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RULED SUBMANIFOLDS WITH HARMONIC GAUSS MAP

Dong-Soo Kim[1], Young Ho Kim[2] and Sun Mi Jung

Abstract. Ruled submanifolds of Minkowski space with harmonic Gauss map are studied. Apart from ruled submanifolds in Euclidean space, ruled submanifolds with degenerate rulings in Minkowski space draw our attention. In particular, we completely classify ruled submanifolds with harmonic Gauss map and we also characterize minimal ruled submanifolds with degenerate rulings by means of harmonic Gauss map.

1. INTRODUCTION

In eighteenth century, the so-called minimal surfaces were introduced when the graph of a certain function minimizes the area among surfaces with the fixed boundary. Since then, the theory of minimal submanifolds has been one of the most interesting topics in differential geometry.

In 1966, T. Takahashi showed: Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into the Euclidean space $\mathbb{E}^m$ and Δ the Laplace operator defined on M. If $\Delta x = \lambda x$ $(\lambda \neq 0)$ holds, then M is a minimal submanifold in a hypersphere of Euclidean space ([17]). Extending this point of view, in the late 1970's B.-Y. Chen introduced the notion of finite type immersion of Riemannian manifolds into Euclidean space ([4, 5]). In particular, minimal submanifolds of Euclidean space can be considered as a spacial case of submanifolds of finite type or those with harmonic immersion. The notion of finite type immersion was extended to submanifolds in

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pseudo-Euclidean space in 1980's: A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}^m_s$ with signature $(m-s, s)$ is said to be of *finite type* if its position vector field x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + \Sigma_{i=1}^k x_i$, where x_0 is a constant map, $x_1, \cdots, x_k$ non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots, k$ ([4, 5]).

Such a notion can be naturally extended to a smooth map defined on submanifolds of pseudo-Euclidean space. A smooth map ϕ on an n-dimensional pseudo-Riemannian submanifold M of $\mathbb{E}^m_s$ is said to be of *finite type* if ϕ is a finite sum of $\mathbb{E}^m_s$-valued eigenfunctions of Δ. We also similarly define a smooth map of *k*-type on M as that of immersion x. A very typical and interesting smooth map on the submanifold M of Euclidean space or pseudo-Euclidean space is the Gauss map. In particular, we say that a differentiable map ϕ is *harmonic* if $\Delta\phi = 0$.

A ruled surface is one of the most natural geometric objects in the classical differential geometry and has been dealt with some geometric conditions ([1, 2, 6, 7, 11, 12, 13, 14, 15, 16]). Due to Beltrami equation, the submanifolds of Euclidean space or Minkowski space with harmonic immersion are the minimal ones.

We now have a question: *Can we completely classify ruled submanifolds in Minkowski space with harmonic Gauss map?*

In this article, we study ruled submanifolds in the Minkowski space $\mathbb{L}^m$ with harmonic Gauss map and we characterize minimal ruled submanifolds with degenerate rulings by means of harmonic Gauss map.

All of geometric objects under consideration are smooth and submanifolds are assumed to be connected unless otherwise stated.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(m-s, s)$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called a *Lorentz-Monkowski* m-space or simply *Minkowski* m-space, which is denoted by $\mathbb{L}^m$. A curve in $\mathbb{L}^m$ is said to be *space-like, time-like* or *null* if its tangent vector field is space-like, time-like or null, respectively. Let $x : M \rightarrow \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, each tangent space of the submanifold is non-degenerate.

Let $(x_1, x_2, \cdots, x_n)$ be a local coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on M induced from that of $\mathbb{E}^m_s$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x_i}(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x_j}).$$

We now choose an adapted local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ in $\mathbb{E}^m_s$ such that $e_1, e_2, \cdots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to M. The Gauss map $G : M \to G(n, m) \subset \mathbb{E}^N$ ($N = {}_mC_n$), $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$, of x is a smooth map which carries a point p in M to an oriented n-plane in $\mathbb{E}^m_s$ which is obtained from the parallel translation of the tangent space of M at p to an n-plane passing through the origin in $\mathbb{E}^m_s$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m_s$.

An indefinite scalar product $\ll \cdot, \cdot \gg$ on $G(n, m) \subset \mathbb{E}^N$ is defined by

$$\ll e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \gg = det(\langle e_{i_l}, e_{j_k} \rangle).$$

Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} | 1 \leq i_1 < \cdots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N_k$ for some positive integer k.

Now, we define a ruled submanifold M in $\mathbb{L}^m$. A non-degenerate $(r + 1)$-dimensional submanifold M in $\mathbb{L}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s, r)$ of $\mathbb{L}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Thus, a parametrization of a ruled submanifold M in $\mathbb{L}^m$ can be given by

$$x = x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I, \ t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \cdots, r$. For each s, $E(s, r)$ is open in Span$\{e_1(s), e_2(s), \cdots, e_r(s)\}$, which is the linear span of linearly independent vector fields $e_1(s)$, $e_2(s)$, $\cdots$, $e_r(s)$ along the curve α. Here we assume $E(s, r)$ are either non-degenerate or degenerate for all s along α. We call $E(s, r)$ the *rulings* and α the *base curve* of the ruled submanifold M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s, r)$ is parallel along α, or *non-cylindrical* otherwise.

Remark 2.1. ([9]).

(1) If the rulings of M are non-degenerate, then the base curve α can be chosen to be orthogonal to the rulings as follows: Let V be a unit vector field on M which is orthogonal to the rulings. Then α can be taken as an integral curve of V.

(2) If the rulings are degenerate, we can choose a null base curve which is transversal to the rulings: Let V be a null vector field on M which is not tangent to the rulings. An integral curve of V can be the base curve.

By solving a system of ordinary differential equations similarly set up in relation to a frame along a curve in $\mathbb{L}^m$ as given in [3], we have

Lemma 2.2. ([10]). *Let $V(s)$ be a smooth l-dimensional non-degenerate distribution in the Minkowski m-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$, where $l \geq 2$ and $m \geq 3$.*

Then, we can choose orthonormal vector fields $e_1(s), \cdots, e_{m-l}(s)$ *along* α *which generate the orthogonal complement* $V^{\perp}(s)$ *satisfying* $e_i'(s) \in V(s)$ *for* $1 \leq i \leq m-l$.

3. Non-degenerate Rulings

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings. By Remark 2.1, the base curve α can be chosen to be orthogonal to the rulings. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = \varepsilon (= \pm 1)$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 2.2, we may choose orthonormal vector fields $e_1(s), \cdots, e_r(s)$ along α satisfying

$$\langle \alpha'(s), e_i(s) \rangle = 0, \ \langle e_i'(s), e_j(s) \rangle = 0, \quad i, j = 1, 2, \cdots, r. \tag{3.1}$$

A parametrization of M is given by

$$x = x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s). \tag{3.2}$$

In this section, we always assume that the parametrization (3.2) satisfies the condition (3.1). Then, M has the Gauss map

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{|q|^{1/2}} (\Phi + \sum_{i=1}^{r} t_i \Psi_i), \tag{3.3}$$

where q is the function of $s, t_1, t_2, \cdots, t_r$ defined by

$$q = \langle x_s, x_s \rangle, \quad \Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r.$$

Now, we separate the cases into two typical types of ruled submanifolds which are cylindrical or non-cylindrical.

Theorem 3.1. *The cylindrical ruled submanifolds in* $\mathbb{L}^m$ *generated by non-degenerate rulings have harmonic Gauss map if and only if* M *is part of an* $(r+1)$*-plane or a cylinder over the base curve* $\alpha(s)$ *which is a plane curve in a degenerate plane given by* $\alpha(s) = s^2\boldsymbol{C} + s\boldsymbol{D}$ *for some constant null vector* $\boldsymbol{C}$ *and a constant space-like unit vector* $\boldsymbol{D}$ *satisfying* $\langle \boldsymbol{C}, \boldsymbol{D} \rangle = 0$.

Proof. Let M be a cylindrical $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings, which is parameterized by (3.2). We may assume that $e_1, e_2, \cdots, e_r$ generating the rulings are constant vectors.

The Laplacian Δ of M is then naturally expressed by

$$\Delta = -\varepsilon\frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r}\varepsilon_i\frac{\partial^2}{\partial t_i^2},$$

where $\varepsilon_i = \langle e_i(s), e_i(s)\rangle = \pm 1$ and the Gauss map G of M is given by

$$G = \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

If we denote by Δ' the Laplacian of α, that is $\Delta' = -\varepsilon\frac{\partial^2}{\partial s^2}$, we have the Laplacian ΔG of the Gauss map

$$\Delta G = \Delta'\alpha' \wedge e_1 \wedge \cdots \wedge e_r. \tag{3.4}$$

We now suppose that the Gauss map G is harmonic, that is $\Delta G = 0$. From (3.4), we have

$$\Delta'\alpha' = 0.$$

The converse is straightforward. ■

We need the following lemmas for later use.

Lemma 3.2. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3.2) *in $\mathbb{L}^m$. Suppose that e_1', e_2',$\cdots, e_r'$ are non-null and some of generators of rulings $e_1, \cdots, e_k$ are constant vector fields along α. Then we have the Laplacian*

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=k+1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\varepsilon_i\frac{\partial^2}{\partial t_i^2}.$$

Proof. The isometric immersion x of M can be put

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{k} t_i e_i(s) + \sum_{j=k+1}^{r} t_j e_j(s).$$

Then, we have

$$x_s = \alpha'(s) + \sum_{j=k+1}^{r} t_j e_j'(s), \quad x_{t_i} = e_i(s)$$

for $i = 1, 2, \cdots, r$. As we introduced in the beginning of this section, the function q is given by

$$q = \langle x_s, x_s\rangle = \varepsilon + \sum_{i=k+1}^{r} 2u_i t_i + \sum_{i,j=k+1}^{r} w_{ij} t_i t_j, \tag{3.5}$$

where $u_i = \langle \alpha', e_i' \rangle$, $w_{ij} = \langle e_i', e_j' \rangle$, $i, j = k+1, \cdots, r$. Note that q is a polynomial in $t = (t_{k+1}, \cdots, t_r)$ with functions in s as coefficients.

Then, the Laplacian Δ is easily obtained by

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=k+1}^{r}\varepsilon_i \frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\varepsilon_i \frac{\partial^2}{\partial t_i^2}. \qquad \blacksquare$$

From now on, for a polynomial $F(t)$ in $t = (t_1, t_2, \cdots, t_r)$, deg $F(t)$ denotes the degree of $F(t)$ in $t = (t_1, t_2, \cdots, t_r)$ unless otherwise stated.

By Lemma 3.2, $\Delta G = 0$ is rewritten as

$$\begin{aligned}(\frac{\partial q}{\partial s})^2(\Phi + \sum_{j=k+1}^{r}\Psi_j t_j) - \frac{3}{2}q\frac{\partial q}{\partial s}(\Phi' + \sum_{j=k+1}^{r}\Psi_j' t_j) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(\Phi + \sum_{j=k+1}^{r}\Psi_j t_j) \\ + q^2(\Phi'' + \sum_{j=k+1}^{r}\Psi_j'' t_j) + \frac{1}{2}q\sum_{i=k+1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2(\Phi + \sum_{j=k+1}^{r}\Psi_j t_j) \\ - \frac{1}{2}q^2\sum_{i=k+1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i - \frac{1}{2}q^2\sum_{i=k+1}^{r}\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}(\Phi + \sum_{j=k+1}^{r}\Psi_j t_j) = 0.\end{aligned} \tag{3.6}$$

To deal with (3.6), we have two possible cases either $\frac{\partial q}{\partial s} \neq 0$ or $\frac{\partial q}{\partial s} = 0$ on some open interval.

Lemma 3.3. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3.2) *in $\mathbb{L}^m$ with harmonic Gauss map. Let $e_1, e_2, \cdots, e_r$ be orthonormal generators of the rulings along α generating the rulings. If e_i' are non-null for $i = 1, 2, \cdots, r$ and some of generators of the rulings $e_1, \cdots, e_k$ are constant vector fields along α, then we have*

$$e_i' = \varepsilon u_i \alpha'.$$

Proof. We will prove this according to the following steps.

Step 1. In this step, we show that $w_{ij} = \varepsilon u_i u_j$ for $i, j = k+1, \cdots, r$.

Case 1. Suppose that $\frac{\partial q}{\partial s} \neq 0$. We may assume $\frac{\partial q}{\partial s} \neq 0$ on an open interval $\mathcal{I}$ for this case. Then, each term of the left side of (3.6) involves $(\frac{\partial q}{\partial s})^2$ or q and thus we have

$$(\frac{\partial q}{\partial s})^2 = q(t)P(t) \tag{3.7}$$

for some polynomial $P(t)$ in t of degree 2 with functions in s as coefficients. Comparing the both sides of (3.7), we can get

$$P(t) = \sum_{i,j}\theta_{ij}t_i t_j$$

for some symmetric functions θ_{ij} of s. Together with this equation and (3.7), we have

$$\varepsilon\theta_{ij} = 4u_i'u_j', \tag{3.8}$$

$$u_i\theta_{jh} + u_j\theta_{hi} + u_h\theta_{ij} = 2(u_i'w_{jh}' + u_j'w_{hi}' + u_h'w_{ij}'), \tag{3.9}$$

$$2(w_{ij}'w_{hl}' + w_{ih}'w_{jl}' + w_{li}'w_{jh}') = w_{ij}\theta_{hl} + w_{ih}\theta_{jl} + w_{jl}\theta_{ih} + w_{li}\theta_{jh} + w_{hl}\theta_{ij} + w_{jh}\theta_{li} \tag{3.10}$$

for $i, j, h, l = k+1, \cdots, r$. From (3.8) and (3.10), we see that $\varepsilon = 1$.

If $u_i' = 0$ for all $i = k+1, \cdots, r$, then $(\frac{\partial q}{\partial s}) = 0$, which is a contradiction. Therefore, there exists $i_0 \in \{k+1, \cdots, r\}$ such that $u_{i_0}' \neq 0$. And, (3.10) yields $(w_{i_0i_0}')^2 = 4w_{i_0i_0}(u_{i_0}')^2$. Also, (3.9) implies $(w_{i_0i_0}')^2 = 4u_{i_0}^2(u_{i_0}')^2$. Thus we have

$$w_{i_0i_0} = u_{i_0}^2 \tag{3.11}$$

for i_0 satisfying $u_{i_0}' \neq 0$.

By replacing h, l with i, j, respectively in (3.9) and (3.10), we get

$$2u_i\theta_{ij} + u_j\theta_{ii} = 4u_i'w_{ij}' + 2u_j'w_{ii}', \tag{3.12}$$

$$4w_{ij}'w_{ij}' + 2w_{ii}'w_{jj}' = 4w_{ij}\theta_{ij} + w_{ii}\theta_{jj} + w_{jj}\theta_{ii}. \tag{3.13}$$

If $u_{i_0}' \neq 0$ and $u_{j_0}' \neq 0$ for some i_0 and j_0, then equation (3.12) with the aid of (3.8) and (3.11) implies

$$w_{i_0j_0}' = u_{i_0}u_{j_0}' + u_{i_0}'u_{j_0}. \tag{3.14}$$

Substituting (3.14) into (3.13), we have

$$w_{i_0j_0} = u_{i_0}u_{j_0} \tag{3.15}$$

for i_0 and j_0.

Suppose that $u_i'(s_0) = 0$ at some s_0 for $i = k+1, \cdots, r$. Let

$$\Lambda_0 = \{i | u_i'(s_0) = 0,\ k+1 \leq i \leq r\}.$$

Then, at s_0, equation (3.7) can be written as

$$\begin{aligned} &4\sum_{i,j\notin\Lambda_0} u_i'u_j't_it_j + 4\sum_{i,j,h\notin\Lambda_0} u_i'w_{jh}'t_it_jt_h + \sum_{i,j,h,l\notin\Lambda_0} w_{ij}'w_{hl}'t_it_jt_ht_l \\ &= (1 + 2\sum_{i=k+1}^{r} u_it_i + \sum_{i,j=k+1}^{r} w_{ij}t_it_j)(\sum_{i,j\notin\Lambda_0}\theta_{ij}t_it_j), \end{aligned} \tag{3.16}$$

from which,

$$u_i(s_0) = 0 \tag{3.17}$$

and

$$w_{ij}(s_0) = 0 \tag{3.18}$$

for $i \in \Lambda_0$ and $j = k+1, \cdots, r$.

Therefore, by (3.15), (3.17) and (3.18), we have

$$w_{ij}(s) = u_i(s)u_j(s) \tag{3.19}$$

for all $i, j = k+1, \cdots, r$ and for all s.

Case 2. Suppose $\frac{\partial q}{\partial s} = 0$ on some open interval U.

On U, all functions u_i and w_{ij} are constants. So, from (3.6) we also have

$$\sum_{i=k+1}^{r} \varepsilon_i (\frac{\partial q}{\partial t_i})^2 = q(t)P_1(t), \tag{3.20}$$

where $P_1(t)$ is a function of t.

Suppose that there exist $j_1, \cdots, j_l \in \{k+1, \cdots, r\}$ such that $(\frac{\partial q}{\partial t_{j_i}})^2$ are not the multiples of $q(t)$ for $i = 1, \cdots, l$. Because of (3.20), we get

$$\sum_{i=1}^{l} \varepsilon_{j_i} (\frac{\partial q}{\partial t_{j_i}})^2 = q(t)P_2(t) \tag{3.21}$$

for some function $P_2(t)$. Since all of $(\frac{\partial q}{\partial t_{j_1}})^2, \cdots, (\frac{\partial q}{\partial t_{j_l}})^2$ are not the multiples of $q(t)$,

$$(\frac{\partial q}{\partial t_{j_i}})^2 = a_{j_i} q(t) + r_{j_i}(t)$$

for some constants a_{j_i} and polynomials $r_{j_i}(t)$ in t with deg $r_{j_i}(t) \le 1$ for $i = 1, \cdots, l$. Then, $\sum_{i=1}^{l} \varepsilon_{j_i} r_{j_i}(t)$ must be a multiple of $q(t)$ because of (3.21), which is a contradiction. Thus, we have

$$(\frac{\partial q}{\partial t_i})^2 = 4\varepsilon u_i^2 q(t)$$

for all $i = k+1, \cdots, r$. If we compare the both sides of the above equation, we easily see that

$$w_{ij} = \varepsilon u_i u_j \tag{3.22}$$

for all $i, j = k+1, \cdots, r$.

Step 2. In this step, we prove that $e_i' = \varepsilon u_i \alpha'$ for $i = k+1, \cdots, r$.

Equations (3.19) and (3.22) imply

$$\langle e_i' - \varepsilon u_i \alpha', e_i' - \varepsilon u_i \alpha' \rangle = 0 \tag{3.23}$$

for all $i = k+1, \cdots, r$.

Suppose that M is Lorentzian. Then, the normal space of M at each point is space-like. By (3.23), we see that the normal components of e_i' vanish and thus we get

$$e_i' = \varepsilon u_i \alpha' \tag{3.24}$$

for all $i = k+1, \cdots, r$.

We now assume that M is space-like. Then, $\varepsilon = 1$. Since $w_{ii} = u_i^2$ for $i = k+1, \cdots, r$, we can put

$$e_i' = u_i \alpha' + \sum_{a=r+1}^{m-1} \lambda_a^i e_a,$$

where $\sum_{a=r+1}^{m-1} \lambda_a^i e_a$ is vanishing or a null vector field along α.

Suppose $\sum_{a=r+1}^{m-1} \lambda_a^i e_a$ is a null vector field along α for some $i = k+1, \cdots, r$. By the hypothesis, u_i is non-zero. In case of $u_i'(s_0) = 0$ at some s_0, if we follow the argument developed in Case 1 of Step 1 above, we see that $u_i(s_0) = 0$, a contradiction. Therefore, $u_i' \neq 0$ for all s. Then, we get

$$\begin{aligned} q = \langle x_s, x_s \rangle &= (1 + \sum_{i=k+1}^{r} t_i u_i)^2 + \sum_{a=r+1}^{m-1} \varepsilon_a (\sum_{i=k+1}^{r} \lambda_a^i t_i)^2 \\ &= (1 + \sum_{i=k+1}^{r} t_i u_i)^2. \end{aligned}$$

Without loss of generality, we may assume that $1 + \sum_{i=k+1}^{r} t_i u_i > 0$. Hence we may put

$$G = \Phi + \frac{1}{\tilde{q}} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} \lambda_a^i t_i) \xi_a,$$

where $\xi_a = e_a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r$ for $a = r+1, \cdots, m-1$ and $\tilde{q}^2 = q$. By straightforward computation we have the Laplacian ΔG of the Gauss map

$$\Delta G = \frac{\sum_{j=k+1}^{r} u_j' t_j}{\tilde{q}^3} \Phi' - \frac{1}{\tilde{q}^2} \Phi'' + \left\{ \frac{\sum_{j=k+1}^{r} u_j'' t_j}{\tilde{q}^4} - \frac{3(\sum_{j=k+1}^{r} u_j' t_j)^2}{\tilde{q}^5} \right\} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} \lambda_a^i t_i) \xi_a$$

$$
\begin{aligned}
&+\frac{3\sum_{j=k+1}^{r} u_j' t_j}{\tilde{q}^4} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} (\lambda_a^i)' t_i)\xi_a + \frac{3\sum_{j=k+1}^{r} u_j' t_j}{\tilde{q}^4} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a' \\
&-\frac{1}{\tilde{q}^3} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} (\lambda_a^i)'' t_i)\xi_a - \frac{2}{\tilde{q}^3} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} (\lambda_a^i)' t_i)\xi_a' \\
&-\frac{1}{\tilde{q}^3} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a'' - \frac{1}{\tilde{q}^3} \sum_{i=k+1}^{r} u_i^2 (\sum_{a=r+1}^{m-1} (\sum_{h=k+1}^{r} \lambda_a^h t_h)\xi_a) \\
&+\frac{1}{\tilde{q}^2} \sum_{i=k+1}^{r} u_i (\sum_{a=r+1}^{m-1} \lambda_a^i \xi_a).
\end{aligned}
$$

Since the Gauss map is harmonic, $\Delta G = 0$ and thus

$$
\begin{aligned}
0 = {}& \tilde{q}^2 (\sum_{j=k+1}^{r} u_j' t_j)\Phi' - \tilde{q}^3 \Phi'' \\
&+ \{\tilde{q}(\sum_{j=k+1}^{r} u_j'' t_j) - 3(\sum_{j=k+1}^{r} u_j' t_j)^2\} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a \\
&+ 3\tilde{q}(\sum_{j=k+1}^{r} u_j' t_j) \sum_{a=r+1}^{m-1} \{(\sum_{i=k+1}^{r} (\lambda_a^i)' t_i)\xi_a + (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a'\} \\
&- \tilde{q}^2 \sum_{a=r+1}^{m-1} \{(\sum_{i=k+1}^{r} (\lambda_a^i)'' t_i)\xi_a - 2(\sum_{i=k+1}^{r} (\lambda_a^i)' t_i)\xi_a' - (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a''\} \\
&- \tilde{q}^2 \sum_{i=k+1}^{r} u_i^2 (\sum_{a=r+1}^{m-1} (\sum_{h=k+1}^{r} \lambda_a^h t_h)\xi_a) + \tilde{q}^3 \sum_{i=k+1}^{r} u_i (\sum_{a=r+1}^{m-1} \lambda_a^i \xi_a).
\end{aligned} \tag{3.25}
$$

In equation (3.25) all the coefficients of terms in t vanish. So, we can see easily that

$$
\Phi'' = \sum_{i=k+1}^{r} u_i (\sum_{a=r+1}^{m-1} \lambda_a^i \xi_a). \tag{3.26}
$$

Considering the coefficient of t_{i_0} for some $i_0 \in \{k+1, \cdots, r\}$, we have

$$
u_{i_0}' \Phi' = \sum_{a=r+1}^{m-1} \{(\lambda_a^{i_0})'' \xi_a - 2(\lambda_a^{i_0})' \xi_a' - \lambda_a^{i_0} \xi_a'' + (\sum_{h=k+1}^{r} u_h^2)\lambda_a^{i_0} \xi_a\}. \tag{3.27}
$$

Using (3.26) and (3.27), (3.25) is rewritten as

$$\begin{aligned}\tilde{q}(\sum_{j=k+1}^{r} u_j''t_j) - 3(\sum_{j=k+1}^{r} u_j't_j)^2\} \sum_{a=r+1}^{m-1} (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a \\ + 3\tilde{q}(\sum_{j=k+1}^{r} u_j't_j) \sum_{a=r+1}^{m-1} \{(\sum_{i=k+1}^{r} (\lambda_a^i)'t_i)\xi_a + (\sum_{i=k+1}^{r} \lambda_a^i t_i)\xi_a'\} = 0.\end{aligned} \tag{3.28}$$

Comparing the coefficients of $t_{i_0}^2$ and $t_{i_0}^3$, we obtain

$$(u_{i_0}')^2 \sum_{a=r+1}^{m-1} \lambda_a^{i_0}\xi_a = 0.$$

Since u_{i_0}' is non-zero, we know that $\lambda_a^{i_0} = 0$ for all $a = r+1, \cdots, m-1$.

Therefore, we have

$$e_i' = u_i\alpha' \tag{3.29}$$

for all $i = k+1, \cdots, r$.

Consequently, by (3.24) and (3.29), we obtain

$$e_i' = \varepsilon u_i\alpha'$$

for all $i = k+1, \cdots, r$. ∎

We now prove that the non-cylindrical ruled submanifold M in $\mathbb{L}^m$ satisfying the conditions of Lemma 3.3 is an $(r+1)$-plane.

Theorem 3.4. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3.2) *in $\mathbb{L}^m$ with harmonic Gauss map. Let $e_1, e_2, \cdots, e_r$ be orthonormal generators of the rulings along the base curve α. If e_i' are non-null for $i = 1, 2, \cdots, r$ and some of generators of the rulings $e_1, \cdots, e_k$ are constant vector fields along α, then M is part of an $(r+1)$-plane in $\mathbb{L}^m$.* *Proof.* By Lemma 3.3, we have

$$\begin{aligned}q = \langle x_s, x_s\rangle &= \langle (1+\varepsilon \sum_{i=k+1}^{r} t_iu_i)\alpha', (1+\varepsilon \sum_{j=k+1}^{r} t_ju_j)\alpha'\rangle \\ &= (1+\varepsilon \sum_{i=k+1}^{r} t_iu_i)^2\langle \alpha', \alpha'\rangle = \varepsilon(1+\varepsilon \sum_{i=k+1}^{r} t_iu_i)^2\end{aligned}$$

and hence $G = \Phi$. From $\Delta G = 0$ we obtain $\Phi'' + \varepsilon\Phi''u_it_i - \varepsilon\Phi'u_i't_i = 0$. Therefore, $\Phi'' = 0$ and hence $\Phi'u_i' = 0$.

If $u_i' \neq 0$ for some $i = k+1, \cdots, r$, we get $\Phi' = 0$.

Suppose that $u_i' \equiv 0$ for all $i = k+1, \cdots, r$. But, $\Phi'' = 0$ implies

$$\alpha''' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{i=k+1}^{r} \alpha'' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r = 0. \tag{3.30}$$

This gives

$$\alpha''' \wedge e_1 \wedge \cdots \wedge e_r \wedge e_j' = 0 \tag{3.31}$$

for all $j = k+1, \cdots, r$. By virtue of (3.31) and Lemma 3.3, we get $\alpha''' \wedge \alpha' = 0$. Together with this fact and (3.30), we have

$$\alpha'' = -\sum_{i=k+1}^{r} \varepsilon_i u_i e_i$$

and hence $\Phi' = 0$.

Therefore, M is part of an $(r+1)$-dimensional plane in $\mathbb{L}^m$. ∎

We now deal with the case that some of generators of rulings have null derivatives.

Lemma 3.5. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3.2) *in $\mathbb{L}^m$ with harmonic Gauss map. If some generators $e_{j_1}, e_{j_2}, \cdots, e_{j_k}$ of the rulings have null derivatives along the base curve α for $j_1 < j_2 < \cdots < j_k \in \{1, 2, \cdots, r\}$, then the Gauss map G has of the form*

$$G = \Phi + \sum_{i=1}^{k} t_{j_i} \Psi_{j_i} \tag{3.32}$$

for the harmonic vectors Φ and Ψ_{j_i}.

Proof. We can rewrite the parametrization (3.2) of M as

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i \neq j_1, j_2, \cdots, j_k} t_i e_i(s) + \sum_{i=1}^{k} t_{j_i} e_{j_i}(s)$$

and its Laplace operator is given by

$$\Delta = \frac{1}{2q^2} \frac{\partial q}{\partial s} \frac{\partial}{\partial s} - \frac{1}{q} \frac{\partial^2}{\partial s^2} - \frac{1}{2q} \sum_{i=1}^{r} \varepsilon_i \frac{\partial q}{\partial t_i} \frac{\partial}{\partial t_i} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2}.$$

Then, there are possible two cases such that either $e_{j_{k+1}}, \cdots, e_{j_r}$ generating the rulings except $e_{j_1}(s), e_{j_2}(s), \cdots, e_{j_k}(s)$ are constant vector fields or $e_i' \neq 0$ for some $i = j_{k+1}, \cdots, j_r$ if $k < r$.

Case 1. Suppose that $e_{j_{k+1}}, \cdots, e_{j_r}$ are constant vector fields.

Subcase 1.1. Let deg $q(t)=0$. In this case, e'_{j_i} are null with $e'_{j_i}(s)\wedge e'_{j_l}(s)=0$ for $i,l=1,2,\cdots,k$ and $\langle \alpha'(s), e'_j(s)\rangle = 0$ for $j=j_1,j_2,\cdots,j_k$. Then M has the Gauss map

$$G=\Phi+\sum_{i=1}^{k} t_{j_i}\Psi_{j_i}.$$

Thus, we have

$$\Delta G=-(\Phi''(s)+\sum_{i=1}^{k} t_{j_i}\Psi''_{j_i}(s)).$$

Hence, Φ and Ψ_{j_i} are harmonic if the Gauss map G is harmonic.

Subcase 1.2. Let deg $q(t)=1$. In this case, $\langle\alpha'(s), e'_{j_i}(s)\rangle \neq 0$ for some j_i $(1\le i\le k)$ and the null vector fields e'_{j_i} satisfy $e'_{j_i}\wedge e'_{j_l}=0$ for $i,l=1,2,\cdots,k$. The Gauss map G of M has the form

$$G=\frac{\tilde G(t)}{(\tilde\varepsilon q)^{1/2}},$$

where deg $\tilde G(t)\le 1$. Computing ΔG and using $\Delta G=0$, we get

$$\begin{aligned}(3.33)\qquad &(\frac{\partial q}{\partial s})^2(\Phi+\sum_{i=1}^{k}\Psi_{j_i}t_{j_i})-\frac{3}{2}q\frac{\partial q}{\partial s}(\Phi'+\sum_{i=1}^{k}\Psi'_{j_i}t_{j_i})-\frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(\Phi+\sum_{i=1}^{k}\Psi_{j_i}t_{j_i})\\ &+q^2(\Phi''+\sum_{i=1}^{k}\Psi''_{j_i}t_{j_i})+\frac{1}{2}q\sum_{i=1}^{k}\varepsilon_{j_i}(\frac{\partial q}{\partial t_{j_i}})^2(\Phi+\sum_{l=1}^{k}\Psi_{j_l}t_{j_l})\\ &-\frac{1}{2}q^2\sum_{i=1}^{k}\varepsilon_{j_i}\frac{\partial q}{\partial t_{j_i}}\Psi_{j_i}=0.\end{aligned}$$

Suppose $u'_{j_i}\equiv 0$ for all $i=1,\cdots,k$. Then, $\frac{\partial q}{\partial s}=0$ and $\frac{\partial^2 q}{\partial s^2}=0$. Together with (3.33) and these facts, we have

$$q(\Phi''+\sum_{i=1}^{k}\Psi''_{j_i}t_{j_i})+\frac{1}{2}\sum_{i=1}^{k}\varepsilon_{j_i}(\frac{\partial q}{\partial t_{j_i}})^2(\Phi+\sum_{l=1}^{k}\Psi_{j_l}t_{j_l})-\frac{1}{2}q\sum_{i=1}^{k}\varepsilon_{j_i}\frac{\partial q}{\partial t_{j_i}}\Psi_{j_i}=0.$$

In this equation, since $u_{j_0}\neq 0$ for some j_0, we can easily see that Ψ''_{j_0} vanishes. Then we obtain two equations as follows:

$$(3.34)\qquad \Phi''+2\varepsilon\sum_{i=1}^{k}\varepsilon_{j_i}u_{j_i}^2\Phi-\sum_{i=1}^{k}\varepsilon_{j_i}u_{j_i}\Psi_{j_i}=0,$$

$$(3.35)\qquad \sum_{i=1}^{k}\Phi''u_{j_i}t_{j_i}+\sum_{l=1}^{k}\sum_{i=1}^{k}\varepsilon_{j_i}u_{j_i}^2t_{j_l}\Psi_{j_l}-\sum_{i,l=1}^{k}\varepsilon_{j_i}u_{j_l}t_{j_l}u_{j_i}\Psi_{j_i}=0.$$

Substituting (3.34) into (3.35), we get

$$\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2 \{\sum_{l=1}^{k} (2\varepsilon u_{j_l} \Phi - \Psi_{j_l}) t_{j_l}\} = 0.$$

If $\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2 \equiv 0$, then there exists e_h for some $h = j_1, j_2, \cdots, j_k$ such that $\varepsilon_h = -1$ since deg $q(t) = 1$. It is a contradiction because of the causal character of e_h. Therefore, $\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2 \neq 0$ on an open interval $\mathcal{J}$. So, we have $2\varepsilon u_{j_i} \Phi = \Psi_{j_i}$ on $\mathcal{J}$, that is,

$$(2\varepsilon u_{j_i} \alpha' - e'_{j_i}) \wedge e_1 \wedge \cdots \wedge e_r = 0.$$

Since $2\varepsilon u_{j_i} \alpha' - e'_{j_i}$ is orthogonal to e_l for each $l = 1, 2, \cdots, r$, $2\varepsilon u_{j_i} \alpha' - e'_{j_i}$ has to be vanishing. But, it is a contradiction because of the characters of α' and e'_{j_i} for all $j_1, \cdots, j_k$.

Hence, there exists a non-zero function u'_{j_0} in some open interval $\mathcal{U}$ for some $j_0 = j_1, j_2, \cdots, j_k$.

On the other hand, equation (3.33) shows that all the coefficients of terms in t vanish. Especially, if we examine the coefficients of $t_{j_0}^3$, $t_{j_0}^2$, $t_{j_0}^1$ and $t_{j_0}^0$, then we have the following four equations:

$$4(u'_{j_0})^2 \Psi_{j_0} - 6u'_{j_0} u_{j_0} \Psi'_{j_0} - 2u''_{j_0} u_{j_0} \Psi_{j_0} + 4u_{j_0}^2 \Psi''_{j_0} = 0, \tag{3.36}$$

$$\begin{aligned} &4(u'_{j_0})^2 \Phi - 6u'_{j_0} u_{j_0} \Phi' - 2u''_{j_0} u_{j_0} \Phi + 4u_{j_0}^2 \Phi'' - 3\varepsilon u'_{j_0} \Psi'_{j_0} - \varepsilon u''_{j_0} \Psi_{j_0} + 4\varepsilon u_{j_0} \Psi''_{j_0} \\ &\qquad + 4(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2) u_{j_0} \Psi_{j_0} - 4(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i} \Psi_{j_i}) u_{j_0}^2 = 0, \end{aligned} \tag{3.37}$$

$$\begin{aligned} &- 3\varepsilon u'_{j_0} \Phi' - \varepsilon u''_{j_0} \Phi + \Psi''_{j_0} + 4\varepsilon u_{j_0} \Phi'' + 4(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2) u_{j_0} \Phi \\ &+ 2\varepsilon(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2) \Psi_{j_0} - 4\varepsilon(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i} \Psi_{j_i}) u_{j_0} = 0, \end{aligned} \tag{3.38}$$

$$\Phi'' + 2\varepsilon(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2) \Phi - (\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i} \Psi_{j_i}) = 0. \tag{3.39}$$

Substituting (3.39) into (3.38), we get

$$-3\varepsilon u'_{j_0} \Phi' - \varepsilon u''_{j_0} \Phi + \Psi''_{j_0} - 4(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2) u_{j_0} \Phi + 2\varepsilon(\sum_{i=1}^{k} \varepsilon_{j_i} u_{j_i}^2) \Psi_{j_0} = 0. \tag{3.40}$$

Putting (3.36) and (3.39) into (3.37), we obtain

$$
\begin{aligned}
&4(u'_{j_0})^2\Phi - 6u'_{j_0}u_{j_0}\Phi' - 2u''_{j_0}u_{j_0}\Phi + 3\varepsilon u'_{j_0}\Psi'_{j_0} + \varepsilon u''_{j_0}\Psi_{j_0} \\
&+4(\sum_{i=1}^{k}\varepsilon_{j_i}u_{j_i}^2)u_{j_0}\Psi_{j_0} - 8\varepsilon u_{j_0}^2(\sum_{i=1}^{k}\varepsilon_{j_i}u_{j_i}^2)\Phi - 4\varepsilon\frac{(u'_{j_0})^2}{u_{j_0}}\Psi_{j_0} = 0.
\end{aligned}
\tag{3.41}
$$

Multiplying $2\varepsilon u_{j_0}$ with (3.40) and substituting the equation obtained in such a way into (3.41), we get $2\varepsilon u_{j_0}\Phi = \Psi_{j_0}$ because u'_{j_0} is non-zero.

Then, one can easily see that $\alpha' \wedge e'_{j_0} = 0$, which is a contradiction. Therefore, we can conclude that no ruled submanifolds with deg $q = 1$ have harmonic Gauss map.

Subcase 1.3. Let deg $q(t) = 2$. Using the similar argument developed in Lemma 3.3, we have

$$\alpha' \wedge e'_i = 0$$

for $i = j_1, \cdots, j_k$, which is a contradiction. Therefore, no ruled submanifolds with deg $q = 2$ have harmonic Gauss map G.

Case 2. Suppose that $e'_i \neq 0$ for some $i = j_{k+1}, \cdots, j_r$.

In this case, we may assume that $e'_i \neq 0$ for all $i = j_{k+1}, \cdots, j_r$, otherwise the ruled submanifold M is a cylinder built over the ruled submanifold parameterized by the base curve α and the rulings generated by e_i's except those constant vector fields. Then, e'_i are non-null for all $i = j_{k+1}, \cdots, j_r$ and deg q= 2.

If we again follow a similar argument in the proof of Lemma 3.3 , we have

$$\alpha' \wedge e'_i = 0$$

for all $i = 1, 2, \cdots, r$. This is a contradiction.

This completes the proof. ■

It is easy to show that if the Gauss map G of a ruled submanifold with non-degenerate rulings in $\mathbb{L}^m$ has of the form (3.32), G is harmonic. Therefore, combining the results of Theorem 3.4 and Proposition 3.5, we conclude

Theorem 3.6. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has harmonic Gauss map if and only if M is part of either an $(r+1)$-plane or a ruled submanifold up to cylinders over a certain submanifold with the parametrization given by*

$$x(s, t_1, t_2, \cdots, t_r) = f(s)\boldsymbol{N} + s\boldsymbol{E} + \sum_{j=1}^{r} t_j(p_j(s)\boldsymbol{N} + \boldsymbol{F}_j)$$

for some smooth functions f and p_j, and some constant vector fields $\boldsymbol{N}, \boldsymbol{E}, \boldsymbol{F}_j$ with $\langle \boldsymbol{E}, \boldsymbol{E}\rangle = 1$, $\langle \boldsymbol{N}, \boldsymbol{N}\rangle = \langle \boldsymbol{N}, \boldsymbol{E}\rangle = \langle \boldsymbol{N}, \boldsymbol{F}_j\rangle = \langle \boldsymbol{E}, \boldsymbol{F}_j\rangle = 0$, and $\langle \boldsymbol{F}_j, \boldsymbol{F}_i\rangle = \delta_{ji}$ for $i, j = 1, 2, \cdots, r$.

Proof. Suppose that M has harmonic Gauss map.

We now suppose that a non-cylindrical ruled submanifold M with non-degenerate rulings is not part of an $(r+1)$-plane and it is parameterized by (3.2). We may also assume that the derivatives of the orthonormal vector fields $e_1, e_2, \cdots, e_r$ defining the rulings never vanish, otherwise M is a cylinder built over those submanifolds. As we see in the proof of Lemma 3.5, only Subcase 1.1 can occur. Therefore, we have $q = 1$ and e'_j are null vector fields with $e'_j(s) \wedge e'_k(s) = 0$ and $\langle \alpha'(s), e'_j(s) \rangle = 0$ for $j, k = 1, 2, \cdots, r$. Then $\Delta G = 0$ implies that

$$\Phi''(s) = 0 \tag{3.42}$$

and

$$\Psi''_j(s) = 0 \tag{3.43}$$

for all $j = 1, 2, \cdots, r$. Since $e'_j \wedge e'_k = 0$ for $j, k = 1, 2, \cdots, r$, we have

$$\Psi''_j = e'''_j \wedge e_1 \wedge \cdots \wedge e_r + \sum_{i=1}^{r} e''_j \wedge \cdots \wedge e'_i \wedge \cdots \wedge e_r = 0$$

for each $j = 1, 2, \cdots, r$. This implies

$$e'''_j \wedge e_1 \wedge \cdots \wedge e_r \wedge e'_l = 0 \tag{3.44}$$

for $j, l = 1, 2, \cdots, r$. Therefore, the vector fields $e'''_j, e_1, \cdots, e_r, e'_l$ are linearly dependent for all s. So, (3.1) and the fact that $e'_j \wedge e'_i = 0$ for $i, j = 1, 2, \cdots r$ imply

$$e'''_j \wedge e'_l = 0$$

for $j, l = 1, 2, \cdots, r$.

Since $e'''_j \wedge e'_k = 0$ and $e'_j \wedge e'_k = 0$ for all $j, k = 1, 2, \cdots r$, we get

$$e''_j \wedge e'_k = 0 \tag{3.45}$$

for all $j, k = 1, 2, \cdots, r$.

On the other hand, (3.42) gives

$$0 = \alpha''' \wedge e_1 \wedge \cdots \wedge e_r + 2 \sum_{i=1}^{r} \alpha'' \wedge \cdots \wedge e'_i \wedge \cdots \wedge e_r + \sum_{i=1}^{r} \alpha' \wedge \cdots \wedge e''_i \wedge \cdots \wedge \cdots \wedge e_r.$$

Thus, we have

$$\alpha''' \wedge e_1 \wedge \cdots \wedge e_r \wedge e'_j = 0$$

and hence

$$\alpha''' \wedge e'_j = 0 \tag{3.46}$$

for all $j = 1, 2, \cdots, r$.

Since $\langle \alpha', \alpha' \rangle = 1$, $\langle \alpha', e'_j \rangle = 0$ and $e''_j \wedge e'_k = 0$ for $j, k = 1, 2, \cdots, r$, we see that

$$\langle \alpha'', \alpha'' \rangle = 0$$

along α. From this, $\alpha'' = 0$ or α'' is null and hence up to translation we may put

$$\alpha(s) = f(s)\mathbf{N} + s\mathbf{E}, \tag{3.47}$$

where **N** is a constant null vector, **E** a constant space-like unit vector satisfying $\langle \mathbf{N}, \mathbf{E} \rangle = 0$ and f a smooth function.

Since $e''_j \wedge e'_i = 0$ and $\alpha'' \wedge e'_j = 0$ for all $i, j = 1, 2, \cdots, r$, we may have

$$e_j = p_j(s)\mathbf{N} + \mathbf{F}_j$$

for some non-zero smooth function p_j and orthonormal space-like constant vector fields $\mathbf{F}_j$ along α satisfying $\langle \mathbf{N}, \mathbf{F}_j \rangle = 0$ for $j = 1, 2, \cdots, r$.

Consequently, up to translation the parametrization (3.2) of M can be put

$$x(s, t_1, t_2, \cdots, t_r) = f(s)\mathbf{N} + s\mathbf{E} + \sum_{j=1}^{r} t_j (p_j(s)\mathbf{N} + \mathbf{F}_j). \tag{3.48}$$

Conversely, for some smooth functions f and p_j defined along α and some constant vector fields $\mathbf{N}, \mathbf{E}, \mathbf{F}_j$ ($j = 1, 2, \cdots, r$) satisfying above conditions, it is easy to show that a non-cylindrical ruled submanifold parameterized by (3.48) satisfies

$$\Delta G = 0.$$

This completes the proof. ■

Remark. In Theorem 3.6, if the base curve α is a straight line and the generators e_i satisfy $e''_i = 0$ along α ($i = 1, 2, \cdots, r$), the ruled submanifold M is minimal.

4. Degenerate Rulings

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with degenerate rulings $E(s, r)$ along a regular curve and let its parametrization be given by $\tilde{x}(s, t)$ where $t = (t_1, t_2, \cdots, t_r)$. Since $E(s, r)$ is degenerate, it can be spanned by a degenerate frame $\{B(s) = e_1(s), e_2(s), \cdots, e_r(s)\}$ such that

$$\langle B(s), B(s) \rangle = \langle B(s), e_i(s) \rangle = 0, \quad \langle e_i(s), e_j(s) \rangle = \delta_{ij}, \quad i, j = 2, 3, \cdots, r.$$

Without loss of generality as Lemma 2.2, we may assume that

$$\langle e_i'(s), e_j(s)\rangle = 0, \quad i, j = 2, 3, \cdots, r.$$

Since the tangent space of M at $\tilde{x}(s,t)$ is a Minkowski $(r+1)$-space which contains the degenerate ruling $E(s,r)$, there exists a tangent vector field A to M which satisfies

$$\langle A(s,t), A(s,t)\rangle = 0, \quad \langle A(s,t), B(s)\rangle = -1, \quad \langle A(s,t), e_i(s)\rangle = 0, \quad i = 2, 3, \cdots, r$$

at $\tilde{x}(s,t)$.

Let $\alpha(s)$ be an integral curve of the vector field A on M. Then we can define another parametrization x of M as follows:

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s),$$

where $\alpha'(s) = A(s)$.

Lemma 4.1. ([9]). *We may assume that* $\langle A(s), B'(s)\rangle = 0$ *for all* s.

Two of the present authors proved the following lemma.

Lemma 4.2. ([10]). *Let* M *be a ruled submanifold with degenerate rulings. Then, the following are equivalent.*

(1) M *is minimal.*

(2) $B'(s)$ *is tangent to* M.

If we put $P = \langle x_s, x_s\rangle$ and $Q = -\langle x_s, x_{t_1}\rangle$, Lemma 4.1 implies

$$P(s,t) = 2\sum_{i=2}^{r} u_i(s)t_i + \sum_{i,j=1}^{r} w_{ij}(s)t_i t_j,$$
$$Q(s,t) = 1 + \sum_{i=2}^{r} v_i(s)t_i,$$

where $v_i(s) = \langle B'(s), e_i(s)\rangle$, $u_i(s) = \langle A(s), e_i'(s)\rangle$, $w_{ij}(s) = \langle e_i'(s), e_j'(s)\rangle$ for $i, j = 1, 2, \cdots, r$. Note that P and Q are polynomials in $t = (t_1, t_2, \cdots, t_r)$ with functions in s as coefficients. Then the Laplacian Δ of M can be expressed as follows:

$$\Delta = \frac{1}{Q^2}\{\frac{\partial \bar{P}}{\partial t_1}\frac{\partial}{\partial t_1} - 2Q\sum_{i=2}^{r} v_i \frac{\partial}{\partial t_i} + 2Q\frac{\partial^2}{\partial s \partial t_1} + \bar{P}\frac{\partial^2}{\partial t_1^2}$$
$$- 2Q\sum_{i=2}^{r} v_i t_1 \frac{\partial^2}{\partial t_1 \partial t_i} - Q^2 \sum_{i=2}^{r}\frac{\partial^2}{\partial t_i^2}\},$$

where $\bar{P} = P - t_1^2\sum_{i=2}^{r} v_i^2$.

By definition of an indefinite scalar product $\ll, \gg$ on $G(r+1, m)$, we may put

$$\ll x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r}, x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \gg = -Q^2.$$

Let $\bar{\varepsilon} =$ sign $Q(t)$. Then we have the Gauss map

$$\begin{aligned} G =& \frac{1}{\bar{\varepsilon}Q} x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \\ =& \frac{1}{\bar{\varepsilon}Q} \{A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + t_1 B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &+ \sum_{i=2}^{r} t_i e_i' \wedge B \wedge e_2 \wedge \cdots \wedge e_r\}. \end{aligned}$$

We now define a G-*kind ruled submanifold* in Minkouski m-space. For a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$, we consider a null frame $\{A(s), B(s) = e_1(s), e_2(s), \cdots, e_{m-1}(s)\}$ along $\tilde{\alpha}(s)$ satisfying

$$\langle A(s), A(s)\rangle = \langle B(s), B(s)\rangle = \langle A(s), e_i(s)\rangle = \langle B(s), e_i(s)\rangle = 0,$$
$$\langle A(s), B(s)\rangle = -1, \quad \langle e_i(s), e_j(s)\rangle = \delta_{ij}, \quad \tilde{\alpha}'(s) = A(s)$$

for $i, j = 2, 3, \cdots, m-1$.

Let $X(s)$ be the matrix $(A(s)\ B(s)\ e_2(s) \cdots e_{m-1}(s))$ consisting of column vectors of $A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_{m-1}(s)$ with respect to the standard coordinate system in $\mathbb{L}^m$. Then we have

$$X^t(s) E X(s) = T,$$

where $X^t(s)$ denotes the transpose of $X(s)$, $E =$diag$(-1,\ 1, \cdots,\ 1)$ and

$$T = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Consider a system of ordinary differential equations

(4.1) $$X'(s) = X(s) M(s),$$

where

$$M(s)=\begin{pmatrix} 0 & 0 & v_2 & v_3 & \cdots & v_r & 0 & 0 & \cdots & 0 \\ 0 & 0 & u_2 & u_3 & \cdots & u_r & u_{r+1} & u_{r+2} & \cdots & u_{m-1} \\ u_2 & v_2 & 0 & 0 & \cdots & 0 & z_{2,r+1} & z_{2,r+2} & \cdots & z_{2,m-1} \\ u_3 & v_3 & 0 & 0 & \cdots & 0 & z_{3,r+1} & z_{3,r+2} & \cdots & z_{3,m-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_r & v_r & 0 & 0 & \cdots & 0 & z_{r,r+1} & z_{r,r+2} & \cdots & z_{r,m-1} \\ u_{r+1} & 0 & -z_{2,r+1} & -z_{3,r+1} & \cdots & -z_{r,r+1} & 0 & 0 & \cdots & 0 \\ u_{r+2} & 0 & -z_{2,r+2} & -z_{3,r+2} & \cdots & -z_{r,r+2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_{m-1} & 0 & -z_{2,m-1} & -z_{3,m-1} & \cdots & -z_{r,m-1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where v_i $(2 \le i \le r)$, u_j $(2 \le j \le m-1)$ and $z_{a,b}$ $(2 \le a \le r,\ \ r+1 \le b \le m-1)$ are some smooth functions of s.

For a given initial condition $X(0) = (A(0)\ B(0)\ e_2(0) \cdots e_{m-1}(0))$ satisfying $X^t(0)EX(0) = T$, there exists a unique solution to $X'(s) = X(s)M(s)$ on the whole domain I of $\tilde{\alpha}(s)$ containing 0. Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X^t(s)EX(s)) = 0$ and hence we have

$$X^t(s)EX(s) = T$$

for all $s \in I$. Therefore, $A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_{m-1}(s)$ form a null frame along a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$ on I. Let $\alpha(s) = \int_0^s A(u)du$.

Then, we can define a parametrization for a ruled submanifold M by

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i e_i(s). \tag{4.2}$$

Definition 4.3. A ruled submanifold M with the parametrization

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i e_i(s), \quad s \in J, \quad t_i \in I_i$$

satisfying (4.1) and (4.2) is called a *G-kind ruled submanifold.*

Remark 4.4. The terminology G-kind ruled submanifolds is named by ruled submanifolds generated by the Gauss map.

We now prove

Theorem 4.5. *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. M has harmonic Gauss map if and only if M is an open portion of a G-kind ruled submanifold.*

Proof. We assume that the ruled submanifold M is parameterized by

$$x(s,t_1,t_2,\cdots,t_r)=\alpha(s)+t_1B(s)+\sum_{i=2}^{r}t_ie_i(s),\quad s\in J,\quad t_i\in I_i$$

such that $\langle A(s),A(s)\rangle=\langle B(s),B(s)\rangle=\langle A(s),e_i(s)\rangle=\langle B(s),e_i(s)\rangle=0$, $\langle A(s), B(s)\rangle=-1$, $\langle e_i(s),e_j(s)\rangle=\delta_{ij}$, and $\langle e_i'(s),e_j(s)\rangle=0$ for $i,j=2,3,\cdots,r$, where J and I_i are some open intervals and $\alpha'(s)=A(s)$. Furthermore, we assume that $\langle A(s),B'(s)\rangle=0$ for all s.

Suppose that M has harmonic Gauss map G. We then have two possible cases according to the degree of Q.

Case 1. Suppose that deg $Q(t)=0$, that is, $Q=1$ and $v_i(s)=0$ for all $i=2,3,\cdots,r$. By definition, we get

$$\begin{aligned}\Delta G=&2\sum_{i=1}^{r}\langle B',e_i'\rangle t_iB'\wedge B\wedge e_2\wedge\cdots\wedge e_r\\&+2B''\wedge B\wedge e_2\wedge\cdots\wedge e_r+2\sum_{i=2}^{r}B'\wedge B\wedge e_2\wedge\cdots\wedge e_i'\wedge\cdots\wedge e_r.\end{aligned}$$

From $\Delta G=0$ we have

$$\langle B',e_i'\rangle B'\wedge B\wedge e_2\wedge\cdots\wedge e_r=0$$

for all $i=1,2,\cdots,r$. From Lemma 4.1, we see that $B'(s)$ must be space-like. Thus, B is a null constant vector field. By using Lemma 4.2, we see that M is minimal.

Let $V(s)=\{A(s),B(s),e_2(s),\cdots,e_r(s)\}$ be a smooth distribution of index 1 along α satisfying $\langle A(s),A(s)\rangle=\langle B(s),B(s)\rangle=\langle A(s),e_i(s)\rangle=\langle B(s),e_i(s)\rangle=0$, $\langle A(s),B(s)\rangle=-1$, $\langle e_i(s),e_j(s)\rangle=\delta_{ij}$, and $\langle e_i'(s),e_j(s)\rangle=0$ for all s and $i,j=2,3,\cdots,r$. Then, by Lemma 2.2, we can choose an orthonormal basis $\{e_{r+1},\cdots, e_{m-1}\}$ for the orthogonal complement $V^{\perp}(s)$ satisfying $e_h'(s)\in V(s)$ for all $h=r+1,\cdots,m-1$. Thus we may put

$$\begin{aligned}&A'(s)=\sum_{i=2}^{m-1}u_i(s)e_i(s),\\&B'(s)=0,\\&e_j'(s)=u_j(s)B(s)+\sum_{a=r+1}^{m-1}(-z_{j,a}(s))e_a(s),\quad j=2,\cdots,r,\\&e_a'(s)=u_h(s)B(s)+\sum_{i=2}^{r}z_{i,a}(s)e_i(s),\quad a=r+1,\cdots,m-1.\end{aligned}$$

For a certain initial condition the above system of linear ordinary differential equations has a unique solution to (4.1) with $v_i = 0$ $(i = 2, \cdots, r)$ and $z_{a,b} = 0$ $(a, b = r+1, r+2, \cdots, m-1)$. The solution defines part of a G-kind ruled submanifold.

Case 2. Suppose that deg $Q(t) = 1$. Let $V(s)$=$\{A(s)$, $B(s)$, $e_2(s)$, $\cdots$, $e_r(s)\}$ be a smooth distribution of index 1 along α. Then we can choose an orthonormal basis $\{e_{r+1}, \cdots, e_{m-1}\}$ for the orthogonal complement $V^{\perp}(s)$ satisfying $e'_h(s) \in V(s)$ for $h = r+1, \cdots, m-1$. Then we may put

$$
\begin{aligned}
A'(s) &= \sum_{i=2}^{m-1} u_i(s)e_i(s), \\
B'(s) &= \sum_{i=2}^{m-1} v_i(s)e_i(s), \\
e'_j(s) &= v_j(s)A(s) + u_j(s)B(s) + \sum_{b=r+1}^{m-1}(-z_{j,b}(s))e_b(s), \quad j = 2, \cdots, r, \\
e'_a(s) &= v_a(s)A(s) + u_a(s)B(s) + \sum_{b=2}^{r} z_{b,a}(s)e_b(s), \quad a = r+1, \cdots, m-1.
\end{aligned}
$$

The straightforward computation provides

$$
\begin{aligned}
\Delta G =& \frac{2\bar{\varepsilon}}{Q^3}\sum_{h=r+1}^{m-1}\{(\sum_{i=1}^{r}\langle B', e'_i\rangle t_i - \sum_{i=2}^{r} v'_i t_i)v_h + v'_h Q\}e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \frac{2\bar{\varepsilon}}{Q^2}\sum_{h=r+1}^{m-1} v_h^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \frac{2\bar{\varepsilon}}{Q^2}\sum_{i=2}^{r}\sum_{h=r+1}^{m-1} v_i v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge A \wedge e_{i+1} \wedge \cdots \wedge e_r \\
&- \frac{2\bar{\varepsilon}}{Q^2}\sum_{i=2}^{r}\sum_{h,l=r+1}^{m-1} v_h z_{i,l} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_l \wedge e_{i+1} \wedge \cdots \wedge e_r.
\end{aligned} \tag{4.3}
$$

Since the Gauss map is harmonic, $\Delta G = 0$. Thus, the functions v_a are vanishing for $a = r+1, \cdots, m-1$. Therefore, M is part of a G-kind ruled submanifold.

Conversely, if M is a G-kind ruled submanifold, then deg $Q \leq 1$. From equation (4.3), we can see that $v_{r+1} = \cdots = v_{m-1} = 0$ implies $\Delta G = 0$. Therefore, a G-kind ruled submanifold M has harmonic Gauss map G. This completes the proof. ■

In [10], two of the present authors set up a characterization of minimal ruled submanifolds in Minkowski space. Together with Theorem 4.5, we have a new characterization of minimal ruled submanifolds with degenerate rulings in Minkowski space.

Theorem 4.6. *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. The following are equivalent:*

(1) *M is minimal.*

(2) *M has harmonic Gauss map.*

(3) *M is an open portion of a G-kind ruled submanifold.*

References

1. C. Baikoussis and D. E. Blair, On the Gauss map of ruled surfaces, *Glasgow Math. J.*, **34** (1992), 355-359.
2. C. Baikoussis, B.-Y. Chen and L. Verstraelen, Ruled surfaces and tubes with finite type Gauss map, *Tokyo J. Math.*, **16** (1993), 341-348.
3. J. M. Barbosa, M. Dajczer and I. P. Jorge, Minimal ruled submanifolds in spaces of constant curvature, *Indiana Univ. Math. J.*, **33** (1984), 531-547.
4. B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore, 1984.
5. B.-Y. Chen, A report on submanifolds of finite type, *Soochow J. Math.*, **22** (1996), 117-337.
6. B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Ruled surfaces of finite type, *Bull. Austral. Math. Soc.*, **42** (1990), 447-453.
7. D.-S. Kim, Ruled surfaces of finite type in Lorentz space-times, *Honam Math. J.*, **31** (2009), 177-183.
8. D.-S. Kim and Y. H. Kim, Finite type ruled hypersurfaces in Lorentz-Minkowski space, *Honam Math. J.*, **30** (2008), 743-748.
9. D.-S. Kim and Y. H. Kim, Some classification results on finite-type ruled submanifolds in a Lorentz-Minkowski space, *Taiwanese J. Math.*, **16** (2012), 1475-1488.
10. D.-S. Kim and Y. H. Kim, Minimal ruled submanifolds in Minkowski space $\mathbb{L}^m$, *J. Geom. Phys.*, **62** (2012), 1893-1902.
11. D.-S. Kim, Y. H. Kim and D. W. Yoon, Extended B-scrolls and their Gauss maps, *Indian J. Pure Appl. Math.*, **33** (2002), 1031-1040.
12. D.-S. Kim, Y. H. Kim and D. W. Yoon, Characterization of generalized B-scrolls and cylinders over finite type curves, *Indian J. Pure Appl. Math.*, **33** (2003), 1523-1532.
13. D.-S. Kim, Y. H. Kim and D. W. Yoon, Finite type ruled surfaces in Lorentz-Minkowski space, *Taiwanese J. Math.*, **11** (2007), 1-13.
14. Y. H. Kim and D. W. Yoon, Ruled surfaces with finite type Gauss map in Minkowski spaces, *Soochow J. Math.*, **26** (2000), 85-96.
15. Y. H. Kim and D. W. Yoon, Classification of ruled surfaces in Minkowski 3-spaces, *J. Geom. Phys.*, **49** (2004), 89-100.

16. Y. H. Kim and D. W. Yoon, On non-developable ruled surfaces in Lorentz-Minkowski 3-spaces, *Taiwanese J. Math.*, **11** (2007), 197-214.

17. T. Takahashi, Minimal immersions of Riemannian manifolds, *J. Math. Soc. Japan*, **18** (1966), 380-385.

Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757
Korea
E-mail: dosokim@chonnam.ac.kr

Young Ho Kim and Sun Mi Jung
Department of Mathematics
Kyungpook National University
Taegu 702-701
Korea
E-mail: yhkim@knu.ac.kr
jung3756@hanmail.net