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COMPLEX DIFFERENTIAL EQUATIONS WITH SOLUTIONS IN THE HARDY SPACES

Li-peng Xiao

Abstract. In this paper, sufficient conditions for the analytic coefficients of the differential equation

$$(f^{(k)})^{n_k} + A_{k-1}(f^{(k-1)})^{n_{k-1}} + \cdots + A_1(f')^{n_1} + A_0 f = 0$$

are found such that all analytic solutions belong to a given $H_p^{\infty}-$ space, or to the Hardy space H^p. The results we obtain are a generalization of some earlier results by Heittokangas, Korhonen and Rättyä.

1. Introduction and Main Results

The growth of solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \tag{1.1}$$

with coefficients A_j in the unit disc has caused the attention and interest of many researchers recently. Nevanlinna theory has been applied for fast-growing analytic solutions [2, 3, 6, 7, 8, 13, 17], but the analysis on slowly growing solutions seems to require a different approach [12, 14, 15, 20, 21].

Pommerenke [20] studied the second-order equation

$$f'' + A(z)f = 0, \tag{1.2}$$

where $A(z)$ is analytic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. He found sufficient conditions for the coefficient function $A(z)$ such that all solutions of (1.2) belong to the Hardy space H^2, the space of analytic functions in $\mathbb{D}$ with square summable Taylor coefficients.

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Theorem 1.1. [20]. *There is a positive constant α with the following property: If the coefficient $A(z)$ of (1.2) is analytic in $\mathbb{D}$ and satisfies*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^3\frac{1-|a|^2}{|1-\overline{a}z|^2}d\sigma(z)\le\alpha,$$

then all solutions of (1.2) belong to H^2.

In the proof of Theorem 1.1, Pommerenke used the classical result by Carleson [4, 5] on bounded measures μ satisfying $\mu(S(I))=O(|I|), |I|\to 0$, where $|I|$ denotes the arc length of a subarc I of the boundary $\mathbb{T}=\{z:|z|=1\}$ and $S(I)=\{z\in\mathbb{D}: z/|z|\in I, 1-|I|\le|z|\}$. These measures are known as Carleson measures and the set $S(I)$ is called the Carleson box based on I. Pommerenke also proved the following formulation of Theorem 1.1 .

Theorem 1.2. [20]. *Let $0<\delta\le 1$. There is a positive constant α with the following property: If the coefficient $A(z)$ of (1.2) is analytic in $\mathbb{D}$ and satisfies*

$$\sup_{|I|\le\delta}\frac{1}{|I|}\int_{S(I)}|A(z)|^2(1-|z|^2)^3d\sigma(z)\le\alpha,$$

then all solutions of (1.2) belong to H^2.

The element of the Lebesgue area measure on $\mathbb{D}$ is denoted by $d\sigma(z)$.

Recently, Heittokangas *et al.* [16] studied equation (1.1) and found sufficient conditions for the analytic coefficients such that all solutions belong to H_p^∞. For $0<p<\infty$, the growth space H_p^∞ consists of those analytic functions f in $\mathbb{D}$, for which

$$\|f\|_{H_p^\infty}=\sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^p<\infty.$$

Theorem 1.3. [16]. *Let $0\le\delta<1$. For every $p>0$ there exists a positive constant α, depending only on p and k, such that if the coefficients $A_j(z)$ of (1.1) are analytic in $\mathbb{D}$ and satisfy*

$$\sup_{|z|\ge\delta}|A_j(z)|(1-|z|^2)^{k-j}\le\alpha,\quad j=0,\cdots,k-1,$$

then all solutions of (1.1) belong to H_p^∞.

Sufficient conditions for the coefficients such that all solutions belong to $\mathcal{D}^p$ were found in [21]. For $0<p<\infty$, the Dirichlet-type space $\mathcal{D}^p$ consists of those analytic functions f in $\mathbb{D}$ for which the integral

$$\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-1}d\sigma(z)$$

converges.

Theorem 1.4. [21]. *Let* $0 \le \delta < 1$. *For every* $0 < p \le 2$, *there is a positive constant* α, *depending only on* p *and* k, *such that if the coefficients* $A_j(z)$ *of (1.1) are analytic in* $\mathbb{D}$ *and satisfy*

$$\sup_{|a|\ge\delta}\int_{\mathbb{D}}|A_0(z)|^p(1-|z|^2)^{pk-1}\frac{1-|a|^2}{|1-\overline{a}z|^2}d\sigma(z)\le\alpha$$

and

$$\sup_{|z|\ge\delta}|A_j(z)|(1-|z|^2)^{k-j}\le\alpha,\quad j=1,\cdots,k-1,$$

then all solutions of (1.1) belong to $\mathcal{D}^p\cap H_p^\infty$.

Sufficient conditions for the coefficients such that all solutions belong to H^p were also found in [21]. For $0 < p < \infty$, the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$\|f\|_{H^p}=\sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}<\infty.$$

See [10] for the theory of Hardy spaces.

Theorem 1.5. [21]. *Let* $0 \le \delta < 1$. *For every* $0 < p < \infty$ *there is a positive constant* α, *depending only on* p *and* k, *such that if the coefficients* $A_j(z)$ *of (1.1) are analytic in* $\mathbb{D}$ *and satisfy*

$$\sup_{|a|\ge\delta}\int_{\mathbb{D}}|A_0(z)|^2(1-|z|^2)^{2k-1}\frac{1-|a|^2}{|1-\overline{a}z|^2}d\sigma(z)\le\alpha$$

and

$$\sup_{|z|\ge\delta}|A_j(z)|(1-|z|^2)^{k-j}\le\alpha,\quad j=1,\cdots,k-1,$$

then all solutions of (1.1) belong to $H^p\cap H_p^\infty$.

Theorem 1.5 has a different formulation in terms of Carleson measures.

Theorem 1.6. [21]. *Let* $0 \le \delta < 1$. *For every* $0 < p < \infty$ *there is a positive constant* α, *depending only on* p *and* k, *such that if the coefficients* $A_j(z)$ *of (1.1) are analytic in* $\mathbb{D}$ *and satisfy*

$$\sup_{|I|\le 1-\delta}\frac{1}{|I|}\int_{S(I)}|A_0(z)|^2(1-|z|^2)^{2k-1}d\sigma(z)\le\alpha$$

and

$$\sup_{|z|\ge\delta}|A_j(z)|(1-|z|^2)^{k-j}\le\alpha,\quad j=1,\cdots,k-1,$$

then all solutions of (1.1) belong to $H^p\cap H_p^\infty$.

The purpose of this study is to find sufficient conditions for the analytic coefficients of the differential equation

$$(f^{(k)})^{n_k} + A_{k-1}(f^{(k-1)})^{n_{k-1}} + \cdots + A_1(f')^{n_1} + A_0 f = 0 \tag{1.3}$$

where $n_j \geq 1$ for all $j = 1, \cdots, k$ and $n_j \leq n_k$ for all $j = 1, \cdots, k-1$, such that all analytic solutions belong to the H_p^∞ space, or to the Dirichlet-type space $\mathcal{D}^p$, or to the Hardy space H^p. The Eq.(1.3) was mentioned in [18]. Using the same idea with Rättyä and Heittokangas, we can obtain some results about the nonlinear differential equation (1.3).

The first result generalizes Theorem 1.3 to the differential equation (1.3).

Theorem 1.7. *Let $0 \leq \delta < 1$. For every $p > 0$ there exists a positive constant α, depending only on p, k and n_k, such that if the coefficients $A_j(z)$ of (1.3) are analytic in $\mathbb{D}$ and satisfy*

$$\sup_{|z|\geq\delta} |A_j(z)|(1-|z|^2)^{n_k(p+k)-n_j(p+j)} \leq \alpha, \quad j = 0, \cdots, k-1, \tag{1.4}$$

where $n_0 = 1$, then all analytic solutions of (1.3) belong to H_p^∞.

The second result of this article, Theorem 1.8 is also a generalization of Theorem 1.4.

Theorem 1.8. *Let $0 \leq \delta < 1$. For every $0 < p \leq 2$, there is a positive constant α, depending only on p, k and n_k, such that if the coefficients $A_j(z)$ of (1.3) are analytic in $\mathbb{D}$ and satisfy*

$$\sup_{|a|\geq\delta} \int_{\mathbb{D}} |A_0(z)|^p (1-|z|^2)^{pk-1} \frac{1-|a|^2}{|1-\overline{a}z|^2} d\sigma(z) \leq \alpha \tag{1.5}$$

and

$$\int_{|z|\geq\delta} |A_j(z)|^{\frac{p}{n_k-n_j}} (1-|z|^2)^{\frac{p(kn_k-jn_j)}{n_k-n_j}-1} d\sigma(z) \leq \alpha, \quad \textit{if} \quad n_j < n_k \tag{1.6}$$

$$\sup_{|z|\geq\delta} |A_j(z)|(1-|z|^2)^{n_k(k-j)} \leq \alpha, \quad \textit{if} \quad n_j = n_k \tag{1.7}$$

for $j = 1, \cdots, k-1$, then all analytic solutions of (1.3) belong to $\mathcal{D}^p \cap H_p^\infty$.

It is worth noticing that in Theorem 1.8 the containment in H_p^∞ follows by Theorem 1.7, since the condition in Theorem 1.7 is weaker than the conditions in Theorem 1.8 by a simply straightforward calculation. Theorem 1.9 below generalizes Theorem 1.5.

Theorem 1.9. *Let* $0 \leq \delta < 1$. *For every* $0 < p < \infty$ *there is a positive constant* α, *depending only on* p, k *and* n_k, *such that if the coefficients* $A_j(z)$ *of (1.3) are analytic in* $\mathbb{D}$ *and satisfy*

$$\sup_{|a|\geq\delta} \int_{\mathbb{D}} |A_0(z)|^2(1-|z|^2)^{2k-1} \frac{1-|a|^2}{|1-\overline{a}z|^2} d\sigma(z) \leq \alpha$$

and

$$\int_{|z|\geq\delta} |A_j(z)|^{\frac{2}{n_k-n_j}} (1-|z|^2)^{\frac{2(kn_k-jn_j)}{n_k-n_j}-2} d\sigma(z) \leq \alpha, \quad \textit{if} \quad n_j < n_k$$

$$\sup_{|z|\geq\delta} |A_j(z)|(1-|z|^2)^{n_k(k-j)} \leq \alpha, \quad \textit{if} \quad n_j = n_k$$

for $j = 1, \cdots, k-1$, *then all analytic solutions of (1.3) belong to* $H^p \cap H_p^\infty$.

Theorem 1.9 has a different formulation similar to Theorem 1.6 in terms of Carleson measures.

Theorem 1.10. *Let* $0 \leq \delta < 1$. *For every* $0 < p < \infty$ *there is a positive constant* α, *depending only on* p *and* k, *such that if the coefficients* $A_j(z)$ *of (1.3) are analytic in* $\mathbb{D}$ *and satisfy*

$$\sup_{|I|\leq 1-\delta} \frac{1}{|I|} \int_{S(I)} |A_0(z)|^2(1-|z|^2)^{2k-1} d\sigma(z) \leq \alpha \tag{1.8}$$

and

$$\int_{|z|\geq\delta} |A_j(z)|^{\frac{2}{n_k-n_j}} (1-|z|^2)^{\frac{2(kn_k-jn_j)}{n_k-n_j}-2} d\sigma(z) \leq \alpha, \quad \textit{if} \quad n_j < n_k \tag{1.9}$$

$$\sup_{|z|\geq\delta} |A_j(z)|(1-|z|^2)^{n_k(k-j)} \leq \alpha, \quad \textit{if} \quad n_j = n_k \tag{1.10}$$

for $j = 1, \cdots, k-1$, *then all analytic solutions of (1.3) belong to* $H^p \cap H_p^\infty$.

2. Lemmas for the Proof of Theorems

Lemma 2.1. ([24]). *Let* f *be an analytic function in* $\mathbb{D}, 1 < \alpha < \infty$ *and* $n \in \mathbb{N}$. *Then the following quantities are comparable:*

(1) $\|f\|_{H^\infty_{\alpha-1}}$,
(2) $\|f\|_{\mathcal{B}^\alpha} + |f(0)|$,
(3) $\sup_{z\in\mathbb{D}} |f^{(n)}(z)|(1-|z|^2)^{n-1+\alpha} + \sum_{j=0}^{n-1} |f^{(j)}(0)|$.

Two quantities A and B are comparable, if there exists a positive constant C such that $C^{-1}B \le A \le CB$. The α-*Bloch spaces* $\mathcal{B}^\alpha$ consist of those functions f, analytic in $\mathbb{D}$, for which

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty, \quad 0<\alpha<\infty.$$

Lemma 2.2. ([10]). *Let $a_n \ge 0$ for $n = 1, \cdots, N$. Then*

$$\left(\sum_{n=1}^N a_n\right)^p \le \left(\sum_{n=1}^N a_n^p\right), \quad 0<p\le 1,$$

and

$$\left(\sum_{n=1}^N a_n\right)^p \le N^{p-1}\left(\sum_{n=1}^N a_n^p\right), \quad 1\le p<\infty.$$

Lemma 2.3. ([19]). *Let f be an analytic function in $\mathbb{D}$, and let $0<p<\infty$. Then,*

$$\begin{aligned}\|f\|_{\mathcal{D}^p}^p &= \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-1} d\sigma(z)\\ &\simeq \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-1} d\sigma(z) + \sum_{j=1}^{n-1} |f^{(j)}(0)|^p, \quad n\ge 2,\end{aligned}$$

where the symbol $\simeq$ means that the quantities on the two sides of the symbol are comparable.

Lemma 2.4. *Let μ be a positive measure on $\mathbb{D}$, and let $0<p\le 2$ and $0\le\delta<1$. Then μ is a bounded Carleson measure if and only if there is a positive constant C, depending only on p, such that*

$$\int_{\mathbb{D}\setminus D(0,\delta)} |f(z)|^p d\mu(z) \le C(\|f\|_{\mathcal{D}^p}^p + |f(0)|^p) \tag{2.1}$$

for all analytic functions f in $\mathbb{D}$, in particular for all $f\in\mathcal{D}^p$. Moreover, if μ is a bounded Carleson measure, then $C = C_1C_2$, where C_1 is a positive constant and

$$C_2 = \sup_{|I|\le 1-\delta} \frac{\mu(S(I))}{|I|} \le 10 \sup_{|a|\ge\delta} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} d\mu(z). \tag{2.2}$$

In (2.1) and from there on $D(0,\delta)$ denotes the Euclidean disc centered at the origin and of radius δ. The case $\delta=0$ of Lemma 2.4 is a consequence of a well-known result by Carleson [4, 5] and the inequality $\|f\|_{H^p} \le C(\|f\|_{D^p} + |f(0)|)$ for $0<p\le 2$ [11, 22, 23]. If $0<\delta<1$, then an application of the case $\delta = 0$ to the measure μ_δ

such that $d\mu_\delta(z) = \chi_{\mathbb{D}\setminus D(0,\delta)} d\mu(z)$ shows that C is of the form $C = C_1 C_2$, where χ_E is the characteristic function of the set E, C_1 is a positive constant and

$$C_2 = \sup_I \frac{\mu(S(I)\setminus D(0,\delta))}{|I|}.$$

Then (2.2) follows by the inequalities

$$\begin{aligned}\sup_I \frac{\mu(S(I)\setminus D(0,\delta))}{|I|} &\le 2 \sup_{|I|\le 1-\delta} \frac{\mu(S(I))}{|I|} \\ &\le 20 \sup_{|a|\ge\delta} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2} d\mu(z).\end{aligned} \tag{2.3}$$

See [16] for a proof of (2.3).

Lemma 2.5. ([1]). *If $0 < p < \infty, n \in \mathbb{N}$ and f is an analytic function in $\mathbb{D}$, then*

$$\|f\|_{H^p}^p \simeq \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2} d\sigma(z) \right)^{p/2} |d\zeta| + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p,$$

where $\Gamma(\zeta) = \{z \in \mathbb{D} : |1 - z\overline{\zeta}| < 1 - |z|^2\}$ is the nontangential approach region with vertex ζ on $\mathbb{T}$, and the constants of the comparison depend only on p and n.

Lemma 2.6. ([9]). *Let $0 < p < \infty$. Let u be an analytic function in $\mathbb{D}$ and let h be a function defined on $\mathbb{D}$. There is a positive constant C, depending only on p, such that*

$$\begin{aligned}&\int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} \frac{|u(z)|^2 |h(z)|^2}{(1-|z|^2)^2} d\sigma(z) \right)^{p/2} |d\zeta| \\ &\le C \|u\|_{H^p}^p \left(\sup_I \frac{1}{|I|} \int_{S(I)} |h(z)|^2 \frac{d\sigma(z)}{1-|z|^2} \right)^{p/2}.\end{aligned}$$

3. Proof of Theorem 1.7

Proof. Let $0 \le \delta < 1, 1/2 \le \rho < 1$, and let f be an analytic solution of (1.3). Denote $f_\rho(z) = f(\rho z)$. Then, by Lemma 2.1, Lemma 2.2, (1.3) and the assumption (1.4),

$$\begin{aligned}\|f_\rho\|_{H_p^\infty} &\le C_1 \left(\sup_{z\in\mathbb{D}} |f^{(k)}(\rho z)|(1-|z|^2)^{p+k} + \sum_{j=0}^{k-1} |f^{(j)}(0)| \right) \\ &= C_1 \sup_{z\in\mathbb{D}} \left| \sum_{j=0}^{k-1} A_j(\rho z)(f^{(j)}(\rho z))^{n_j} \right|^{\frac{1}{n_k}} (1-|z|^2)^{p+k} \\ &\quad + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|\end{aligned}$$

$$
\begin{aligned}
&\leq C_1 \sup_{z\in\mathbb{D}} \sum_{j=0}^{k-1} |A_j(\rho z)|^{\frac{1}{n_k}} |f^{(j)}(\rho z)|^{\frac{n_j}{n_k}} (1-|z|^2)^{p+k} \\
&\quad + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)| \\
&\leq C_1 \sup_{|z|\geq\delta} \sum_{j=0}^{k-1} |A_j(\rho z)|^{\frac{1}{n_k}} |f^{(j)}(\rho z)|^{\frac{n_j}{n_k}} (1-|z|^2)^{p+k} \\
&\quad + C_1 C_2 + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)| \\
&\leq C_1 \sum_{j=0}^{k-1} \|f_\rho\|_{H_p^\infty}^{\frac{n_j}{n_k}} \left(\sup_{|z|\geq\delta} |A_j(\rho z)|(1-|z|^2)^{n_k(p+k)-n_j(p+j)} \right)^{\frac{1}{n_k}} \\
&\quad + C_1 C_2 + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)| \\
&\leq C_1 \left(\sum_{j=0}^{k-1} \|f_\rho\|_{H_p^\infty}^{\frac{n_j}{n_k}} \alpha^{\frac{1}{n_k}} + C_2 + \sum_{j=0}^{k-1} |f^{(j)}(0)| \right),
\end{aligned}
$$

where C_1 is a positive constant, depending only on p and k,

$$
C_2 = \sup_{0\leq|z|<\delta} \sum_{j=0}^{k-1} |A_j(\rho z)|^{\frac{1}{n_k}} |f^{(j)}(\rho z)|^{\frac{n_j}{n_k}} (1-|z|^2)^{p+k}.
$$

Without loss of generality, we can assume that $\|f_\rho\|_{H_p^\infty} > 1$, for otherwise the conclusion is clearly established. It follows that

$$
\|f_\rho\|_{H_p^\infty} \leq C_1(\alpha^{\frac{1}{n_k}} k \|f_\rho\|_{H_p^\infty} + C_2 + \sum_{j=0}^{k-1} |f^{(j)}(0)|),
$$

or

$$
\|f_\rho\|_{H_p^\infty}(1 - C_1 \alpha^{\frac{1}{n_k}} k) \leq C_1 C_2 + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|.
$$

The assertion is obtained by choosing α sufficiently small and letting $\rho \to 1^-$. This completes the proof of Theorem 1.7. ∎

4. Proof of Theorem 1.8

Proof. By Theorem 1.7 it suffices to show that all analytic solutions belong to $\mathcal{D}^p$ under the assumption (1.5)-(1.7). Let $0 \leq \delta < 1, 1/2 \leq \rho < 1$, and let f be an analytic solution of (1.3). Then, by Lemma 2.3 and (1.3),

$$
\begin{aligned}
\|f_\rho\|^p_{\mathcal{D}^p} &\le C_1\left(\int_{\mathbb{D}}|f^{(k)}(\rho z)|^p(1-|z|^2)^{pk-1}d\sigma(z)+\sum_{j=1}^{k-1}|f^{(j)}(0)|^p\right)\\
&= C_1\left(\int_{\mathbb{D}}|\sum_{j=0}^{k-1}A_j(\rho z)(f^{(j)}(\rho z))^{n_j}|^{\frac{p}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z)+C_2\right)\\
&\le C_3\left(\sum_{j=0}^{k-1}\int_{\mathbb{D}}|A_j(\rho z)|^{\frac{p}{n_k}}|f^{(j)}(\rho z)|^{\frac{n_jp}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z)+C_2\right)\\
&= C_3\sum_{j=0}^{k-1}\int_{\mathbb{D}\setminus D(0,\delta)}|A_j(\rho z)|^{\frac{p}{n_k}}|f^{(j)}(\rho z)|^{\frac{n_jp}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z)\\
&\quad +C_3C_2+C_3C_4\\
&= C_3\int_{\mathbb{D}\setminus D(0,\delta)}|A_0(\rho z)|^{\frac{p}{n_k}}|f(\rho z)|^{\frac{p}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z)\\
&\quad +C_3\sum_{j=1}^{k-1}\int_{\mathbb{D}\setminus D(0,\delta)}|A_j(\rho z)|^{\frac{p}{n_k}}|f^{(j)}(\rho z)|^{\frac{n_jp}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z)\\
&\quad +C_3C_2+C_3C_4\\
&= C_3(I_0+\sum_{j=1}^{k-1}I_j+C_2+C_4),
\end{aligned}
\tag{4.1}
$$

where C_1 is a positive constant, depending only on p and k, $C_2=\sum_{j=1}^{k-1}|f^{(j)}(0)|^p$, C_3 is a positive constant, depending only on p,k and n_k,

$$
\begin{aligned}
C_4 &= \sum_{j=0}^{k-1}\int_{D(0,\delta)}|A_j(\rho z)|^{\frac{p}{n_k}}|f^{(j)}(\rho z)|^{\frac{n_jp}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z),\\
I_0 &= \int_{\mathbb{D}\setminus D(0,\delta)}|A_0(\rho z)|^{\frac{p}{n_k}}|f(\rho z)|^{\frac{p}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z),\\
I_j &= \int_{\mathbb{D}\setminus D(0,\delta)}|A_j(\rho z)|^{\frac{p}{n_k}}|f^{(j)}(\rho z)|^{\frac{n_jp}{n_k}}(1-|z|^2)^{pk-1}d\sigma(z),
\end{aligned}
$$

for $j=1,\cdots,k-1$.

By Lemma 2.4 and the Hölder inequality, we have

$$
\begin{aligned}
I_0 &= \int_{\mathbb{D}\setminus D(0,\delta)} |A_0(\rho z)|^{\frac{p}{n_k}} |f(\rho z)|^{\frac{p}{n_k}} (1-|z|^2)^{pk-1} d\sigma(z) \\
&\le \left(\int_{\mathbb{D}\setminus D(0,\delta)} |A_0(\rho z)|^p |f(\rho z)|^p (1-|z|^2)^{pk-1} d\sigma(z) \right)^{\frac{1}{n_k}} \\
&\quad \times \left(\int_{\mathbb{D}\setminus D(0,\delta)} (1-|z|^2)^{pk-1} d\sigma(z) \right)^{1-\frac{1}{n_k}} \\
&\le (C_5 C_6 (\|f_\rho\|^p_{\mathcal{D}^p} + |f(0)|^p))^{\frac{1}{n_k}} C_7,
\end{aligned} \tag{4.2}
$$

where C_5 is an absolute positive constant,

$$
C_6 = \sup_{|a|\ge\delta} \int_{\mathbb{D}} |A_0(\rho z)|^p (1-|z|^2)^{pk-1} \frac{1-|a|^2}{|1-\overline{a}z|^2} d\sigma(z),
$$

$$
C_7 = \left(\int_{\mathbb{D}\setminus D(0,\delta)} (1-|z|^2)^{pk-1} d\sigma(z) \right)^{1-\frac{1}{n_k}}.
$$

For $I_j \quad (j = 1, \cdots, k-1)$, when $n_j = n_k$, by the assumption (1.7) and Lemma 2.3, we have,

$$
\begin{aligned}
I_j &= \int_{\mathbb{D}\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{p}{n_k}} |f^{(j)}(\rho z)|^p (1-|z|^2)^{pk-1} d\sigma(z) \\
&\le \alpha^{\frac{p}{n_k}} \int_{\mathbb{D}\setminus D(0,\delta)} |f^{(j)}(\rho z)|^p (1-|z|^2)^{pj-1} d\sigma(z) \\
&\le C_8 \alpha^{\frac{p}{n_k}} \|f_\rho\|^p_{\mathcal{D}^p},
\end{aligned} \tag{4.3}
$$

where $C_8 > 1$ is a positive constant, depending only on p and k.

When $n_j < n_k$, by assumption (1.6) and the Hölder inequality, we have,

$$
\begin{aligned}
I_j &= \int_{\mathbb{D}\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{p}{n_k}} |f^{(j)}(\rho z)|^{\frac{n_j p}{n_k}} (1-|z|^2)^{pk-1} d\sigma(z) \\
&\le \left(\int_{\mathbb{D}\setminus D(0,\delta)} |f^{(j)}(\rho z)|^p (1-|z|^2)^{pj-1} d\sigma(z) \right)^{\frac{n_j}{n_k}} \\
&\quad \times \left(\int_{\mathbb{D}\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{p}{n_k-n_j}} (1-|z|^2)^{\frac{p(kn_k-jn_j)}{n_k-n_j}-1} d\sigma(z) \right)^{1-\frac{n_j}{n_k}} \\
&\le \left(\int_{\mathbb{D}\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{p}{n_k-n_j}} (1-|z|^2)^{\frac{p(kn_k-jn_j)}{n_k-n_j}-1} d\sigma(z) \right)^{1-\frac{n_j}{n_k}} C_8^{\frac{n_j}{n_k}} \|f_\rho\|^{\frac{n_j}{n_k}p}_{\mathcal{D}^p}.
\end{aligned} \tag{4.4}
$$

Without loss of generality, we can assume that $\|f_\rho\|_{\mathcal{D}^p}^p > 1$, then combing (4.1)-(4.4), and letting $\rho \to 1^-$, the assumptions (1.5),(1.6) yield

$$\begin{aligned}&\|f\|_{\mathcal{D}^p}^p \left(1 - C_3 C_5^{\frac{1}{n_k}} \alpha^{\frac{1}{n_k}} C_7 - (k-1) C_3 C_8 \alpha^{\frac{1}{n_k}\min\{p,1\}}\right)\\ \le& C_3 C_5^{\frac{1}{n_k}} \alpha^{\frac{1}{n_k}} |f(0)|^{\frac{p}{n_k}} C_7 + C_2 C_3 + C_3 C_4,\end{aligned}$$

from which the assertion follows. This completes the proof of Theorem 1.8. ∎

5. Proof of Theorem 1.9 and Theorem 1.10

It suffices to prove Theorem 1.10 since Theorem 1.9 then follows by the second inequality in (2.2).

Proof. By Theorem 1.7, it suffices to show that all solutions belong to H^p under the assumptions (1.8)-(1.10). Let $0 \le \delta < 1, 1/2 \le \rho < 1$, and let f be an analytic solution of (1.3). Then, by Lemma 2.2, Lemma 2.5, Eq. (1.3),

$$\begin{aligned}
&\|f_\rho\|_{H^p}^p \\
\le\, & C_1 \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} |f^{(k)}(\rho z)|^2 (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
=\, & C_1 \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} \Big|\sum_{j=0}^{k-1} A_j(\rho z)(f^{(j)}(\rho z))^{n_j}\Big|^{\frac{2}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \\
& + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
\le\, & C_2 \sum_{j=0}^{k-1} \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} |A_j(\rho z)|^{\frac{2}{n_k}} |f^{(j)}(\rho z)|^{\frac{2n_j}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \qquad (5.1)\\
& + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
=\, & C_2 \sum_{j=1}^{k-1} \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{2}{n_k}} |f^{(j)}(\rho z)|^{\frac{2n_j}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \\
& + C_2 \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_0(\rho z)|^{\frac{2}{n_k}} |f(\rho z)|^{\frac{2}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \\
& + C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|^p + C_3,
\end{aligned}$$

where C_1 and C_2 are positive constants, depending only on p, k and n_k,

$$C_3 = C_2 \sum_{j=0}^{k-1} \int_{\mathbb{T}} \left(\int_{D(0,\delta) \bigcap \Gamma(\zeta)} |A_j(\rho z)|^{\frac{2}{n_k}} |f^{(j)}(\rho z)|^{\frac{2n_j}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta|.$$

To deal with the second term in (5.1), denote

$$J_0 := \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta) \setminus D(0,\delta)} |A_0(\rho z)|^{\frac{2}{n_k}} |f(\rho z)|^{\frac{2}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta|.$$

Then by the Hölder inequality and Lemma 2.6,

$$\begin{aligned} J_0 &\le \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_0(\rho z)|^2 |f(\rho z)|^2 (1-|z|^2)^{2k-2} d\sigma(z) \right)^{\frac{p}{2n_k}} \\ &\quad \times \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} (1-|z|^2)^{2k-2} d\sigma(z)) \right)^{(1-\frac{1}{n_k})\frac{p}{2}} |d\zeta| \\ &\le C_4 \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_0(\rho z)|^2 |f(\rho z)|^2 (1-|z|^2)^{2k-2} d\sigma(z) \right)^{\frac{p}{2n_k}} |d\zeta| \\ &\le C_5 \left(\int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_0(\rho z)|^2 |f(\rho z)|^2 (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \right)^{\frac{1}{n_k}}, \end{aligned} \tag{5.2}$$

where C_4, C_5 are positive constants.

Choosing $u(z) = f_\rho(z)$ and $h(z) = |A_0(\rho z)|(1-|z|^2)^k \chi_{\mathbb{D}\setminus D(0,\delta)}(z)$ in Lemma 2.6, it follows that there are positive constants C_6, C_7, depending only on p, such that

$$\begin{aligned} &\int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_0(\rho z)|^2 |f(\rho z)|^2 (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \\ &\le C_6 \|f_\rho\|_{H^p}^p \left(\sup_I \frac{1}{|I|} \int_{S(I)\setminus D(0,\delta)} |A_0(\rho z)|^2 (1-|z|^2)^{2k-1} d\sigma(z) \right)^{p/2} \\ &\le C_7 \|f_\rho\|_{H^p}^p \left(\sup_{|I|\le 1-\delta} \frac{1}{|I|} \int_{S(I)} |A_0(\rho z)|^2 (1-|z|^2)^{2k-1} d\sigma(z) \right)^{p/2}, \end{aligned} \tag{5.3}$$

where the last inequality follows by (2.3).

To deal with the first sum in (5.1), denote

$$J_j := \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{2}{n_k}} |f^{(j)}(\rho z)|^{\frac{2n_j}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta|,$$

for $j = 1, \cdots, k-1$.

When $n_j = n_k$, by (1.10) and Lemma 2.5,

$$
\begin{aligned}
J_j &= \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{2}{n_k}} |f^{(j)}(\rho z)|^2 (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \\
&\le \int_{\mathbb{T}} \alpha^{\frac{2}{n_k}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |f^{(j)}(\rho z)|^2 (1-|z|^2)^{2j-2} d\sigma(z) \right)^{p/2} |d\zeta| \qquad (5.4)\\
&\le C_8 \alpha^{\frac{p}{n_k}} \|f_\rho\|_{H^p}^p,
\end{aligned}
$$

where C_8 is a positive constant, depending only on p and k.

When $n_j < n_k$, by the Hölder inequality and Lemma 2.5,

$$
\begin{aligned}
J_j &= \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{2}{n_k}} |f^{(j)}(\rho z)|^{\frac{2n_j}{n_k}} (1-|z|^2)^{2k-2} d\sigma(z) \right)^{p/2} |d\zeta| \\
&\le \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |f^{(j)}(\rho z)|^2 (1-|z|^2)^{2j-2} d\sigma(z) \right)^{\frac{pn_j}{2n_k}} \\
&\quad \times \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{2}{n_k-n_j}} (1-|z|^2)^{\frac{2(kn_k-jn_j)}{n_k-n_j}-2} d\sigma(z) \right)^{\frac{p}{2}(1-\frac{n_j}{n_k})} |d\zeta| \\
&= \int_{\mathbb{T}} C_9^{1-\frac{n_j}{n_k}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |f^{(j)}(\rho z)|^2 (1-|z|^2)^{2j-2} d\sigma(z) \right)^{\frac{pn_j}{2n_k}} |d\zeta| \qquad (5.5)\\
&\le \left(\int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |f^{(j)}(\rho z)|^2 (1-|z|^2)^{2j-2} d\sigma(z) \right)^{\frac{p}{2}} |d\zeta| \right)^{\frac{n_j}{n_k}} \\
&\quad \times \left(\int_{\mathbb{T}} C_9 |d\zeta| \right)^{1-\frac{n_j}{n_k}} \\
&\le \left(C_8 \|f_\rho\|_{H^p}^p \right)^{\frac{n_j}{n_k}} \left(\int_{\mathbb{T}} C_9 |d\zeta| \right)^{1-\frac{n_j}{n_k}},
\end{aligned}
$$

where

$$
C_9 = \left(\int_{\Gamma(\zeta)\setminus D(0,\delta)} |A_j(\rho z)|^{\frac{2}{n_k-n_j}} (1-|z|^2)^{\frac{2(kn_k-jn_j)}{n_k-n_j}-2} d\sigma(z) \right)^{\frac{p}{2}}.
$$

Without loss of generality, we can assume that $\|f_\rho\|_{H^p}^p > 1$, then combining (5.1)-(5.5) and letting $\rho \to 1^-$, the assumption (1.8) and (1.9),

$$\|f\|_{H^p}^p \left(1 - 2\pi(k-1)C_2C_8\alpha^{\frac{p}{2n_k}} - C_2C_5C_7^{\frac{1}{n_k}}\alpha^{\frac{p}{2n_k}}\right)$$
$$\leq C_1 \sum_{j=0}^{k-1} |f^{(j)}(0)|^p + C_3,$$

from which the assertion follows by choosing α sufficiently small. This completes the proof of Theorem 1.10. ∎

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Li-peng Xiao
Institute of Mathematics and Informations
Jiangxi Normal University
Nanchang 330022
P. R. China
E-mail: lipeng_xiao08@yahoo.com