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# LOCAL AND GLOBAL OPTIMALITY CONDITIONS FOR DC INFINITE OPTIMIZATION PROBLEMS

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**Abstract.** We consider the optimality conditions for the DC (difference of two convex functions) optimization problem with the objective and constraint functions given as DC functions. Adopting convexification technique, the local and global KKT type conditions for this optimization problem are defined. By using properties of the subdifferentials of the involved functions, some sufficient and/or necessary conditions for these two types of optimality conditions are provided.

## 1. INTRODUCTION

Let $X$ be a locally convex Hausdorff topological vector space, $C$ be a nonempty convex subset of $X$, $T$ be an arbitrary (possibly infinite) index set and $h, h_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, for each $t \in T$, be proper functions. Consider the following optimization problem

$$\begin{array}{lll} & \text{Minimize} & h(x), \\ (1.1) & \text{s. t.} & h_t(x) \leq 0, \ t \in T, \\ & & x \in C. \end{array}$$

Since many problems in optimization and approximation theory such as linear semi-infinite optimization and the best approximation with restricted ranges can be recast into the form (1.1), more and more papers treating this kind of problems have appeared during the last decades, see for example [2, 3, 9, 10, 12, 13, 15, 18, 19, 20, 21, 22, 23] and the references therein.

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Usually for the optimality conditions for problem (1.1), one seeks conditions ensuring the following equivalence:

$$
\begin{aligned}
&[h(x_0) = \min_{x\in A} h(x)] \\
\text{(1.2)} \quad &\Longleftrightarrow \big[\exists \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}, \text{ s.t. } 0 \in \partial h(x_0) + N_C(x_0) + \sum_{t\in T(x_0)} \lambda_t \partial h_t(x_0)\big],
\end{aligned}
$$

where $A := \{x \in C : h_t(x) \leq 0, \ \forall\, t \in T\}$ is the solution set of the system (1.1), $x_0 \in \mathrm{dom} h \cap A$ and $T(x_0) := \{t \in T : \ h_t(x_0) = 0\}$. We say that the family $\{\delta_C; h_t : t \in T\}$ satisfies the KKT condition if (1.2) holds for each point in $\mathrm{dom} h \cap A$. Since the backward direction of the equivalence in (1.2) is easy to verify, the family $\{\delta_C; h_t : t \in T\}$ satisfies the KKT condition if and only if the forward direction of (1.2) holds at each point in $\mathrm{dom} h \cap A$. KKT type conditions are fundamental and important in both convex optimization and nonconvex optimization, and the literature on these areas is very rich, see for example [2, 3, 4, 5, 10, 22].

Recently, the DC (difference of two convex functions) optimization problem, that is, the involved functions $h$ and/or $h_t$, $t \in T$, in problem (1.1) are DC functions, has received much attention and been extensively studied by many authors, see, e.g., [1, 4, 5, 6, 7, 8, 11, 14, 27]. The reason is, as pointed out in [4], that DC programming problems are of high importance from both optimization theory and applications points of view. Moreover, by assuming that $h := f - g$ is the difference of two proper lower semicontinuous (l.s.c., for short) convex functions, each $h_t$ with $t \in T$ is a proper l.s.c. convex function and that $C$ is a closed convex set, Dinh, Mordukhovich and Nghia [4] derived the necessary optimality conditions for local solutions to (1.1) as well as necessary and sufficient optimality conditions for global solutions to (1.1) under the following closedness qualification condition (CQC) introduced there:

$$
\mathrm{epi}\, f^* + \mathrm{epi}\, \delta_C^* + \mathrm{cone}\left(\bigcup_{t\in T} \mathrm{epi}\, f_t^*\right) \quad \text{is weak}^* \text{ closed.}
$$

But to the best of our knowledge, not many results are known to provide characterizations for the KKT conditions for the DC optimization problem with both the objective and constraint functions be DC functions. Taking inspiration from this, we study in the present paper the KKT conditions for this kind of DC optimization problem and we do not impose any topological assumption on the set $C$ and the involved functions. Let $h := f - g$ and $h_t := f_t - g_t$, for each $t \in T$, be DC functions, where $f, g, f_t, g_t : X \to \overline{\mathbb{R}}$, for each $t \in T$, are proper convex functions. Define the primal problem by

$$
\text{(1.3)} \qquad (P) \quad
\begin{array}{ll}
\text{Minimization} & f(x) - g(x), \\
\text{s. t.} & f_t(x) - g_t(x) \leq 0, \ t \in T, \\
& x \in C.
\end{array}
$$

Our interest here is the investigation of the sufficient and/or necessary conditions for the optimality conditions for problem (1.3). Let $A$ denote the solution set of the system (1.3), that is

$$A := \{x \in C : f_t(x) - g_t(x) \leq 0,\ \forall\, t \in T\}. \tag{1.4}$$

To avoid the triviality in our study for $(P)$, we assume throughout the paper that $\operatorname{dom}(f-g) \cap A \neq \emptyset$. Let $x_0$ be a global minimizer of problem (1.3). In the case when $g$ and $g_t$ are subdifferentiable at $x_0$, the standard convexification technique can be applied. In fact, in this case, $x_0$ is also a global minimizer of the following problem

$$(P_{(u^*,v^*)}) \quad \begin{array}{ll} \text{Minimize} & f(x) - \langle u^*, x\rangle + g^*(u^*), \\ \text{s. t.} & f_t(x) - \langle v_t^*, x\rangle + g_t^*(v_t^*) \leq 0,\ t \in T, \\ & x \in C, \end{array} \tag{1.5}$$

where $u^* \in \partial g(x_0)$ and $v^* = (v_t^*)_{t\in T} \in \prod_{t\in T} \partial g_t(x_0)$. Let $\partial H(x_0) := \partial g(x_0) \times \prod_{t\in T} \partial g_t(x_0)$. Note that for each $(u^*, v^*) \in \partial H(x_0)$, the problem $(P_{(u^*,v^*)})$ is a convex optimization problem. Then, we can define the global KKT condition for problem $(P_{(u^*,v^*)})$ (applied $\{h, h_t : t \in T\}$ to the system $\{f - u^* + g^*(u^*), f_t - v_t^* + g_t^*(v_t^*) : t \in T\}$ in (1.2)) as follows:

$$\begin{aligned} & x_0 \text{ is a global minimizer of problem } (P_{(u^*,v^*)}) \\ \Longleftrightarrow\ & [\exists \lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}, \text{s.t.}\ u^* + \textstyle\sum_{t\in T_{v^*}(x_0)} \lambda_t v_t^* \in \partial f(x_0) \\ & +N_C(x_0) + \textstyle\sum_{t\in T_{v^*}(x_0)} \lambda_t \partial f_t(x_0)], \end{aligned} \tag{1.6}$$

where $T_{v^*}(x_0) := \{t \in T : f_t(x_0) - \langle v_t^*, x_0\rangle + g_t^*(v_t^*) = 0\}$. Moreover, it is easy to verify that for each $(u^*, v^*) \in \partial H(x_0)$,

$$T(x_0) := \{t \in T : f_t(x_0) - g_t(x_0) = 0\} = T_{v^*}(x_0).$$

This reformulation motivates us to define the following (convexification) global KKT condition at $x_0 \in \operatorname{dom}(f-g) \cap A$ for problem (1.3):

$$\begin{gathered} x_0 \text{ is a global minimizer of problem (1.3)} \\ \Longleftrightarrow [\forall (u^*, v^*) \in \partial H(x_0),\ \exists \lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}, \text{ s.t.} \\ u^* + \textstyle\sum_{t\in T(x_0)} \lambda_t v_t^* \in \partial f(x_0) + N_C(x_0) + \sum_{t\in T(x_0)} \lambda_t \partial f_t(x_0)]. \end{gathered}$$

Similarly, we define the local KKT condition at $x_0 \in \operatorname{dom}(f-g) \cap A$ for problem (1.3) as the following implication:

$$\begin{gathered} x_0 \text{ is a local minimizer of problem (1.3)} \\ \Longrightarrow [\forall (u^*, v^*) \in \partial H(x_0),\ \exists \lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}, \text{ s.t.} \\ u^* + \textstyle\sum_{t\in T(x_0)} \lambda_t v_t^* \in \partial f(x_0) + N_C(x_0) + \sum_{t\in T(x_0)} \lambda_t \partial f_t(x_0)]. \end{gathered}$$

Constraint qualifications involving subdifferentials have been studied and extensively used, see, e.g., [3, 5, 12, 17, 22]. The aim in the present paper is to use these constraint qualifications (or their variations) to provide some sufficient conditions for the local KKT condition and complete characterizations for the global KKT condition for the DC programming problem (1.3). Most of results obtained in this paper seem new and are proper extensions of the results in [4, 5] for the case when $g_t = 0$ and those in [10] for the special case when $g = g_t = 0, t \in T$.

The paper is organized as follows. The next section contains the necessary notation and preliminary results. Some sufficient conditions for the local KKT condition for problem (1.3) are provided in Section 3 and an equivalent condition for the global KKT condition for problem (1.3) is given in Section 4.

## 2. Notation and Preliminary Results

The notation used in the present paper is standard (cf. [28]). In particular, we assume throughout the whole paper that $X$ is a real locally convex space and let $X^*$ denote the dual space of $X$. For $x \in X$ and $x^* \in X^*$, we write $\langle x^*, x\rangle$ for the value of $x^*$ at $x$, that is, $\langle x^*, x\rangle := x^*(x)$. Let $Z$ be a set in $X$, the closure of $Z$ is denoted by $\operatorname{cl} Z$. The dual $X^*$ is endowed with the weak$^*$-topology. Thus if $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak$^*$-closure of $W$. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The normal cone of $Z$ at $z_0 \in Z$ is denoted by $N_Z(z_0)$ and is defined by

$$N_Z(z_0) := \{x^* \in X^* : \langle x^*, z - z_0\rangle \leq 0 \text{ for all } z \in Z\}.$$

Following [16], we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)_{t\in T}$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}_+^{(T)}$ denote the nonnegative cone in $\mathbb{R}^{(T)}$, that is

$$\mathbb{R}_+^{(T)} := \{(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \geq 0 \text{ for each } t \in T\}.$$

The indicator function $\delta_Z$ of a nonempty set $Z$ is defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $f : X \to \bar{\mathbb{R}}$ be a proper function. The effective domain, conjugate function and epigraph of $f$ are denoted by $\operatorname{dom} f$, $f^*$ and $\operatorname{epi} f$ respectively; they are defined by

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\},$$

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : \ x \in X\} \quad \text{for each } x^* \in X^*,$$

and

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : \ f(x) \leq r\}.$$

The subdifferential of $f$ at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) := \{x^* \in X^* : \ f(x) + \langle x^*, y - x\rangle \leq f(y) \text{ for each } y \in X\}. \tag{2.1}$$

By [28, Theorems 2.3.1 and 2.4.2 (iii)], the Young-Fenchel inequality below holds

$$f(x) + f^*(x^*) \geq \langle x^*, x\rangle \text{ for each pair } (x, x^*) \in X \times X^* \tag{2.2}$$

and the Young equality holds

$$f(x) + f^*(x^*) = \langle x^*, x\rangle \quad \text{if and only if } x^* \in \partial f(x). \tag{2.3}$$

In particular,

$$N_Z(x) = \partial \delta_Z(x) \quad \text{for each } x \in Z. \tag{2.4}$$

Furthermore, if $g : X \to \mathbb{R}$ is a proper convex function such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, then

$$\partial f(a) + \partial g(a) \subseteq \partial(f+g)(a) \quad \text{for each } a \in \operatorname{dom} f \cap \operatorname{dom} g. \tag{2.5}$$

Let $\phi : X \to [-\infty, +\infty]$ be an extended real-valued function. Recall from [25], one also can see [24, page 90], that the Fréchet subdifferential of $\phi$ at a point $x_0$ with $|\phi(x_0)| < \infty$, is defined by

$$\hat{\partial}\phi(x_0) := \{x^* \in X^* : \liminf_{x \to x_0} \frac{\phi(x) - \phi(x_0) - \langle x^*, x - x_0\rangle}{\|x - x_0\|} \geq 0\}. \tag{2.6}$$

Then it follows from the definition that

$$\partial\phi(x_0) \subseteq \hat{\partial}\phi(x_0) \quad \text{ for each } x_0 \text{ with } |\phi(x_0)| < \infty. \tag{2.7}$$

Particularly, in the case when $\phi$ is a convex function, then for each $x_0 \in \operatorname{dom}\phi$, $\hat{\partial}\phi(x_0)$ coincides with the subdifferential $\partial\phi(x_0)$ in the sense of convex analysis. Moreover, by the definition, we have the following implication

$$x_0 \text{ is a local minimizer of } \phi \Longrightarrow 0 \in \hat{\partial}\phi(x_0). \tag{2.8}$$

Let $\varphi : X \to [-\infty, +\infty]$ be another extended real-valued function. Assume that both $\phi$ and $\varphi$ are finite at some point $x_0$ and that $\hat{\partial}\varphi(x_0) \neq \emptyset$. Then, by [25, Theorem 3.1],

$$\hat{\partial}(\phi - \varphi)(x_0) \subseteq \bigcap_{u^* \in \partial\hat{\varphi}(x_0)} (\hat{\partial}\phi(x_0) - u^*). \tag{2.9}$$

Given a set $\Omega \subset X$ and a point $x_0 \in \Omega$, the Fréchet normal cone $\hat{N}_\Omega(x_0)$ to $\Omega$ at $x_0$ is defined by

$$\hat{N}_\Omega(x_0) := \{x^* \in X^* : \limsup_{x \xrightarrow{\Omega} x_0} \frac{\langle x^*, x - x_0\rangle}{\|x - x_0\|} \leq 0\}, \tag{2.10}$$

where $x \xrightarrow{\Omega} x_0$ means $x \to x_0$ with $\{x\} \subseteq \Omega$. Clearly, by definition, one can easily observe that

$$\hat{N}_\Omega(x_0) = \hat{\partial}\delta_\Omega(x_0) \quad \text{for each } x_0 \in \Omega.$$

For two sets $\Omega_1$, $\Omega_2$ in $X$ with $\Omega_2 \subseteq \Omega_1$, the following property is well known and easy to verify

$$\hat{N}_{\Omega_1}(x) \subseteq \hat{N}_{\Omega_2}(x) \quad \text{for each } x \in \Omega_2. \tag{2.11}$$

Moreover, in the case when $\Omega$ is a convex subset, then for each $x_0 \in \Omega$, $\hat{N}_\Omega(x_0)$ agrees with the normal cone $N_\Omega(x_0)$ in the sense of convex analysis.

## 3. Local Optimality Condition

Unless explicitly stated otherwise, let $f, g, T, C, \{f_t, g_t : t \in T\}$ and $A$ be as in Section 1; namely, $T$ is an index set, $C \subseteq X$ is a convex set, $f, g, f_t, g_t, t \in T$, are proper convex functions on $X$ such that $f - g$ and $f_t - g_t, t \in T$, are proper, and $A$ is the solution set of the following system:

$$x \in C; \;\; f_t(x) - g_t(x) \leq 0 \quad \text{for each} \quad t \in T. \tag{3.1}$$

To avoid the triviality, we always assume that $\operatorname{dom}(f - g) \cap A \neq \emptyset$. Throughout the whole paper, following [28, page 39], we adapt the convention that $(+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty$ and $0 \cdot (\infty) = 0$. Then, we have that

$$\emptyset \neq \operatorname{dom} f \subseteq \operatorname{dom} g \quad \text{and} \quad \emptyset \neq \operatorname{dom} f_t \subseteq \operatorname{dom} g_t \quad \text{for each } t \in T. \tag{3.2}$$

For simplicity, we denote

$$\partial H(x) := \partial g(x) \times \prod_{t \in T} \partial g_t(x) \quad \text{for each } x \in X, \tag{3.3}$$

where $\prod_{t\in T} \partial g_t(x) := \{(v_t^*)_{t\in T} : v_t^* \in \partial g_t(x) \,\forall\, t \in T\}$. For the whole paper, any elements $\lambda \in \mathbb{R}_+^{(T)}$ and $v^* \in \prod_{t\in T} \partial g_t(x)$ are understood as $\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $v^* = (v_t^*)_{t\in T} \in \prod_{t\in T} \partial g_t(x)$, respectively. Given $x_0 \in X$, let $T(x_0)$ be the active index set at $x_0$, that is,

$$T(x_0) := \{t \in T : \; f_t(x_0) - g_t(x_0) = 0\}.$$

**Definition 3.1.** Let $x_0 \in \mathrm{dom}(f-g) \cap A$. Consider the following statements:

(i) $x_0$ is a local minimizer of problem (1.3).

(ii) $\forall (u^*, v^*) \in \partial H(x_0)$, $\exists \lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$ such that

$$u^* + \sum_{t \in T(x_0)} \lambda_t v_t^* \in \partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t \partial f_t(x_0). \tag{3.4}$$

The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is said to satisfy the local KKT condition at $x_0$ if (i) $\Rightarrow$ (ii). We say that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the local KKT condition if it satisfies the local KKT condition at each point in $\mathrm{dom}(f-g) \cap A$.

In this section, we give some sufficient conditions to ensure the local optimality condition for DC infinite optimization problem (1.3). For this, we introduce the following definition. For a family of subsets $\{S_t : t \in T\}$ of $X$, we adapt the convention that $\bigcap_{t \in \emptyset} S_t = X$.

**Definition 3.2.** Let $x_0 \in \mathrm{dom}\,(f-g) \cap A$. The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is said to have the Fréchet-$(BCQ)$ ($F$-$(BCQ)$ in brief) at $x_0$ if

$$\begin{aligned} &\hat{\partial}\,(f - g + \delta_A)(x_0) \\ &\subseteq \bigcap_{(u^*, v^*) \in \partial H(x_0)} \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \Big(\partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t (\partial f_t(x_0) - v_t^*) - u^*\Big). \end{aligned} \tag{3.5}$$

Moreover, we say the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ has the $F$-$(BCQ)$ if it has the $F$-$(BCQ)$ at each point $x \in \mathrm{dom}\,(f-g) \cap A$.

**Remark 3.1.** Note that in the special case when $g = g_t = 0, t \in T$, then $f - g + \delta_A$ is a convex function and $\partial g(x) = \partial g_t(x) = \{0\}$ for each $x \in \mathrm{dom} f \cap A$ and $t \in T$. Thus, (3.5) reduces to

$$\partial(f + \delta_A)(x_0) \subseteq \partial f(x_0) + N_C(x_0) + \mathrm{cone}\big(\bigcup_{t \in T(x_0)} \partial f_t(x_0)\big). \tag{3.6}$$

This constraint qualification was called the $(BCQ)_f$ at $x_0$ for the family $\{\delta_C; f_t : t \in T\}$. If (3.6) holds for each $x \in \mathrm{dom} f \cap A$, then the family $\{\delta_C; f_t : t \in T\}$ is said to have the $(BCQ)_f$, which was introduced in [10] to study the optimality condition (of KKT type) for the problem of the form (1.1) with the system $\{h, h_t : t \in T\}$ be replaced by $\{f, f_t : t \in T\}$.

**Proposition 3.1.** *Suppose that $g_t = 0$ for each $t \in T$. If the family $\{\delta_C; f_t : t \in T\}$ has the $(BCQ)_f$, then the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ has the $F$-$(BCQ)$.*

*Proof.* Let $x_0 \in \operatorname{dom}(f-g) \cap A$. By assumption, (3.6) holds. By definition, we need to show

$$\hat{\partial}\,(f-g+\delta_A)(x_0) \subseteq \bigcap_{u^* \in \partial g(x_0)} \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} (\partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t \partial f_t(x_0) - u^*) \tag{3.7}$$

as $\partial g_t(x_0) = \{0\}$ for each $t \in T$. Note that if $\partial g(x_0) = \emptyset$ or $\hat{\partial}(f-g+\delta_A)(x_0) = \emptyset$, then (3.7) holds automatically. Below we assume that $\partial g(x_0) \neq \emptyset$ and $\hat{\partial}(f-g+\delta_A)(x_0) \neq \emptyset$. Take $p \in \hat{\partial}(f-g+\delta_A)(x_0)$. Then, by (2.9) and note that $f+\delta_A$ and $g$ are convex, one has

$$p \in \bigcap_{u^* \in \partial g(x_0)} (\partial(f+\delta_A)(x_0) - u^*).$$

This, together with (3.6), implies that

$$p \in \bigcap_{u^* \in \partial g(x_0)} \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} (\partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t \partial f_t(x_0) - u^*).$$

Thus, (3.7) holds as $p \in \hat{\partial}(f-g+\delta_A)(x_0)$ is arbitrary. The proof is complete. ■

**Theorem 3.1.** *Let $x_0 \in \operatorname{dom}(f-g) \cap A$. If the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ has the $F$-$(BCQ)$ at $x_0$, then it satisfies the local KKT condition at $x_0$.*

*Proof.* Suppose that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ has the $F$-$(BCQ)$ at $x_0$. Then, (3.5) holds. Let $x_0$ be a local minimizer of problem (1.3). Then, by (2.8),

$$0 \in \hat{\partial}(f-g+\delta_A)(x_0).$$

Combining this with (3.5) yields that

$$0 \in \bigcap_{(u^*, v^*) \in \partial H(x_0)} \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \big(\partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t (\partial f_t(x_0) - v_t^*) - u^*\big),$$

which means that for each $(u^*, v^*) \in \partial H(x_0)$ there exists $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that

$$u^* + \sum_{t \in T(x_0)} \lambda_t v_t^* \in \partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t \partial f_t(x_0).$$

Hence, the proof is complete. ■

Below we aim to the study the local KKT condition via convexification techniques. We first give some notation. For each $u^* \in \operatorname{dom} g^*$ and $v^* = (v_t^*)_{t \in T} \in \prod_{t \in T} \operatorname{dom} g_t^*$, we define the convex function $F^{u^*} : X \to \overline{\mathbb{R}}$ by

$$F^{u^*}(x) := f(x) - \langle u^*, x \rangle + g^*(u^*), \quad \forall\, x \in X, \tag{3.8}$$

and the convex function family $\{F_t^{v^*} : t \in T\}$ with each $F_t^{v^*} : X \to \overline{\mathbb{R}}$ is defined by

(3.9) $$F_t^{v^*}(x) := f_t(x) - \langle v_t^*, x\rangle + g_t^*(v_t^*), \quad \forall\, x \in X.$$

Then, by [28, Theorem 2.4.2 (vi)], we have that

$$\partial F^{u^*}(x) = \partial f(x) - u^*, \quad \forall\, x \in \mathrm{dom} f$$

and for each $t \in T$,

(3.10) $$\partial F_t^{v^*}(x) = \partial f_t(x) - v_t^*, \quad \forall\, x \in \mathrm{dom} f_t.$$

Furthermore, we use $A_{v^*}$ to denote the solution set of the following inequality system:

(3.11) $$x \in C, \;\; F_t^{v^*}(x) \le 0, \;\; t \in T.$$

Note that by (2.2), one has

(3.12) $$f - g \le F^{u^*} \quad \text{and} \quad f_t - g_t \le F_t^{v^*} \quad \text{for each } t \in T.$$

Hence, we obtain that

(3.13) $$A_{v^*} \subseteq A.$$

Moreover, for each $x_0 \in X$, let $T_{v^*}(x_0)$ denote the active index set of the system (3.11) at $x_0$, that is,

(3.14) $$T_{v^*}(x_0) := \{t \in T : F_t^{v^*}(x_0) = 0\}.$$

Since for each $v^* \in \prod_{t\in T} \partial g_t(x_0)$,

(3.15) $$F_t^{v^*}(x_0) = f_t(x_0) - g_t(x_0) \quad \text{for each} \quad t \in T$$

(see (2.3)), it follows that

(3.16) $$T_{v^*}(x_0) = T(x_0).$$

**Theorem 3.2.** *Let $x_0 \in \mathrm{dom}(f-g)\cap A$. Suppose that, for each $v^* \in \prod_{t\in T} \partial g_t(x_0)$, the family $\{\delta_C;\; F_t^{v^*} :\; t \in T\}$ has the $(BCQ)_f$ at $x_0$. Then the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the local KKT condition at $x_0$.*

*Proof.* Let $x_0$ be a local minimizer of (1.3). Take $(u^*, v^*) \in \partial H(x_0)$. Then, one can observe from (3.12), (3.13) and (3.15) that $x_0$ is also a local minimizer of the following optimization problem:

$$\text{(3.17)} \qquad \begin{array}{ll} \text{Minimize} & F^{u^*}(x) \\ \text{s.t.} & x \in A_{v^*}. \end{array}$$

Note that $F^{u^*}$ is a convex function and that $A_{v^*}$ is a convex subset of $X$. It follows that $x_0$ is also a global minimizer of problem (3.17). Then, by the first order optimality condition (see [28, Theorem 2.5.7]), one has that

$$\text{(3.18)} \qquad 0 \in \partial(F^{u^*} + \delta_{A_{v^*}})(x_0) = \partial(f + \delta_{A_{v^*}})(x_0) - u^*.$$

Moreover, by assumption that the family $\{\delta_C;\ F_t^{v^*}\ :\ t \in T\}$ has the $(BCQ)_f$ at $x_0$ and also note (3.16), we have

$$\partial(f + \delta_{A_{v^*}})(x_0) \subseteq \partial f(x_0) + N_C(x_0) + \text{cone}(\bigcup_{t \in T(x_0)} \partial F_t^{v^*}(x_0)).$$

Combining this with (3.18) and note (3.10), one can obtain that

$$u^* \in \partial f(x_0) + N_C(x_0) + \text{cone}(\bigcup_{t \in T(x_0)} (\partial f_t(x_0) - v_t^*)).$$

This means that there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$u^* \in \partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t(\partial f_t(x_0) - v_t^*),$$

which is equivalent to that

$$u^* + \sum_{t \in T(x_0)} \lambda_t v_t^* \in \partial f(x_0) + N_C(x_0) + \sum_{t \in T(x_0)} \lambda_t \partial f_t(x_0).$$

Therefore, by the arbitraryness of $(u^*, v^*) \in \partial H(x_0)$, the result is seen to hold and the proof is complete. ■

The following corollary follows directly from Theorem 3.2 or from Proposition 3.1 and Theorem 3.1.

**Corollary 3.1.** *Let $x_0 \in \text{dom}(f - g) \cap A$. Suppose that $g_t = 0$ for each $t \in T$ and that the family $\{\delta_C; f_t : t \in T\}$ has the $(BCQ)_f$ at $x_0$. If $x_0$ is a local minimizer of problem* (1.3)*, then the following inclusion holds:*

$$\text{(3.19)} \qquad \partial g(x_0) \subseteq \partial f(x_0) + N_C(x_0) + \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \sum_{t \in T(x_0)} \lambda_t \partial f_t(x_0).$$

**Remark 3.2.** In the special case when $f, g, f_t$, $t \in T$ are l.s.c., $C$ is closed and $g_t = 0$ for each $t \in T$, Dinh, Mordukhovich and Nghia [4] introduced the following closedness qualification condition (CQC):

$$\operatorname{epi} f^* + \operatorname{epi} \delta_C^* + \operatorname{cone}\left(\bigcup_{t\in T} \operatorname{epi} f_t^*\right) \quad \text{is weak}^* \text{ closed}$$

to establish the necessary optimality condition (3.19) for the DC infinite program (1.3). By [9, Corollary 3.4], if $f, f_t, t \in T$ are l.s.c. and $C$ is closed, then (CQC) is equivalent to the following conical $(EHP)_f$ for the family $\{\delta_C; f_t : t \in T\}$:

$$\operatorname{epi}(f+\delta_A)^* = \operatorname{epi} f^* + \operatorname{epi} \delta_C^* + \operatorname{cone}\left(\bigcup_{t\in T} \operatorname{epi} f_t^*\right),$$

which was introduced in [9, Definition 3.1]. Moreover, by [10, Proposition 3.1], we know that the conical $(EHP)_f$ is stronger than $(BCQ)_f$. Consequently, in the case when $f, g, f_t$, $t \in T$ are l.s.c. and $C$ is closed, if we replace the $(BCQ)_f$ by $(CQC)$, then the necessary optimality condition in Corollary 3.1 still holds. Thus, Corollary 3.1 extends the result [4, Theorem 5.2] to the case when the involved functions are not necessarily l.s.c. and the involved set is not necessarily closed.

Recall from [10, Definition 3.1] that the family $\{\delta_C; f_t : t \in T\}$ is said to have the $(BCQ)$ at some point $x_0 \in \tilde{A} := \{x \in C : f_t(x) \le 0, t \in T\}$ if

$$N_{\tilde{A}}(x_0) = N_C(x_0) + \operatorname{cone}(\bigcup_{t\in \tilde{T}(x_0)} \partial f_t(x_0)),$$

where $\tilde{T}(x_0) = \{t \in T : f_t(x_0) = 0\}$, and we say that the family $\{\delta_C; f_t : t \in T\}$ has the $(BCQ)$ if it has the $(BCQ)$ at each point in $\tilde{A}$. Moreover, for a proper function $h : X \to \overline{\mathbb{R}}$ and a nonempty subset $\Omega$ of $X$, following [25], we say that $h$ is Fréchet decomposable on $\Omega$ at $x_0 \in \Omega$ if

$$\hat{\partial}(h+\delta_\Omega)(x_0) \subseteq \hat{\partial} h(x_0) + \hat{N}_\Omega(x_0). \tag{3.20}$$

It happens, for example, when $h$ is Fréchet differentiable at $x_0 \in \Omega$, or when $h$ is a proper convex function and $\Omega$ is a convex set such that $h$ is continuous at some point in $\operatorname{dom} h \cap \Omega$ (cf. [24, Theorem 3.16] and [10, Lemma 2.1]).

**Corollary 3.2.** *Let $x_0 \in \operatorname{dom}(f-g)\cap A$. Suppose that, for each $v^* \in \prod_{t\in T} \partial g_t(x_0)$, the family $\{\delta_C;\ F_t^{v^*} :\ t \in T\}$ has the $(BCQ)$ at $x_0$ and that $f$ is Fréchet decomposable on $A_{v^*}$ at $x_0$. Then the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the local KKT condition at $x_0$.*

*Proof.* Let $v^* = (v_t^*)_{t\in T} \in \prod_{t\in T} \partial g_t(x_0)$. By Theorem 3.2, it suffices to show that the family $\{\delta_C;\ F_t^{v^*} :\ t \in T\}$ has the $(BCQ)_f$ at $x_0$, that is

$$\partial(f+\delta_{A_{v^*}})(x_0) \subseteq \partial f(x_0) + N_C(x_0) + \operatorname{cone}(\bigcup_{t\in T_{v^*}(x_0)} \partial F_t^{v^*}(x_0)). \tag{3.21}$$

Since $f$ is Fréchet decomposable on $A_{v^*}$ at $x_0$, also note that $f + \delta_{A_{v^*}}$ is a convex function and $A_{v^*}$ is a convex set, it follows that

$$\partial(f + \delta_{A_{v^*}})(x_0) \subseteq \partial f(x_0) + N_{A_{v^*}}(x_0). \tag{3.22}$$

Moreover, by the assumption that the family $\{\delta_C;\ F_t^{v^*} : t \in T\}$ has the $(BCQ)$ at $x_0$, we have

$$N_{A_{v^*}}(x_0) = N_C(x_0) + \text{cone}\Big(\bigcup_{t \in T_{v^*}(x_0)} \partial F_t^{v^*}(x_0)\Big). \tag{3.23}$$

Thus, (3.21) follows immediately from (3.22) and (3.23), which completes the proof. ■

**Proposition 3.2.** *Let $x_0 \in \text{dom}(f-g) \cap A$. Suppose that $f$ is Fréchet decomposable on $A$ at $x_0$ and that for each $v^* = (v_t^*) \in \prod_{t\in T} \partial g_t(x_0)$, the family $\{\delta_C;\ F_t^{v^*} : t \in T\}$ has the $(BCQ)$ at $x_0$. Then the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the local KKT condition at $x_0$.*

*Proof.* Let $x_0$ be a local minimizer of problem (1.3). Then, by (2.8), (2.9) and the given assumption that $f$ is Fréchet decomposable on $A$ at $x_0$, one has

$$\begin{aligned} 0 \in \hat{\partial}(f - g + \delta_A)(x_0) &\subseteq \bigcap_{u^* \in \partial g(x_0)} (\hat{\partial}(f + \delta_A)(x_0) - u^*) \\ &\subseteq \bigcap_{u^* \in \partial g(x_0)} (\partial f(x_0) + \hat{N}_A(x_0) - u^*) \end{aligned} \tag{3.24}$$

(note that $f$ is convex). We will show that

$$\hat{N}_A(x_0) \subseteq N_C(x_0) + \bigcap_{v^* \in \prod_{t\in T} \partial g_t(x_0)} \left( \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \Big( \sum_{t \in T(x_0)} \lambda_t(\partial f_t(x_0) - v_t^*)\Big) \right). \tag{3.25}$$

Granting this and (3.24) imply that

$$0 \in \bigcap_{(u^*, v^*) \in \partial H(x_0)} \left( \partial f(x_0) + N_C(x_0) + \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \Big( \sum_{t \in T(x_0)} \lambda_t(\partial f_t(x_0) - v_t^*)\Big) - u^* \right),$$

which is equivalent to (3.4); and hence the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the local KKT condition at $x_0$. It remains to show (3.25). Take $x^* \in \hat{N}_A(x_0)$ and $v^* \in \prod_{t\in T} \partial g_t(x_0)$. Note (3.15), we have $x_0 \in A_{v^*}$. Then, by (3.13) and (2.11),

$$x^* \in \hat{N}_{A_{v^*}}(x_0) = N_{A_{v^*}}(x_0), \tag{3.26}$$

where the equality holds because $A_{v^*}$ is a convex subset of $X$. Thus, by the given assumption that the family $\{\delta_C;\ F_t^{v^*}:\ t\in T\}$ has the $(BCQ)$ at $x_0$, we have that

$$x^*\in N_{A_{v^*}}(x_0)=N_C(x_0)+\bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\Big(\sum_{t\in T_{v^*}(x_0)}\lambda_t\partial F_t^{v^*}(x_0)\Big)$$
$$=N_C(x_0)+\bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\Big(\sum_{t\in T(x_0)}\lambda_t(\partial f_t(x_0)-v_t^*)\Big), \tag{3.27}$$

where the second equality holds thanks to (3.10) and (3.16). Thus, (3.25) is proved as $x^*\in\hat{N}_A(x_0)$ and $v^*\in\prod_{t\in T}\partial g_t(x_0)$ are arbitrary, which completes the proof. ∎

The following proposition gives a sufficient condition ensuring the local KKT condition in the case when $T$ is a finite index set. Before it, we first give some notation. Let $\mathrm{cont}h$ denote the set of all points at which $h$ is continuous, that is,

$$\mathrm{cont}h=\{x\in X:\ h \text{ is continuous at } x\}.$$

Given $v^*\in\prod_{t\in T}\partial g_t(x_0)$, we write

$$T_L:=\{t\in T: F_t^{v^*}\text{ is an affine function}\}\quad\text{and}\quad T_N:=T\setminus T_L. \tag{3.28}$$

**Proposition 3.3.** *Let $T=\{1,2,\cdots,m\}$ be a finite index set and let $x_0\in\mathrm{dom}(f-g)\cap A$. Suppose that*

$$\bigcap_{t\in T(x_0)}(\mathrm{cont}\,f_t)\cap(\mathrm{cont}\,f)\cap C\neq\emptyset, \tag{3.29}$$

*and that for each $v^*\in\prod_{t\in T}\partial g_t(x_0)$, there exists $\bar{x}\in\mathrm{ri}\,C$ such that*

$$\begin{cases} F_t^{v^*}(\bar{x})\le 0, & t\in T_L,\\ F_t^{v^*}(\bar{x})<0, & t\in T_N.\end{cases} \tag{3.30}$$

*If $C\cap(\cap_{t\in T}\mathrm{dom}\,f_t)$ is finite dimensional, then the family $\{f,g,\delta_C;f_t,g_t:t\in T\}$ satisfies the local KKT condition at $x_0$.*

*Proof.* Suppose that $C\cap(\cap_{t\in T}\mathrm{dom}\,f_t)$ is finite dimensional. To prove this proposition, by Theorem 3.2, we need only to show that for each $v^*\in\prod_{t\in T}\partial g_t(x_0)$, the family $\{\delta_C;\ F_t^{v^*}:\ t\in T\}$ has the $(BCQ)_f$ at $x_0$, i.e., the following inclusion holds:

$$\partial(f+\delta_{A_{v^*}})(x_0)\subseteq\partial f(x_0)+N_C(x_0)+\mathrm{cone}\Big(\bigcup_{t\in T_{v^*}(x_0)}\partial F_t^{v^*}(x_0)\Big). \tag{3.31}$$

Let $v^*\in\prod_{t\in T}\partial g_t(x_0)$. Consider the following convex optimization problem:

$$(P)\qquad \inf_{x\in A_{v^*}} f(x)$$

and its corresponding Lagrangian dual problem:

$$\begin{array}{ll}\text{Maximize} & \inf_{x\in C}\{f(x)+\sum_{t\in T}\lambda_t F_t^{v^*}(x)\}\\ \text{s.t.} & \lambda\in\mathbb{R}_+^{(T)}.\end{array}$$

Let $v(P)$ denote the optimal objective value of problem $(P)$. If $v(P)=-\infty$, then $\partial(f+\delta_{A_{v^*}})(x_0)=\emptyset$, and hence (3.31) holds automatically. Below we assume that $v(P)\in\mathbb{R}$. Let $Y_0$ denote the subspace spanned by $C\cap(\cap_{t\in T}\mathrm{dom}\, f_t)$. Then $Y_0$ is finite dimensional and $A_{v^*}$ is a convex subset of $Y_0$. Moreover, by the assumption, there exists $\bar{x}\in \mathrm{ri}C$ such that (3.30) holds. Thus, [26, Theorem 28.2] is applicable in $Y_0$ to get that

$$\inf_{x\in A_{v^*}}\{f(x)-\langle p,x\rangle\}=\max_{\lambda\in\mathbb{R}_+^{(T)}}\inf_{x\in C}\{f(x)-\langle p,x\rangle+\sum_{t\in T}\lambda_t F_t^{v^*}(x)\}\quad \text{for each } p\in X^*.$$

Then, by [9, Theorem 5.2] and [10, Proposition 3.1], one has that

$$\partial(f+\delta_{A_{v^*}})(x_0)=\bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\partial(f+\delta_C+\sum_{t\in T_{v^*}(x_0)}\lambda_t F_t^{v^*})(x_0). \tag{3.32}$$

Note that for each $t\in T$, $\mathrm{cont}\, F_t^{v^*}=\mathrm{cont}\, f_t$. Thus, by (3.16) and (3.29),

$$\bigcap_{t\in T_{v^*}(x_0)}(\mathrm{cont}\, F_t^{v^*})\cap(\mathrm{cont} f)\cap C\neq\emptyset.$$

Then, by [28, Theorem 2.4.2 (vi) and Theorem 2.8.7 (iii)], we can obtain that for each $\lambda\in\mathbb{R}_+^{(T)}$,

$$\partial\big(f+\delta_C+\sum_{t\in T_{v^*}(x_0)}\lambda_t F_t^{v^*}\big)(x_0)=\partial f(x_0)+N_C(x_0)+\sum_{t\in T_{v^*}(x_0)}\lambda_t\partial F_t^{v^*}(x_0).$$

This, together with (3.32), implies that (3.31) holds. The proof is complete. ■

## 4. GLOBAL OPTIMALITY CONDITION

Throughout this section, the notations $f,g,C,\{f_t,g_t:t\in T\},A$ and $T$ are as explained at the beginning of Section 3. The main aim of this section is to study the global optimality condition for DC infinite optimization problem (1.3).

**Definition 4.3.** Let $x_0\in\mathrm{dom}(f-g)\cap A$. Consider the following statements:

(i) $x_0$ is a global minimizer of problem (1.3).

(ii) $\forall(u^*,v^*)\in\partial H(x_0)$, $\exists\lambda=(\lambda_t)\in\mathbb{R}_+^{(T)}$ such that

$$u^*+\sum_{t\in T(x_0)}\lambda_t v_t^*\in\partial f(x_0)+N_C(x_0)+\sum_{t\in T(x_0)}\lambda_t\partial f_t(x_0). \tag{4.1}$$

The family $\{f,g,\delta_C;f_t,g_t:t\in T\}$ is said to satisfy the global KKT condition at $x_0$ if (i) $\Leftrightarrow$ (ii). We say that the family $\{f,g,\delta_C;f_t,g_t:t\in T\}$ satisfies the global KKT condition if it satisfies the global KKT condition at each point in $\mathrm{dom}(f-g)\cap A$.

In view of the first order optimality condition (see [28, Theorem 2.5.7]), the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the global KKT condition at $x_0 \in \operatorname{dom}(f-g) \cap A$ if and only if the following equivalence holds:

$$
\begin{aligned}
&0 \in \partial(f-g+\delta_A)(x_0) \Longleftrightarrow 0 \in \bigcap_{(u^*,v^*)\in\partial H(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial f(x_0) \\
&+N_C(x_0)+\sum_{t\in T(x_0)} \lambda_t(\partial f_t(x_0)-v_t^*) - u^*\big).
\end{aligned} \tag{4.2}
$$

**Definition 4.4.** Let $x_0 \in \operatorname{dom}(f-g) \cap A$. The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is said to have

(i) the $(BCQ)$ at $x_0$ if

$$
\begin{aligned}
&\partial(f-g+\delta_A)(x_0) \\
&= \bigcap_{(u^*,v^*)\in\partial H(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial f(x_0)+N_C(x_0)+\sum_{t\in T(x_0)} \lambda_t(\partial f_t(x_0)-v_t^*) - u^*\big);
\end{aligned} \tag{4.3}
$$

(ii) the $(BCQ)$ if it has the $(BCQ)$ at each point in $\operatorname{dom}(f-g) \cap A$.

**Remark 4.3.** In the special case when $g = g_t = 0, t \in T$, the $(BCQ)$ for the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ reduces to the $(BCQ)_f$ for the family $\{\delta_C; f_t : t \in T\}$.

**Proposition 4.4.** *Let $g_t = 0$ for each $t \in T$ and let $x_0 \in \operatorname{dom}(f-g)\cap A$. Suppose that the family $\{\delta_C; f_t : t \in T\}$ has the $(BCQ)_f$ at $x_0$. Then*

$$
\begin{aligned}
&\partial(f-g+\delta_A)(x_0) \\
&\subseteq \bigcap_{u^*\in\partial g(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial f(x_0)+N_C(x_0)+\sum_{t\in T(x_0)} \lambda_t\partial f_t(x_0) - u^*\big).
\end{aligned} \tag{4.4}
$$

*Proof.* If $\partial g(x_0) = \emptyset$, then (4.4) holds automatically. Below we assume that $\partial g(x_0) \neq \emptyset$. Take $p \in \partial(f-g+\delta_A)(x_0)$. Then, by (2.7), we have that $p \in \hat{\partial}(f-g+\delta_A)(x_0)$ and hence, by Proposition 3.1,

$$
p \in \bigcap_{u^*\in\partial g(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial f(x_0)+N_C(x_0)+\sum_{t\in T(x_0)} \lambda_t\partial f_t(x_0) - u^*\big).
$$

Therefore, (4.4) holds as $p$ is arbitrary. The proof is complete. ■

**Theorem 4.1.** *Let $x_0 \in \operatorname{dom}(f-g)\cap A$. Then the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ has the $(BCQ)$ at $x_0$ if and only if for each $p \in X^*$, the family $\{f+p, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the global KKT condition at $x_0$.*

*Proof.* For each $p \in X^*$, the family $\{f+p, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the global KKT condition at $x_0$ if and only if for each $p \in X^*$,

$$0 \in \partial(f+p-g+\delta_A)(x_0) \Longleftrightarrow 0 \in \bigcap_{(u^*,v^*)\in\partial H(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial(f+p)(x_0)$$
$$+N_C(x_0) + \sum_{t\in T(x_0)} \lambda_t(\partial f_t(x_0) - v_t^*) - u^*\big),$$

or, equivalently, for each $p \in X^*$,

$$0-p \in \partial(f-g+\delta_A)(x_0) \Longleftrightarrow -p \in \bigcap_{(u^*,v^*)\in\partial H(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial f(x_0)$$
$$+N_C(x_0) + \sum_{t\in T(x_0)} \lambda_t(\partial f_t(x_0) - v_t^*) - u^*\big),$$

which is the same with

$$\partial(f-g+\delta_A)(x_0)$$
$$= \bigcap_{(u^*,v^*)\in\partial H(x_0)} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \big(\partial f(x_0) + N_C(x_0) + \sum_{t\in T(x_0)} \lambda_t(\partial f_t(x_0) - v_t^*) - u^*\big).$$

Hence, the result is seen to hold. ■

**Corollary 4.3.** *If the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *has the* $(BCQ)$*, then it satisfies the global KKT condition.*

Note Remark 4.3, the following corollary is a direct consequence of Theorem 4.1, which was given in [10, Theorem 4.1].

**Corollary 4.4.** *Let* $g = g_t = 0$*,* $t \in T$ *and let* $x_0 \in \mathrm{dom} f \cap A$*. Then the family* $\{\delta_C; f_t : t \in T\}$ *has the* $(BCQ)_f$ *at* $x_0$ *if and only if the following equivalence holds for each* $p \in X^*$*:*

$$[f(x_0) + \langle p, x_0\rangle = \min\nolimits_{x\in A}(f(x) + \langle p, x\rangle)]$$
$$\Longleftrightarrow [\exists\lambda \in \mathbb{R}_+^{(T)},\ \text{s.t. } 0 \in \partial(f+p)(x_0) + N_C(x_0) + \textstyle\sum_{t\in T(x_0)} \lambda_t \partial f_t(x_0)].$$

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