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SOME CLASSIFICATIONS OF RULED SUBMANIFOLDS IN MINKOWSKI SPACE AND THEIR GAUSS MAP

Dong-Soo Kim[1], Young Ho Kim[2] and Sun Mi Jung

Abstract. Ruled submanifolds of Minkowski space with finite-type Gauss map are studied. Not having a parallel in Euclidean space, ruled submanifolds with degenerate rulings in Minkowski space drew our attention. We show that if non-cylindrical ruled submanifolds with non-degenerate rulings or ruled submanifolds with degenerate rulings have finite-type Gauss map, the Gauss map is one of the following: (1) harmonic; (2) of the so-called finite rank; (3) of null 2-type. For ruled submanifolds with degenerate rulings, we set up a relationship between finite-type immersions and immersions with finite-type Gauss map and introduce new examples of ruled submanifolds with degenerate rulings. We also characterize minimal ruled submanifolds with degenerate rulings in terms of finite-type Gauss map.

1. INTRODUCTION

The theory of minimal submanifolds is one of interesting topics in differential geometry. Especially, they are characterized by the relationship between the immersion and the Laplace operator defined on them. For example, the Veronese surface in 4-dimensional unit sphere is a minimal surface whose immersion is formed with eigenfunctions of the Laplace operator on the sphere. In 1966, T. Takahashi showed: Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into the Euclidean space $\mathbb{E}^m$ and Δ the Laplace operator defined on M. If $\Delta x = \lambda x$ holds, then M is a minimal submanifold in Euclidean space or a minimal submanifold in a hypersphere of Euclidean space ([22]).

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Extending this point of view, in the late 1970's B.-Y. Chen introduced the notion of finite-type immersion of Riemannian manifolds into Euclidean space ([5, 6]). In particular, minimal submanifolds of Euclidean space can be considered as a special case of submanifolds of finite-type. The notion of finite-type immersion was extended to submanifolds in pseudo-Euclidean space in 1980's: A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}^m_s$ with signature $(m-s, s)$ is said to be of *finite-type* if its position vector field x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + \Sigma^k_{i=1} x_i$, where x_0 is a constant map and $x_1, \cdots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots, k$ ([5, 6]). Furthermore, M is said to be of *k*-type if all eigenvalues λ_1, λ_2, $\cdots$, λ_k are mutually different. If one of λ_1, λ_2, $\cdots$, λ_k is zero, M is said to be of *null* k-type. Note that the immersion of a minimal submanifold of pseudo-Euclidean space is harmonic.

Such a notion can be naturally extended to a smooth map defined on submanifolds of pseudo-Euclidean space. A smooth map ϕ on an n-dimensional pseudo-Riemannian submanifold M of $\mathbb{E}^m_s$ is said to be of *finite-type* if ϕ is a finite sum of $\mathbb{E}^m_s$-valued eigenfunctions of Δ. We also similarly define a smooth map of *k*-type on M as that of immersion x. A very typical and interesting smooth map on the submanifold M of Euclidean space or pseudo-Euclidean space is the Gauss map.

A ruled surface is one of the most natural geometric objects in the classical differential geometry and has been studied under various geometric conditions ([2, 3, 7, 10, 16, 17, 18, 19, 20, 21]). The only minimal ruled surfaces in Euclidean 3-space are the planes and the helicoids. In [4], J. M. Barbosa et al. generalized the theory of minimal ruled surfaces to minimal ruled submanifolds and showed that those of Euclidean space are the so-called generalized helicoids. In [7] B.-Y. Chen et al. proved that a ruled surface of finite-type in an m-dimensional Euclidean space is an open part of either a cylinder over a curve of finite-type or a helicoid in $\mathbb{E}^3$. It follows that a ruled surface of finite-type in $\mathbb{E}^3$ is a part of a plane, a circular cylinder or a helicoid. F. Dillen extended these results to ruled submanifolds in Euclidean space with finite-type immersion ([9]). Also, ruled surfaces and submanifolds with finite-type Gauss map were studied in [1, 2, 3, 8].

On the other hand, for the ruled surfaces with null rulings in Minkowski m-space, two of the present authors and D. W. Yoon defined the extended B-scroll and the generalized B-scroll which are generalizations of a usual B-scroll in 3-dimensional Minkowski space and they completely classified the family of ruled surfaces of Minkowski space with finite-type Gauss map ([16, 17]). In [10, 11, 12, 13, 18] the ruled surfaces and the ruled submanifolds of finite-type immersion in Minkowski space were studied and classification theorems of such ruled surfaces and ruled submanifolds were given.

Very recently the authors classified ruled submanifolds with harmonic Gauss map in Minkowski space and characterized minimal ruled submanifolds in Minkowski space with harmonic Gauss map ([15]).

We pose a natural question: *Classify ruled submanifolds in Minkowski space with finite-type Gauss map.*

In this article, we completely classify ruled submanifolds in the Minkowski space $\mathbb{L}^m$ with finite-type Gauss map and introduce a relationship between finite-type immersions and immersions with finite-type Gauss map for ruled submanifolds when they have degenerate rulings. We characterize non-cylindrical ruled submanifolds of non-degenerate rulings with finite-type Gauss map. We also set up a characterization between ruled submanifold with degenerate rulings and the BS-kind ruled submanifold which is the extension of B-scroll in 3-dimensional Minkowski space, and that of minimal ruled submanifold with degenerate rulings is made by means of finite-type Gauss map.

All of geometric objects under consideration are smooth and submanifolds are assumed to be connected unless otherwise stated.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(m-s, s)$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called a *Lorentz-Minkowski* m-space or simply *Minkowski* m-space, which is denoted by $\mathbb{L}^m$. A curve in $\mathbb{L}^m$ is said to be *space-like, time-like* or *null* if its tangent vector field is space-like, time-like or null, respectively. Let $x : M \to \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, each tangent space of the submanifold is non-degenerate.

Let $(x_1, x_2, \cdots, x_n)$ be a local coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on M induced from that of $\mathbb{E}^m_s$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}}\sum_{i,j}\frac{\partial}{\partial x_i}(\sqrt{|\mathcal{G}|}g^{ij}\frac{\partial}{\partial x_j}).$$

We now choose an adapted local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ in $\mathbb{E}^m_s$ such that $e_1, e_2, \cdots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to M. The Gauss map $G : M \to G(n,m) \subset \mathbb{E}^N$ ($N = {}_mC_n$), $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$, of x is a smooth map which carries a point p in M to an oriented n-plane in $\mathbb{E}^m_s$ which is obtained from the parallel translation of the tangent space of M at p to an n-plane passing through the origin in $\mathbb{E}^m_s$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m_s$.

An indefinite scalar product $\ll\cdot,\cdot\gg$ on $G(n,m) \subset \mathbb{E}^N$ is defined by

$$\ll e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \gg = det(\langle e_{i_l}, e_{j_k}\rangle).$$

Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} | 1 \leq i_1 < \cdots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N_k$ for some positive integer k.

Now, we define a ruled submanifold M in $\mathbb{L}^m$. A non-degenerate $(r+1)$-dimensional submanifold M in $\mathbb{L}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s,r)$ of $\mathbb{L}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Thus, a parametrization of a ruled submanifold M in $\mathbb{L}^m$ can be given by

$$x = x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I,\ t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \cdots, r$. For each s, $E(s,r)$ is open in Span$\{e_1(s), e_2(s), \cdots, e_r(s)\}$, which is the linear span of linearly independent vector fields $e_1(s)$, $e_2(s)$, $\cdots$, $e_r(s)$ along the curve α. Here we assume $E(s,r)$ are either non-degenerate or degenerate for all s along α. We call $E(s,r)$ the *rulings* and α the *base curve* of the ruled submanifold M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s,r)$ is parallel along α, or *non-cylindrical* otherwise.

Remark 2.1. ([13, 15]).

(1) If the rulings of M are non-degenerate, then the base curve α can be chosen to be orthogonal to the rulings as follows: Let V be a unit vector field on M which is orthogonal to the rulings. Then α can be taken as an integral curve of V.

(2) If the rulings are degenerate, we can choose a null base curve which is transversal to the rulings: Let V be a null vector field on M which is not tangent to the rulings. An integral curve of V can be the base curve.

By solving a system of ordinary differential equations similarly set up in relation to a frame along a curve in $\mathbb{L}^m$ as given in [4], we have

Lemma 2.2. ([14, 15]). *Let $V(s)$ be a smooth l-dimensional non-degenerate distribution in the Minkowski m-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$, where $l \geq 2$ and $m \geq 3$. Then, we can choose orthonormal vector fields $e_1(s), \cdots, e_{m-l}(s)$ along α which generate the orthogonal complement $V^{\perp}(s)$ satisfying $e_i'(s) \in V(s)$ for $1 \leq i \leq m-l$.*

Remark 2.3. We would like to point out one thing for an n-plane in $\mathbb{L}^m$, which is usually a cylindrical ruled submanifold, but it is also regarded as a non-cylindrical ruled submanifold as well by a change of rulings.

3. Ruled Submanifolds with Non-degenerate Rulings

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings. By Remark 2.1, the base curve α can be chosen to be orthogonal to the rulings. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = \varepsilon(= \pm 1)$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 2.2, we may choose orthonormal vector fields $e_1(s), \cdots, e_r(s)$ along α satisfying

$$\langle \alpha'(s), e_i(s) \rangle = 0,\ \langle e_i'(s), e_j(s) \rangle = 0, \quad i, j = 1, 2, \cdots, r. \tag{3.1}$$

A parametrization of M is given by

$$x = x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s). \tag{3.2}$$

In this section, we always assume that the parametrization (3.2) satisfies the condition (3.1). Then, M has the Gauss map

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{|q|^{1/2}} (\Phi + \sum_{i=1}^{r} t_i \Psi_i), \tag{3.3}$$

where q is the function of $s, t_1, t_2, \cdots, t_r$ defined by

$$q = \langle x_s, x_s \rangle, \quad \Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r.$$

Now, we separate the cases into two typical types of ruled submanifolds which are cylindrical or non-cylindrical.

Theorem 3.1. *The cylindrical ruled submanifolds in $\mathbb{L}^m$ generated by non-degenerate rulings have finite-type Gauss map if and only if they are cylinders over a curve of finite-type.*

Proof. Let M be a cylindrical $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings, which is parameterized by (3.2). We may assume that $e_1, e_2, \cdots, e_r$, generating the rulings, are constant vectors.

Then the Laplacian Δ of M is expressed by

$$\Delta = -\varepsilon \frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2},$$

where $\varepsilon_i = \langle e_i(s), e_i(s) \rangle = \pm 1$ and the Gauss map G of M is given by

$$G = \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

If we denote by Δ' the Laplacian of α, that is $\Delta' = -\varepsilon\frac{\partial^2}{\partial s^2}$, we have the Laplacian ΔG of the Gauss map

$$\Delta G = \Delta'\alpha' \wedge e_1 \wedge \cdots \wedge e_r. \tag{3.4}$$

We now suppose that the Gauss map G is of finite-type. Then there exist real numbers $c_1, \cdots, c_k$ such that

$$\Delta^{k+1}G + c_1\Delta^k G + \cdots + c_k\Delta G = 0.$$

By using (3.4), we have

$$\Delta'^{k+1}\alpha' + c_1\Delta'^k\alpha' + \cdots + c_k\Delta'\alpha' = 0,$$

which implies that α is of finite-type.

The converse is straightforward. ■

In [13], two of the present authors defined the notion of finite rank k for the isometric immersions of non-cylindrical ruled submanifolds with non-degenerate rulings. Similarly we define the term of being of finite rank for Gauss map:

Definition 3.2. Let M be a non-cylindrical $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with non-degenerate rulings. Let $e_1, e_2, \cdots, e_r$ be orthonormal vector fields which span the rulings. We assume that e'_{j_1}, $e'_{j_2}, \cdots, e'_{j_k}$ are null vector fields and $e_{j_{k+1}}$, $e_{j_{k+2}}, \cdots$, e_{j_r} constant vector fields for some integer k $(1 \le k \le r)$. In this case, the Gauss map G is said to be *of finite rank* k if the vectors $\alpha' \wedge e_1 \wedge \cdots \wedge e_r$ and $e'_{j_i} \wedge e_1 \wedge \cdots \wedge e_r$ are of the same finite-type with the same eigenvalues for $i = 1, \cdots, k$.

We need the following lemmas for later use.

Lemma 3.3. ([15]). *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3.2) *in $\mathbb{L}^m$. Suppose that e'_1, $e'_2, \cdots, e'_r$ are non-null and some of generators of rulings $e_1, \cdots, e_k$ are constant vector fields along α. Then we have the Laplacian*

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=k+1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\varepsilon_i\frac{\partial^2}{\partial t_i^2}.$$

From now on, for a polynomial $F(t)$ in $t = (t_1, t_2, \cdots, t_r)$, deg $F(t)$ denotes the degree of $F(t)$ in $t = (t_1, t_2, \cdots, t_r)$ unless otherwise stated.

Lemma 3.4. ([13]). *Let $P(t)$ be a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions of s as the coefficients and deg $P(t) = d$. Then, for $l \in \mathbb{R}\backslash\{0\}$ we have*

$$\Delta(\frac{P(t)}{q^l}) = \frac{\tilde{P}(t)}{q^{l+3}},$$

where $\tilde{P}(t)$ is a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions of s as the coefficients and deg $\tilde{P}(t) \leq d + 4$.

Lemma 3.5. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parametrized by* (3.2) *in $\mathbb{L}^m$ with finite-type Gauss map. Let $e_1, e_2, \cdots, e_r$ be orthonormal generators of the rulings along α generating the rulings. If e_i' are non-null for $i = 1, 2, \cdots, r$ and some of generators of the rulings $e_1, \cdots, e_k$ are constant vector fields along α, then the Gauss map is harmonic.*

Proof. Suppose that the ruled submanifold M has finite-type Gauss map G. Since $e_1', e_2', \cdots, e_r'$ are non-null and M is non-cylindrical, we see that deg $q(t) = 2$. Let $\tilde{\varepsilon}$ =sign q. Since $e_1, \cdots, e_k$ are constant vector fields along α, the Gauss map G has the form

$$G = \frac{\tilde{G}(t)}{(\tilde{\varepsilon} q)^{1/2}},$$

where $\tilde{G}(t) = \Phi + \Sigma_{i=k+1}^r t_i \Psi_i$ with deg $\tilde{G}(t) \leq 1$. By using Lemma 3.3 and Lemma 3.4, we get

$$\Delta^j G = \frac{G_j(t)}{(\tilde{\varepsilon} q)^{(1/2)+3j}}, \ j = 1, 2, \cdots,$$

where $G_j(t)$ is a polynomial in t with functions in s as coefficients and deg $G_j(t) \leq 1 + 4j$. If j goes up by one, the degree of the numerator of $\Delta^j G$ goes up by at most 4 while that of the denominator goes up by 6. Thus, $\Delta^{i+1} G + c_1 \Delta^i G + \cdots + c_i \Delta G = 0$ can never hold for some positive integer i unless

$$\Delta G = 0. \qquad \blacksquare$$

In [15], the present authors proved that on a non-cylindrical ruled submanifold M in Minkowski space $\mathbb{L}^m$ with harmonic Gauss map, if the derivatives $e_1', ..., e_r'$ of generators of rulings are non-null, then $e_1', ..., e_r'$ are parallel to the tangent vector field of the base curve α after a long computation. Using such a property, they prove that the Gauss map G is constant along the base curve α. Together with Remark 2.3, they obtain

Theorem 3.6. ([15]). *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parametrized by* (3.2) *in $\mathbb{L}^m$ with finite-type Gauss map. Let $e_1, e_2, \cdots, e_r$ be orthonormal generators of the rulings along the base curve α. If e_i' are non-null for $i = 1, 2, \cdots, r$ and some of generators of the rulings $e_1, \cdots, e_k$ are constant vector fields along α, then M is an open part of an $(r+1)$-plane in $\mathbb{L}^m$.*

We now deal with the case that some of generators of rulings have null derivatives. In this case, authors showed that no ruled submanifolds with deg $q(t) = 1$ or deg $q(t) = 2$ have harmonic Gauss map ([15]). Using this, we readily prove

Proposition 3.7. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parametrized by* (3.2) *in $\mathbb{L}^m$ with finite-type Gauss map. If some generators $e_{j_1}, e_{j_2}, \cdots, e_{j_k}$ of the rulings have null derivatives along the base curve α for $j_1 < j_2 < \cdots < j_k \in \{1, 2, \cdots, r\}$, then the Gauss map G has finite rank k, where $1 \le k \le r$.*

Proof. We can rewrite the parametrization (3.2) of M as

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i \neq j_1, j_2, \cdots, j_k} t_i e_i(s) + \sum_{i=1}^{k} t_{j_i} e_{j_i}(s)$$

and its Laplace operator is given by

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r} \varepsilon_i \frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2}.$$

Then, there are possible two cases such that either $e_{j_{k+1}}, \cdots, e_{j_r}$ generating the rulings except $e_{j_1}(s), e_{j_2}(s), \cdots, e_{j_k}(s)$ are constant vector fields or $e_i' \neq 0$ for some $i = j_{k+1}, \cdots, j_r$ if $k < r$.

Case 1. Suppose that $e_{j_{k+1}}, \cdots, e_{j_r}$ are constant vector fields.

Subcase 1.1. Let deg $q(t) = 0$. In this case, e_{j_i}' are null with $e_{j_i}'(s) \wedge e_{j_l}'(s) = 0$ for $i, l = 1, 2, \cdots, k$ and $\langle \alpha'(s), e_j'(s) \rangle = 0$ for $j = j_1, j_2, \cdots, j_k$. Then M has the Gauss map

$$G = \Phi + \sum_{i=1}^{k} t_{j_i} \Psi_{j_i}.$$

Thus, we have

$$\Delta^l G = -(\Phi^{(2l)}(s) + \sum_{i=1}^{k} t_{j_i} \Psi_{j_i}^{(2l)}(s))$$

for $l = 1, 2, 3, \cdots$. Hence, Φ and Ψ_{j_i} are of the same finite-type with the same eigenvalues if the Gauss map G is of finite-type. In other words, the Gauss map G is of finite rank k.

Subcase 1.2. Let deg $q(t) = 1$. In this case, $\langle \alpha'(s), e_{j_i}'(s) \rangle \neq 0$ for some j_i $(1 \le i \le k)$ and the null vector fields e_{j_i}' satisfy $e_{j_i}' \wedge e_{j_l}' = 0$ for $i, l = 1, 2, \cdots, k$. The Gauss map G of M has the form

$$G = \frac{\tilde{G}(t)}{(\tilde{\varepsilon} q)^{1/2}},$$

where deg $\tilde{G}(t) \leq 1$. Thanks to Lemma 3.4, we have

$$\Delta^j G = \frac{G_j(t)}{(\tilde{\varepsilon}q)^{(1/2)+3j}},\ j = 1, 2, \cdots,$$

where $G_j(t)$ is a polynomial in t with functions in s as coefficients and deg $G_j(t) \leq 1 + 2j$.

If j goes up by one, the degree of numerator of $\Delta^j G$ goes up by at most 2 while that of the denominator goes up by 3. Thus, if G is of finite-type, we must have $\Delta G = 0$, i.e., G is harmonic. Therefore, we can conclude that no such ruled submanifolds with deg $q = 1$ have finite-type Gauss map.

Subcase 1.3. Let deg $q(t) = 2$. If we again use Lemma 3.4, we get the j-th Laplacian $\Delta^j G$ as

$$\Delta^j G = \frac{\tilde{G}_j(t)}{(\tilde{\varepsilon}q)^{1/2+3j}},\quad j = 1, 2, 3, \cdots$$

for a polynomial $\tilde{G}_j(t)$ in $t = (t_1, t_2, \cdots, t_r)$ with deg $\tilde{G}_j(t) \leq 1 + 4j$. Using the similar argument developed in Lemma 3.5, we have

$$\Delta G = 0.$$

Therefore, no such ruled submanifolds with deg $q = 2$ have finite-type Gauss map G.

Case 2. Suppose that $e_i' \neq 0$ for some $i = j_{k+1}, \cdots, j_r$.

In this case, we may assume that $e_i' \neq 0$ for all $i = j_{k+1}, \cdots, j_r$, otherwise the ruled submanifold M is a cylinder built over the ruled submanifold parametrized by the base curve α and the rulings generated by e_i's except those constant vector fields. Then, e_i' are non-null for all $i = j_{k+1}, \cdots, j_r$ and deg q= 2.

If we again follow a similar argument in the proof of Lemma 3.5, we have

$$\Delta G = 0 \quad \text{and} \quad \alpha' \wedge e_i' = 0$$

for all $i = 1, 2, \cdots, r$. This is a contradiction.

This completes the proof. ■

It is easy to show that if the Gauss map G of a ruled submanifold with non-degenerate rulings in $\mathbb{L}^m$ has finite rank k $(1 \leq k \leq r)$, G is of finite-type. Therefore, combining the results of Theorem 3.6 and Proposition 3.7, we conclude

Theorem 3.8. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has finite-type Gauss map G if and only if either M is an open part of an $(r+1)$-dimensional plane or the Gauss map G is of finite rank k for some k $(1 \leq k \leq r)$.*

Corollary 3.9. ([15]). *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has harmonic Gauss map if and only if M is an open part of either an $(r+1)$-plane or a ruled submanifold up to cylinders over a certain submanifold with the parametrization given by*

$$x(s,t_1,t_2,\cdots,t_r)=f(s)\boldsymbol{N}+s\boldsymbol{E}+\sum_{j=1}^{r}t_j(p_j(s)\boldsymbol{N}+\boldsymbol{F}_j)$$

for some smooth functions f and p_j, and some constant vector fields $\boldsymbol{N},\boldsymbol{E},\boldsymbol{F}_j$ with $\langle\boldsymbol{E},\boldsymbol{E}\rangle=1$, $\langle\boldsymbol{N},\boldsymbol{N}\rangle=\langle\boldsymbol{N},\boldsymbol{E}\rangle=\langle\boldsymbol{N},\boldsymbol{F}_j\rangle=\langle\boldsymbol{E},\boldsymbol{F}_j\rangle=0$, and $\langle\boldsymbol{F}_j,\boldsymbol{F}_i\rangle=\delta_{ji}$ for $i,j=1,2,\cdots,r$.

Remark. ([15]). In Corollary 3.9, if the base curve α is a straight line and the generators e_i satisfy $e_i''=0$ along α $(i=1,2,\cdots,r)$, the ruled submanifold M is minimal.

4. Classifications of Ruled Submanifolds with Degenerate Rulings

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with degenerate rulings $E(s,r)$ along a regular curve and let its parametrization be given by $\tilde{x}(s,t)$ where $t=(t_1,t_2,\cdots,t_r)$. Since $E(s,r)$ is degenerate, it can be spanned by a degenerate frame $\{B(s)=e_1(s),e_2(s),\cdots,e_r(s)\}$ such that

$$\langle B(s),B(s)\rangle=\langle B(s),e_i(s)\rangle=0,\quad \langle e_i(s),e_j(s)\rangle=\delta_{ij},\quad i,j=2,3,\cdots,r.$$

Without loss of generality as in Lemma 2.2, we may assume that

$$\langle e_i'(s),e_j(s)\rangle=0,\quad i,j=2,3,\cdots,r.$$

Since the tangent space of M at $\tilde{x}(s,t)$ is a Minkowski $(r+1)$-space which contains the degenerate ruling $E(s,r)$, there exists a tangent vector field A to M which satisfies

$$\langle A(s,t),A(s,t)\rangle=0,\quad \langle A(s,t),B(s)\rangle=-1,\quad \langle A(s,t),e_i(s)\rangle=0,\quad i=2,3,\cdots,r$$

at $\tilde{x}(s,t)$.

Let $\alpha(s)$ be an integral curve of the vector field A on M. Then we can define another parametrization x of M as follows:

$$x(s,t_1,t_2,\cdots,t_r)=\alpha(s)+\sum_{i=1}^{r}t_ie_i(s),$$

where $\alpha'(s)=A(s)$.

Lemma 4.1. ([13]). *We may assume that* $\langle A(s), B'(s)\rangle = 0$ *for all* s.

Two of the present authors proved the following lemma.

Lemma 4.2. ([14]). *Let* M *be a ruled submanifold with degenerate rulings. Then, the following are equivalent.*
(1) M *is minimal.*
(2) $B'(s)$ *is tangent to* M.

If we put $P = \langle x_s, x_s\rangle$ and $Q = -\langle x_s, x_{t_1}\rangle$, Lemma 4.1 implies

$$\begin{aligned} P(s,t) &= 2\sum_{i=2}^{r} u_i(s)t_i + \sum_{i,j=1}^{r} w_{ij}(s)t_it_j, \\ Q(s,t) &= 1 + \sum_{i=2}^{r} v_i(s)t_i, \end{aligned}$$

where $v_i(s) = \langle B'(s), e_i(s)\rangle$, $u_i(s) = \langle A(s), e_i'(s)\rangle$, $w_{ij}(s) = \langle e_i'(s), e_j'(s)\rangle$ for $i, j = 1, 2, \cdots, r$. Note that P and Q are polynomials in $t = (t_1, t_2, \cdots, t_r)$ with functions in s as coefficients. Then the Laplacian Δ of M can be expressed as follows:

$$\begin{aligned} \Delta = \frac{1}{Q^2}\{\frac{\partial \bar{P}}{\partial t_1}\frac{\partial}{\partial t_1} &-2Q\sum_{i=2}^{r} v_i\frac{\partial}{\partial t_i} + 2Q\frac{\partial^2}{\partial s\partial t_1} + \bar{P}\frac{\partial^2}{\partial t_1^2} \\ &-2Q\sum_{i=2}^{r} v_it_1\frac{\partial^2}{\partial t_1\partial t_i} - Q^2\sum_{i=2}^{r}\frac{\partial^2}{\partial t_i^2}\}, \end{aligned}$$

where $\bar{P} = P - t_1^2\sum_{i=2}^{r} v_i^2$.

By definition of an indefinite scalar product $\ll, \gg$ on $G(r+1, m)$, we may put

$$\ll x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r}, x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \gg = -Q^2.$$

Let $\bar{\varepsilon} = \text{sign } Q(t)$. Then we have the Gauss map

$$\begin{aligned} G =& \frac{1}{\bar{\varepsilon}Q}x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \\ =& \frac{1}{\bar{\varepsilon}Q}\{A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + t_1B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &+ \sum_{i=2}^{r} t_ie_i' \wedge B \wedge e_2 \wedge \cdots \wedge e_r\}. \end{aligned}$$

Depending upon the degree of Q, we can have two cases.

Case 1. Suppose that deg $Q(t) = 0$, that is, $Q = 1$ and $v_i(s) = 0$ for all

$i = 2, 3, \cdots, r$. By definition, we get

$$\Delta G = 2\sum_{i=1}^{r}\langle B', e_i'\rangle t_i B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r$$
$$+ 2B'' \wedge B \wedge e_2 \wedge \cdots \wedge e_r + 2\sum_{i=2}^{r} B' \wedge B \wedge e_2 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r.$$

It is straightforward to get

$$\Delta^k G$$
$$= \{2^k\beta^{k-1}\sum_{i=1}^{r}\langle B', e_i'\rangle t_i + 2^k(\beta^{k-1})'\}B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r$$
$$+ 2^k\beta^{k-1}B'' \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \sum_{i=2}^{r} 2^k\beta^{k-1}B' \wedge B \wedge e_2 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r$$

for $k \geq 2$, where we have put $\beta = \langle B', B'\rangle$.

We now suppose that the Gauss map G is of finite-type. Then, there exist some constants $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that

$$\Delta^{k+1}G + \lambda_1\Delta^k G + \cdots + \lambda_k \Delta G = 0 \tag{4.1}$$

for some positive integer k.

Subcase 1. Let $\beta = 0$. From Lemma 4.1, we see that $B'(s)$ must be space-like. Thus, $B'(s) = 0$ and hence $\Delta G = 0$, i.e., G is harmonic. Taking account of Lemma 4.2, we see that M is minimal.

Subcase 2. Let $\beta \neq 0$. In this case, $B'(s)$ is normal to M and hence M is not minimal due to Lemma 4.2. But, (4.1) implies

$$\sum_{j=1}^{k+1}\lambda_{(k+1)-j}\{2^j\beta^{j-1}(\sum_{i=1}^{r}\langle B', e_i'\rangle t_i) + 2^j(\beta^{j-1})'\}B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r$$
$$+ (\sum_{j=1}^{k+1}\lambda_{(k+1)-j}2^j\beta^{j-1})B'' \wedge B \wedge e_2 \wedge \cdots \wedge e_r$$
$$+ \sum_{i=2}^{r}(\sum_{j=1}^{k+1}\lambda_{(k+1)-j}2^j\beta^{j-1})B' \wedge B \wedge e_2 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r = 0.$$

Considering the causal characters of vector fields, we have

$$\sum_{j=1}^{k+1}\lambda_{(k+1)-j}2^j\beta^{j-1} = 0.$$

Then, β is a solution of an algebraic equation

$$2\lambda_k + 2^2\lambda_{k-1}z + 2^3\lambda_{k-2}z^2 + \cdots + 2^{k+1}z^k = 0$$

with real coefficients and thus β is a real constant. It is easy to show that

$$\Delta^2 G = 2\beta\Delta G.$$

From this, we can show that the Gauss map G is of null 2-type.

Case 2. Suppose that deg $Q(t) = 1$. Then there exists some $j = 2, 3, \cdots, r$ such that v_j is non-vanishing, that is, $\langle B', e_j\rangle \neq 0$. The Laplacian of G is then obtained by

$$\Delta G = \frac{P_1(t)}{Q^3}$$

for some polynomial $P_1(t)$ in t with functions in s as coefficients of deg $P_1(t) \leq 1$. It gives

$$\Delta^j G = \frac{P_j(t)}{Q^{2j+1}}$$

with deg $P_j(t) \leq 1$ for $j \geq 2$. Thus, for some positive integer j, $\Delta^{j+1}G + \lambda_1\Delta^j G + \cdots + \lambda_j\Delta G = 0$ never occurs unless $\Delta G = 0$. Therefore, G is harmonic.

Consequently, we have

Theorem 4.3. *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. If M has finite-type Gauss map G, then G is harmonic or of null 2-type.*

In [13], two of the present authors defined *a ruled submanifold of the B-scroll kind* in Minkowski m-space and in [15], the present authors defined a *G-kind ruled submanifold* in Minkowski m-space. For a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$, we consider a null frame $\{A(s), B(s) = e_1(s), e_2(s), \cdots, e_{m-1}(s)\}$ along $\tilde{\alpha}(s)$ satisfying

$$\langle A(s), A(s)\rangle = \langle B(s), B(s)\rangle = \langle A(s), e_i(s)\rangle = \langle B(s), e_i(s)\rangle = 0,$$
$$\langle A(s), B(s)\rangle = -1, \quad \langle e_i(s), e_j(s)\rangle = \delta_{ij}, \quad \tilde{\alpha}'(s) = A(s)$$

for $i, j = 2, 3, \cdots, m-1$.

Let $X(s)$ be the matrix $(A(s)\ B(s)\ e_2(s) \cdots e_{m-1}(s))$ consisting of column vectors of $A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_{m-1}(s)$ with respect to the standard coordinate system in $\mathbb{L}^m$. Then we have

$$X^t(s)EX(s) = T,$$

where $X^t(s)$ denotes the transpose of $X(s)$, $E =$diag$(-1,\ 1, \cdots,\ 1)$ and

$$T = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Consider a system of ordinary differential equations

$$X'(s) = X(s)M(s), \tag{4.2}$$

where

$$M(s) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{\beta} & 0 & \cdots & 0 \\ 0 & 0 & u_2 & u_3 & \cdots & u_r & u_{r+1} & u_{r+2} & \cdots & u_{m-1} \\ u_2 & 0 & 0 & 0 & \cdots & 0 & z_{2,r+1} & z_{2,r+2} & \cdots & z_{2,m-1} \\ u_3 & 0 & 0 & 0 & \cdots & 0 & z_{3,r+1} & z_{3,r+2} & \cdots & z_{3,m-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_r & 0 & 0 & 0 & \cdots & 0 & z_{r,r+1} & z_{r,r+2} & \cdots & z_{r,m-1} \\ u_{r+1} & \sqrt{\beta} & -z_{2,r+1} & -z_{3,r+1} & \cdots & -z_{r,r+1} & 0 & z_{r+1,r+2} & \cdots & z_{r+1,m-1} \\ u_{r+2} & 0 & -z_{2,r+2} & -z_{3,r+2} & \cdots & -z_{r,r+2} & -z_{r+1,r+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_{m-1} & 0 & -z_{2,m-1} & -z_{3,m-1} & \cdots & -z_{r,m-1} & -z_{r+1,m-1} & 0 & \cdots & 0 \end{pmatrix},$$

where u_j $(2 \le j \le m-1)$ and $z_{a,b}$ $(2 \le a \le r+1, \quad r+1 \le b \le m-1)$ are some smooth functions of s and β is a non-negative constant.

For a given initial condition $X(0) = (A(0)\ B(0)\ e_2(0)\cdots e_{m-1}(0))$ satisfying $X^t(0)EX(0) = T$, there exists a unique solution to $X'(s) = X(s)M(s)$ on the whole domain I of $\tilde{\alpha}(s)$ containing 0. Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X^t(s)EX(s)) = 0$ and hence we have

$$X^t(s)EX(s) = T$$

for all $s \in I$. Therefore, $A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_{m-1}(s)$ form a null frame along a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$ on I. Let $\alpha(s) = \int_0^s A(u)du$.

Then, we can define a parametrization for a ruled submanifold M by

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i e_i(s). \tag{4.3}$$

Definition 4.4. A ruled submanifold parametrized by (4.3) in the Minkowski m-space $\mathbb{L}^m$ with degenerate rulings satisfying (4.2) is called a *ruled submanifold of the B-scroll kind* or simply a *BS-kind ruled submanifold* ([13]). In particular, if $\beta = 0$ and $z_{a,b} = 0$ for $a, b = r+1, r+2, \cdots, m-1$, we call M an *extended BS-kind ruled submanifold* and if $\beta \neq 0$, M is called a *generalized BS-kind ruled submanifold*.

Remark 4.5. ([13]). If $r = 1$ and $m = 3$, then a BS-kind ruled surface is just an ordinary B-scroll. If $m \geq 4$, either an extended B-scroll or a generalized B-scroll in $\mathbb{L}^m$ is a BS-kind ruled submanifold. A time-like ruled surface over a null curve with degenerate rulings in $\mathbb{L}^m$ is called a null scroll.

We now consider another matrix $\tilde{M}(s)$:

$$\tilde{M}(s) =$$

$$\begin{pmatrix} 0 & 0 & v_2 & v_3 & \cdots & v_r & 0 & 0 & \cdots & 0 \\ 0 & 0 & u_2 & u_3 & \cdots & u_r & u_{r+1} & u_{r+2} & \cdots & u_{m-1} \\ u_2 & v_2 & 0 & 0 & \cdots & 0 & z_{2,r+1} & z_{2,r+2} & \cdots & z_{2,m-1} \\ u_3 & v_3 & 0 & 0 & \cdots & 0 & z_{3,r+1} & z_{3,r+2} & \cdots & z_{3,m-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_r & v_r & 0 & 0 & \cdots & 0 & z_{r,r+1} & z_{r,r+2} & \cdots & z_{r,m-1} \\ u_{r+1} & 0 & -z_{2,r+1} & -z_{3,r+1} & \cdots & -z_{r,r+1} & 0 & 0 & \cdots & 0 \\ u_{r+2} & 0 & -z_{2,r+2} & -z_{3,r+2} & \cdots & -z_{r,r+2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_{m-1} & 0 & -z_{2,m-1} & -z_{3,m-1} & \cdots & -z_{r,m-1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.4}$$

where v_i $(2 \leq i \leq r)$, u_j $(2 \leq j \leq m-1)$ and $z_{a,b}$ $(2 \leq a \leq r, \quad r+1 \leq b \leq m-1)$ are some smooth functions of s.

Then the matrix $X(s)$ satisfying

$$X'(s) = X(s)\tilde{M}(s) \tag{4.5}$$

generates a null frame along a null curve $\tilde{\alpha}$ in $\mathbb{L}^m$ on I by a similar argument above, that is, we can have a null frame $\{A(s),\ B(s),\ e_2(s) \cdots e_{m-1}(s)\}$ along $\tilde{\alpha}$. Again, let $\alpha(s) = \int_0^s A(u)du$.

Definition 4.6. A ruled submanifold M with the parametrization

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i e_i(s), \quad s \in J, \quad t_i \in I_i$$

satisfying (4.4) and (4.5) is called a *G-kind ruled submanifold* ([15]).

Remark 4.7. ([15]). An extended BS-kind ruled submanifold is a special case of G-kind ruled submanifold.

In [14], two of the present authors set up a characterization of minimal ruled submanifolds in Minkowski space and in [15], the present authors had a new characterization of minimal ruled submanifolds with degenerate rulings in Minkowski space by means of Gauss map as follows:

Theorem 4.8. ([15]). *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. The following are equivalent:*
(1) *M is minimal.*
(2) *M has harmonic Gauss map.*
(3) *M is an open portion of a G-kind ruled submanifold.*

By generalizing the above theorem, we now prove

Theorem 4.9. *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. M has finite-type Gauss map if and only if M is an open portion of a generalized BS-kind ruled submanifold or a G-kind ruled submanifold.*

Proof. We assume that the ruled submanifold M is parametrized by

$$x(s,t_1,t_2,\cdots,t_r)=\alpha(s)+t_1B(s)+\sum_{i=2}^{r}t_ie_i(s),\quad s\in J,\quad t_i\in I_i$$

such that $\langle A(s),A(s)\rangle=\langle B(s),B(s)\rangle=\langle A(s),e_i(s)\rangle=\langle B(s),e_i(s)\rangle=0$, $\langle A(s), B(s)\rangle=-1$, $\langle e_i(s),e_j(s)\rangle=\delta_{ij}$, and $\langle e_i'(s),e_j(s)\rangle=0$ for $i,j=2,3,\cdots,r$, where J and I_i are some open intervals and $\alpha'(s)=A(s)$. Furthermore, we assume that $\langle A(s),B'(s)\rangle=0$ for all s.

Suppose that M has finite-type Gauss map G. We then have two possible cases according to the degree of Q.

Case 1. Suppose that deg $Q(t)=0$. First, if $B'(s)\wedge B(s)\equiv 0$ for all s, B is a null constant vector field and the Gauss map G is harmonic according to Theorem 4.3. By Lemma 4.2, we see that M is minimal.

Let $V(s)=\{A(s),B(s),e_2(s),\cdots,e_r(s)\}$ be a smooth distribution of index 1 along α satisfying $\langle A(s),A(s)\rangle=\langle B(s),B(s)\rangle$=$\langle A(s),e_i(s)\rangle$= $\langle B(s),e_i(s)\rangle=0$, $\langle A(s),B(s)\rangle=-1$, $\langle e_i(s),e_j(s)\rangle=\delta_{ij}$, and $\langle e_i'(s),e_j(s)\rangle=0$ for all s and $i,j=2,3,\cdots,r$. Then, by Lemma 2.2, we can choose an orthonormal basis $\{e_{r+1},\cdots, e_{m-1}\}$ for the orthogonal complement $V^\perp(s)$ satisfying $e_h'(s)\in V(s)$ for all $h=r+1,\cdots,m-1$. Thus we may put

$$\begin{aligned}
&A'(s)=\sum_{i=2}^{m-1}u_i(s)e_i(s),\\
&B'(s)=0,\\
&e_j'(s)=u_j(s)B(s)+\sum_{a=r+1}^{m-1}(-z_{j,a}(s))e_a(s),\quad j=2,\cdots,r,
\end{aligned}$$

$$e'_a(s) = u_a(s)B(s) + \sum_{i=2}^{r} z_{i,a}(s)e_i(s), \quad a = r+1, \cdots, m-1.$$

For a certain initial condition the above system of linear ordinary differential equations has a unique solution to (4.2) with $z_{a,b} = 0$ $(a, b = r+1, r+2, \cdots, m-1)$. The solution defines part of an extended BS-kind ruled submanifold, which is a special case of G-kind ruled submanifold.

Next, we consider the case $B'(s) \wedge B(s) \neq 0$. Then, $B'(s)$ is a non-zero normal vector field to M. From Theorem 4.3, we see that β is a positive constant. Let $e_{r+1}(s) = \frac{1}{\sqrt{\beta}}B'(s)$. Let $V(s)$ be the vector space spanned by $A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_r(s)$, $e_{r+1}(s)$ along α. Then we have an orthonormal vector fields $e_{r+2}(s)$, $\cdots$, $e_{m-1}(s)$ which generate the orthogonal complement $V^{\perp}(s)$ satisfying $e'_h(s) \in V(s)$ for $h = r+2, \cdots, m-1$. Let $u_i(s) = \langle A'(s), e_i(s)\rangle$ for $i = 2, 3, \cdots, m-1$ and $z_{a,b}(s) = \langle e_a(s), e'_b(s)\rangle$ for $a, b = 2, 3, \cdots, m-1$. Then we have

$$\begin{aligned}
A'(s) &= \sum_{i=2}^{m-1} u_i(s)e_i(s), \\
B'(s) &= \sqrt{\beta}e_{r+1}(s), \\
e'_j(s) &= u_j(s)B(s) + \sum_{b=r+1}^{m-1} (-z_{j,b}(s))e_b(s), \quad j = 2, \cdots, r, \\
e'_{r+1}(s) &= \sqrt{\beta}A(s) + u_{r+1}(s)B(s) + \sum_{a=2}^{r} z_{a,r+1}(s)e_a(s) \\
&\quad + \sum_{b=r+2}^{m-1} (-z_{r+1,b}(s))e_b(s), \\
e'_h(s) &= u_h(s)B(s) + \sum_{a=2}^{r+1} z_{a,h}(s)e_a(s), \quad h = r+2, \cdots, m-1.
\end{aligned}$$

If we give a certain initial condition, then the above system of linear ordinary differential equations has a unique solution to (4.2), which defines part of a generalized BS-kind ruled submanifold.

Case 2. Suppose that deg $Q(t) = 1$. Let $V(s)$=$\{A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_r(s)\}$ be a smooth distribution of index 1 along α. Then we can choose an orthonormal basis $\{e_{r+1}$, $\cdots$, $e_{m-1}\}$ for the orthogonal complement $V^{\perp}(s)$ satisfying $e'_h(s) \in V(s)$ for $h = r+1, \cdots, m-1$. Then we may put

$$A'(s) = \sum_{i=2}^{m-1} u_i(s)e_i(s),$$

$$B'(s) = \sum_{i=2}^{m-1} v_i(s)e_i(s),$$

$$e_j'(s) = v_j(s)A(s) + u_j(s)B(s) + \sum_{b=r+1}^{m-1}(-z_{j,b}(s))e_b(s), \quad j = 2, \cdots, r,$$
$$e_a'(s) = v_a(s)A(s) + u_a(s)B(s) + \sum_{b=2}^{r} z_{b,a}(s)e_b(s), \quad a = r+1, \cdots, m-1.$$

Theorem 4.3 gives that if G is of finite-type, G is harmonic, i.e., $\Delta G = 0$. The straightforward computation provides

$$\begin{aligned}\Delta G =& \frac{2\bar{\varepsilon}}{Q^3}\sum_{h=r+1}^{m-1}\{(\sum_{i=1}^{r}\langle B', e_i'\rangle t_i - \sum_{i=2}^{r} v_i' t_i)v_h + v_h' Q\}e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r\\ &+ \frac{2\bar{\varepsilon}}{Q^2}\sum_{h=r+1}^{m-1} v_h^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r\\ &+ \frac{2\bar{\varepsilon}}{Q^2}\sum_{i=2}^{r}\sum_{h=r+1}^{m-1} v_i v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge A \wedge e_{i+1} \wedge \cdots \wedge e_r\\ &- \frac{2\bar{\varepsilon}}{Q^2}\sum_{i=2}^{r}\sum_{h,l=r+1}^{m-1} v_h z_{i,l} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_l \wedge e_{i+1} \wedge \cdots \wedge e_r.\end{aligned} \tag{4.6}$$

Thus, $\Delta G = 0$ yields that the functions v_a are vanishing for $a = r+1, \cdots, m-1$. Therefore, M is part of a G-kind ruled submanifold.

Conversely, for a generalized BS-kind ruled submanifold M, the function Q is a constant function with value 1 and we obtain $\Delta^2 G = 2\beta\Delta G$, that is, the Gauss map G of M is of null 2-type. If M is a G-kind ruled submanifold, then deg $Q \leq 1$. From equation (4.6), we can see that $v_{r+1} = \cdots = v_{m-1} = 0$ implies $\Delta G = 0$. Therefore, a G-kind ruled submanifold M has harmonic Gauss map G.

This completes the proof. ∎

From Theorem 4.9, we immediately have

Corollary 4.10. *There exist no ruled submanifolds with degenerate rulings that have k-type Gauss map ($k \geq 3$).*

If we put together the results of [13] and Theorem 4.9, we obtain the following theorem which shows the relationship between finite-type immersions and immersions with finite-type Gauss map when ruled submanifolds have degenerate rulings.

Theorem 4.11. *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. Then, the following are equivalent:*
(1) *M is an open part of a generalized BS-kind ruled submanifold.*
(2) *The Gauss map of M is of null 2-type.*
(3) *M is of null 2-type immersion.*

Remark 4.12. (1) In Theorem 4.3 of [13], two of the present authors proved that a ruled submanifold M of $\mathbb{L}^m$ with degenerate rulings is of finite-type if and only if it is of either 1-type or null 2-type. But, unfortunately, they dropped the case of being minimal when stating Theorem 4.5 and thus we would like to correct the statement of Theorem 4.5 in [13] as follows:

Let M be a ruled submanifold of $\mathbb{L}^m$ with degenerate rulings. Then, M is of finite-type if and only if it is either minimal or a BS-kind ruled submanifold.

(2) The BS-kind ruled submanifold in (1) is the generalized BS-kind ruled submanifold in the sense discussed in this paper.

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Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757
Korea
E-mail: dosokim@chonnam.ac.kr

Young Ho Kim and Sun Mi Jung
Department of Mathematics
Kyungpook National University
Taegu 702-701
Korea
E-mail: yhkim@knu.ac.kr
jung3756@hanmail.net