such that $x(t) > 0$ for any $t \geq T$.

Let $y(t) = -\dfrac{v(t)x'(t)}{x(t)}$, for $t \geq T$, then y satisfies the Riccati equation

$$y'(t) = \frac{y^2(t)}{v(t)} + \lambda v(t). \tag{2.2}$$

Since $\int_{t_0}^{\infty} v(t)\,dt = \infty$ and $y'(t) = \frac{y^2(t)}{v(t)} + \lambda v(t) \geq \lambda v(t)$, $y(t) \geq \lambda(V(t) - V(T)) + y(T)$, we may assume $y(t) > 0$, for $t \geq T$. By (2.2), we have $y'(t) \geq 2\sqrt{\lambda}\, y(t)$, and hence

$$y(t) \geq y(T) \exp\left(2\sqrt{\lambda}(t - T)\right). \tag{2.3}$$

Furthermore, equation (2.2) gives

$$\frac{y'(t)}{y^2(t)} = \frac{1}{v(t)} + \frac{\lambda}{y^2(t)} \geq \frac{1}{v(t)},$$

and using Hölder's inequality,

$$-\frac{1}{y(t)} + \frac{1}{y(t-1)} \geq \int_{t-1}^{t} \frac{1}{v(\tau)}\,d\tau \geq \frac{1}{\int_{t-1}^{t} v(\tau)\,d\tau} \geq \frac{1}{\int_{T}^{t} v(\tau)\,d\tau} = \frac{1}{V(t)}.$$

It follows from (2.3) that

$$\begin{aligned} 0 &< \frac{1}{y(t)} \\ &\leq \frac{1}{y(T)\exp 2\sqrt{\lambda}(t-T-1)} - \frac{1}{V(t)} \\ &= \frac{\exp 2\sqrt{\lambda}(T+1)}{y(T)V(t)} \left[\frac{V(t)}{\exp\left(2\sqrt{\lambda}t\right)} - \frac{y(T)}{\exp\left(2\sqrt{\lambda}(T+1)\right)} \right], \end{aligned}$$

hence

$$\frac{V(t)}{\exp\left(2\sqrt{\lambda}t\right)} - \frac{y(T)}{\exp\left(2\sqrt{\lambda}(T+1)\right)} > 0. \tag{2.4}$$

Since $\liminf_{t\to\infty} \dfrac{\log V(t)}{t} = a$, there exists a sequence t_n, $t_n \to \infty$, such that $\lim_{n\to\infty} \dfrac{\log V(t_n)}{t_n} = a$, thus $V(t_n) = \exp\left((a+o(1))t_n\right)$. Then

$$\frac{V(t_n)}{\exp\left(2\sqrt{\lambda}t_n\right)} = \exp\left((a - 2\sqrt{\lambda} + o(1))t_n\right) \ \to \ 0$$

as $n \to \infty$. This contradicts to (2.4). ■

The estimate is sharp, as the following example shows. Let $v(t) = \exp(a\,t)$, for some nonnegative constant a. Then $\liminf_{t\to\infty} \dfrac{\log V(t)}{t} = a$. In this case, the general solution of (2.1) is given explicitly by

$$\begin{aligned} x(t) &= \exp\left(-\frac{a}{2}\ t\right)(c_1 \cosh(\frac{\sqrt{a^2-4\lambda}}{2}\ t) + c_2 \sinh(-\frac{\sqrt{a^2-4\lambda}}{2}\ t)), \\ x(t) &= \exp\left(-\frac{a}{2}\ t\right)(c_1 + c_2 t) \ \text{ and} \\ x(t) &= \exp\left(-\frac{a}{2}\ t\right)(c_1 \cos(\frac{\sqrt{4\lambda-a^2}}{2}\ t) + c_2 \sin(\frac{\sqrt{4\lambda-a^2}}{2}\ t)), \end{aligned}$$

for $\lambda < \frac{a^2}{4}$, $\lambda = \frac{a^2}{4}$ and $\lambda > \frac{a^2}{4}$, respectively. The equation (2.1) is nonoscillatory if $\sqrt{\lambda} \leq \frac{a}{2}$, and oscillatory if $\sqrt{\lambda} > \frac{a}{2}$, respectively.

3. Volume Growth and Upper Bound Estimates

Using the oscillation criteria of the preceding section, it is natural to obtain the following upper bound of the first eigenvalue of punctured manifolds. Let M be a Riemannian manifold. For a fixed point $p \in M$, let $V(r)$ be the volume of the geodesic ball $B(r)$ in M with radius r centered at p.

Theorem 3.1. *Let M be an n-dimensional complete Riemannian manifold with infinite volume, and Ω be a compact subset in M. Then* $\lambda_1(M\backslash\Omega) \leq \frac{1}{4}(\liminf_{r\to\infty}\frac{\log V(r)}{r})^2$.

Proof. Being a compact subset, there is a r_0 such that $\Omega \subset B(r_0)$. Let $v(r)$ be the area of the geodesic sphere $\partial B(r)$. Then the volume of the geodesic ball $B(r)$ is given by $V(r) = \int_0^r v(r)\, dr$. Since M has infinite volume, $\displaystyle\int_{r_0}^{\infty} v(r)\, dr = \infty$.

Without loss of generality we can assume that $\liminf_{r\to\infty}\frac{\log V(r)}{r} = a$, for some $a \geq 0$. Applying theorem 2.1, for any λ, $\sqrt{\lambda} > \frac{a}{2}$, all solutions of (2.1) have arbitrary large zeros on $[r_0, \infty)$. Thus there exists a nontrivial solution x of (2.1), r_1 and r_2 such that $r_2 > r_1 \geq r_0$ and $x(r_1) = x(r_2) = 0$. As a trial function, let $\phi = x \circ r$. Then

$$\begin{aligned}
0 &\leq \lambda_1(M \setminus \Omega) \\
&\leq \lambda_1(B(r_2) \setminus B(r_1)) \\
&\leq \frac{\displaystyle\int_{B(r_2)\backslash B(r_1)} |\nabla\phi|^2\, dV}{\displaystyle\int_{B(r_2)\backslash B(r_1)} \phi^2\, dV} \\
&= \frac{\displaystyle\int_{r_1}^{r_2} (x'(r))^2 v(r)\, dr}{\displaystyle\int_{r_1}^{r_2} x(r)^2 v(r)\, dr} \\
&= -\frac{\displaystyle\int_{r_1}^{r_2} (v(r)x'(r))' x(r)\, dr}{\displaystyle\int_{r_1}^{r_2} (x(r))^2 v(r)\, dr} \\
&= \lambda.
\end{aligned}$$

Since λ is an arbitrary positive constant, $\sqrt{\lambda} > \frac{a}{2}$, it follows that $\lambda_1(M \setminus \Omega) \leq \frac{1}{4} a^2$. ∎

4. Ricci Curvature and Upper Bound Estimates

The volume growth of geodesic balls are closely related to lower bounds on Ricci curvature, such as the classical Bonnet and Myers' theorem, Myers' theorem and Bishop's theorem. In this section, we proceed with the estimate for the first eigenvalue of the Laplacian on the punctured manifold $M \setminus \Omega$ in terms of a lower bound of Ricci curvature. We start with a simple observation of the asymptotic behavior of the differential equation

$$y''(t) - k^2(t)y(t) = 0, \tag{4.1}$$

with the given initial condition $y(0) = 0$ and $y'(0) = 1$, where k is a positive continuous function on $[0, \infty)$, and $k(t)$ tends to a nonnegative limit K as t tends to infinity. The solution y is positive for all $t > 0$. Indeed, suppose otherwise, if $y(t_1) = 0$ for some $t_1 > 0$, then y has a positive local maximum at $\xi \in (0, t_1)$, which contradicts the fact that $y''(\xi) = k^2(\xi)y(\xi) > 0$. Furthermore, since $y''(t) = k^2(t)y(t) > 0$, $y'(t)$ is increasing, $y'(t) \geq y'(0) = 1$, and therefore $y(t)$ tends to infinity as t tends to infinity. Because of $y(t)$ is increasing to infinity, by l'Hospital's rule, we have

$$\lim_{t\to\infty} \frac{(y'(t))^2}{y^2(t)} = \lim_{t\to\infty} \frac{2y'(t)y''(t)}{2y(t)y'(t)} = \lim_{t\to\infty} k^2(t) = K^2.$$

It consequently follows that

$$\lim_{t\to\infty} \frac{y'(t)}{y(t)} = K. \tag{4.2}$$

That is,

$$y(t) = C\ \exp(Kt + \int_{t_0}^{t} \varepsilon(s)\ ds),$$

where $\varepsilon(t)$ tends to zero as t tends to infinity. Using (4.6) and l'Hospital's rule, we see that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\log \displaystyle\int_{t_0}^{t} y^{n-1}(s)\, ds}{t} &= \lim_{t\to\infty} \frac{y^{n-1}(t)}{\displaystyle\int_{t_0}^{t} y^{n-1}(s)\, ds} \\
&= \lim_{t\to\infty} \frac{(n-1)y^{n-2}(t)y'(t)}{y^{n-1}(t)} \\
&= \lim_{t\to\infty} (n-1)\frac{y'(t)}{y(t)} \\
&= (n-1)K,
\end{aligned}$$

hence

$$\lim_{t\to\infty} \frac{\log \int_{t_0}^{t} y^{n-1}(s)\, ds}{t} = (n-1)K. \tag{4.3}$$

When k is restricted to the C^1 class, there is a another approach to obtaining an upper bound of y. The starting point is to observe that (4.5) may be considered as a system of equations $z' = Bz$, where

$$z(t) = \begin{bmatrix} \frac{1}{2}y(t) + \frac{1}{2k(t)}y'(t) \\ \frac{1}{2}y(t) - \frac{1}{2k(t)}y'(t) \end{bmatrix},$$

and B is the symmetric matrix

$$B(t) = \begin{bmatrix} -\dfrac{k'(t)}{2k(t)} + k(t) & \dfrac{k'(t)}{2k(t)} \\ \dfrac{k'(t)}{2k(t)} & -\dfrac{k'(t)}{2k(t)} - k(t) \end{bmatrix}.$$

Using a result of [4], the Euclidean norm $\|z\|$ of z is controlled by the integral of the greatest eigenvalue of $B(t)$. More precisely,

$$\sqrt{\frac{y^2(t)}{2} + \frac{(y'(t))^2}{2k^2(t)}} \le C \exp \int_{t_0}^{t} \frac{-k'(s) + \sqrt{(k'(s))^2 + 4k^4(s)}}{2k(s)}\, ds,$$

and hence

$$y(t) \le C \exp \int_{t_0}^{t} \frac{-k'(s) + \sqrt{(k'(s))^2 + 4k^4(s)}}{2k(s)}\, ds. \tag{4.4}$$

The main theorem we want to prove in this section is the following

Theorem 4.1. *Let M be an n-dimensional complete Riemannian manifold with infinite volume, and Ω be a compact subset in M. If Ricci curvature is bounded from below by $Ric \ge -\,(n-1)k^2(r)$, where k is a nonnegative continuous function, then* $\lambda_1(M \setminus \Omega) \le \dfrac{(n-1)^2}{4}\left(\limsup_{r\to\infty} k(r)\right)^2.$

Proof. We may assume that $\limsup_{r\to\infty} k(r) = K$ for some nonnegative constant K. For simplicity, we replace $k(r)$ by the truncated function $\tilde{k}(r) = \max\{k(r), K\} + \varepsilon$, where ε is a positive constant. Then $\tilde{k}$ is a positive continuous function on $[0,\infty)$, $\tilde{k}(t)$ tends to a positive limit $K+\varepsilon$ as t tends to infinity, and $Ric \ge -\,(n-1)\tilde{k}^2(r)$ still holds true.

For a fixed point p of M, let $J(r,\theta)\, d\theta$ be the area element of the boundary $\partial B(r)$. Then the volume of the geodesic ball $B(r)$ with radius r is given by

$$V(r) = \int_{S^{n-1}} \int_0^{\min\{r, \mathrm{foc}(\theta)\}} J(s,\theta)\, ds\, d\theta,$$

where $\mathrm{foc}(\theta)$ is the distance from p to the focal point of p along the minimal geodesic in $\theta-$direction. Let $y(r)$ be the solution of the initial value problem (4.5) with k replaced by $\tilde{k}$. According to Bishop volume comparison theorem [5], $J(r,\theta) \le y^{n-1}(r)$ whenever (r,θ) is within cut-locus of p. Therefore, $V(r) \le \int_{S^{n-1}} \int_0^r y^{n-1}(s)\, ds\, d\theta = \omega_{n-1} \int_0^r y^{n-1}(s)\, ds$, where ω_{n-1} is the area of the standard sphere S^{n-1}. Using theorem 3.1 and (4.7), we have

$$
\begin{aligned}
0 \le \lambda_1(M \setminus \Omega) &\le \frac{1}{4}\left(\liminf_{r\to\infty} \frac{\log V(r)}{r}\right)^2 \\
&= \frac{1}{4}\left[\liminf_{r\to\infty} \frac{\log \int_{S^{n-1}} \int_0^{\min\{r, \mathrm{foc}(\theta)\}} J(s,\theta)\, ds\, d\theta}{r}\right]^2 \\
&\le \frac{1}{4}\left[\liminf_{r\to\infty} \frac{\log \int_{S^{n-1}} \int_0^{r} y^{n-1}(s)\, ds\, d\theta}{r}\right]^2 \\
&= \frac{1}{4}\left[\lim_{r\to\infty} \frac{\log(\omega_{n-1} \int_0^{r} y^{n-1}(s)\, ds)}{r}\right]^2 \\
&= \frac{1}{4}\left[\lim_{r\to\infty} \frac{\log \int_0^{r} y^{n-1}(s)\, ds}{r}\right]^2 \\
&= \frac{1}{4}\left[\lim_{r\to\infty} \frac{\log \int_{r_0}^{r} y^{n-1}(s)\, ds}{r}\right]^2 \\
&= \frac{1}{4}(n-1)^2(K+\epsilon)^2,
\end{aligned}
$$

for arbitrary ϵ. Letting ϵ tend to zero, we complete the proof of theorem. ■

As a subsequence of (4.8), an argument like the above, we obtain the same upper bound estimate of $\lambda_1(M \setminus \Omega)$ of theorem 4.1 if k is a C^1 monotonic function.

If M_c is an n dimensional simply connected complete Riemannian manifold with constant negative sectional curvature $-\, c^2$, then $\lambda_1(M_c) = \frac{(n-1)^2}{4}\, c^2$ (see [1]). Since $\lambda_1(M_c) \le \lambda_1(M_c \setminus \Omega)$, we can apply theorem 4.1 to conclude that $\lambda_1(M_c \setminus \Omega) = \frac{(n-1)^2}{4}\, c^2$.

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