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ROUGH SINGULAR INTEGRALS SUPPORTED BY SUBMANIFOLDS IN TRIEBEL-LIZORKIN SPACES AND BESOVE SPACES

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Abstract. This paper is devoted to studying the singular integral operators associated to polynomial mappings as well as the corresponding compound submanifolds. By imposing a restrictive condition on the kernels of the operators in the radial direction, the boundedness for such operators on Triebel-Lizorkin spaces and Besov spaces are established, provided that the kernels satisfy a rather weak size condition on the unit sphere, which is distinct from the Hardy space functions. Some previous results are essentially improved and generalized.

1. INTRODUCTION

Let $\mathbb{R}^n,\ n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $\Omega \in L^1(S^{n-1})$ be a homogeneous function of degree zero and satisfy

$$\int_{S^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1.1}$$

For $d \geq 1$, let $\mathcal{P} = (P_1, \cdots, P_d)$ and $\deg(\mathcal{P}) = \max\{\deg(P_j) : 1 \leq j \leq d\}$, where P_j is a real-valued polynomial in $\mathbb{R}^n$ for $1 \leq j \leq d$. For a suitable function h defined on $\mathbb{R}^+ = \{t \in \mathbb{R} :\ t > 0\}$, we define the singular integrals $T_{h,\Omega,\mathcal{P}}$ associated to polynomial mappings $\mathcal{P}$ in $\mathbb{R}^d$ by

$$T_{h,\Omega,\mathcal{P}}(f)(x) := p.v. \int_{\mathbb{R}^n} f(x - \mathcal{P}(y)) \frac{\Omega(y)h(|y|)}{|y|^n} dy, \quad x \in \mathbb{R}^d. \tag{1.2}$$

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As is well known, the operators $T_{h,\Omega,\mathcal{P}}$ belong to the class of singular radon transforms. The L^p-mapping properties of $T_{h,\Omega,\mathcal{P}}$ were first given by Stein (see [17], [18, pp. 513-517]) under the stronger assumption that $\Omega \in C^1(S^{n-1})$ and $h(t) \equiv 1$. Subsequently, the investigation on the boundedness of $T_{h,\Omega,\mathcal{P}}$ on function spaces abstracted many attentions, for examples see [2, 4, 10, 16] et al. In particular, Fan and Pan[10] showed that $T_{h,\Omega,\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^d)$ for p with satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ if $\Omega \in H^1(S^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, where $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere (see [5, 15]) and $\Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ denotes the set of all measurable functions h on $\mathbb{R}^+$ satisfying the condition

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} = \sup_{R>0} \Big(R^{-1} \int_0^R |h(t)|^\gamma dt\Big)^{1/\gamma} < \infty.$$

It is easy to check that $\Delta_\infty(\mathbb{R}^+) = L^\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $0 < \gamma_1 < \gamma_2 < \infty$.

In 2010, Chen, Ding and Liu[4] generalized the result of [10] to the Triebel-Lizorkin spaces and Besov spaces, which contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. The homogeneous Triebel-Lizorkin space $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ and homogeneous Besov space $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ are defined, respectively, by

$$\begin{aligned} \dot{F}_\alpha^{p,q}(\mathbb{R}^d) &:= \Big\{f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \\ &= \Big\|\Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q}|\Psi_i * f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} < \infty\Big\} \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \dot{B}_\alpha^{p,q}(\mathbb{R}^d) &:= \Big\{f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} \\ &= \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q}\|\Psi_i * f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} < \infty\Big\}, \end{aligned} \tag{1.4}$$

where $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty\,(p \neq \infty)$, $\mathcal{S}'(\mathbb{R}^d)$ denotes the tempered distribution class on $\mathbb{R}^d$, $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ satisfies the conditions: $0 \leq \phi(x) \leq 1$; $\mathrm{supp}(\phi) \subset \{x : \ 1/2 \leq |x| \leq 2\}$; $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$. It is well known that

$$\dot{F}_0^{p,2}(\mathbb{R}^d) = L^p(\mathbb{R}^d) \tag{1.5}$$

for any $1 < p < \infty$, see [9, 13, 19] for more properties of $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ and $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$. Chen, Ding and Liu's result in [4] can be stated as follows:

Theorem A. (see [4]). *Let* $\alpha \in \mathbb{R}$ *and* $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$. *Suppose that* $\Omega \in H^1(S^{n-1})$ *and satisfies* (1.1). *Then there exists a constant* $C > 0$ *such that*

(i) *for* $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$ *and* $f \in \dot{F}_\alpha^{p,q}(\mathbb{R}^d)$,

$$\|T_{h,\Omega,\mathcal{P}}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq C\|\Omega\|_{H^1(S^{n-1})}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)};$$

(ii) *for* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, $1 < q < \infty$ *and* $f \in \dot{B}_\alpha^{p,q}(\mathbb{R}^d)$,

$$\|T_{h,\Omega,\mathcal{P}}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} \leq C\|\Omega\|_{H^1(S^{n-1})}\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)}.$$

The constant $C = C(n, d, h, p, q, \alpha, \deg(\mathcal{P}))$ *is independent of the coefficients of* P_j *for* $1 \leq j \leq d$.

On the other hand, for $\mathcal{P}(y) = (y_1, y_2, \cdots, y_d)$ and $n = d$, we denote $T_{h,\Omega,\mathcal{P}}$ by $T_{h,\Omega}$ which has been studied by many authors (see [1, 6, 11, 12, 14] etc.). In 2006, Al-Qassem[1] showed that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ provided that $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ and $h \in \mathcal{H}_\gamma(\mathbb{R}^+)$ for some $1 < \gamma \leq \infty$ (also see [12] for the generalization in non-isotropic setting). Here $\mathcal{H}_\gamma(\mathbb{R}^+)$, $\gamma > 0$, is the set of all measurable functions h on $\mathbb{R}^+$ satisfying

$$\|h\|_{\mathcal{H}_\gamma(\mathbb{R}^+)} = \Big(\int_0^\infty |h(t)|^\gamma dt/t\Big)^{1/\gamma} < \infty,$$

and $L(\log^+ L)^\alpha(S^{n-1})$, $\alpha > 0$, denote the space of all those functions Ω on S^{n-1}, which satisfy

$$\int_{S^{n-1}} |\Omega(\theta)| \log^\alpha(2 + |\Omega(\theta)|) d\sigma(\theta) < \infty.$$

It is easy to check that for $0 < \gamma < \infty$, $\mathcal{H}_\gamma(\mathbb{R}^+) \subsetneq \Delta_\gamma(\mathbb{R}^+)$ and $\mathcal{H}_\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. Also, the following proper inclusions hold:

$$L(\log^+ L)^\beta(S^{n-1}) \subsetneq L(\log^+ L)^\alpha(S^{n-1}), \quad \text{if } 0 < \alpha < \beta; \tag{1.6}$$

$$L(\log^+ L)^\alpha(S^{n-1}) \subsetneq H^1(S^{n-1}), \quad \text{for any } \alpha \geq 1; \tag{1.7}$$

$$L(\log^+ L)^\alpha(S^{n-1}) \nsubseteq H^1(S^{n-1}), \quad \text{for any } 0 < \alpha < 1. \tag{1.8}$$

Recently, Le[14] generalized the result of [1] as follows.

Theorem B. (see [14]). *Let* $\alpha \in \mathbb{R}$ *and* $1 < p < \infty$. *Suppose that* $\Omega \in L(\log^+ L)^{v_q}(S^{n-1})$ *and satisfies* (1.1). *Then* $T_{h,\Omega}$ *is bounded on* $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ *provided that one of the following conditions holds:*

(i) $v_q = 1/q$, $h \in \mathcal{H}_{q'}(\mathbb{R}^+)$ *and* $1 < q \le 2$;

(ii) $v_q = 1/2$, $h \in \mathcal{H}_2(\mathbb{R}^+)$ *and* $q > 2$.

Comparing Theorem A with Theorem B, a natural question is the following:

Question. Is $T_{h,\Omega,\mathcal{P}}$ bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ if $\Omega \in L(\log^+ L)^\alpha(S^{n-1})$ for some $\alpha \in (0,1)$ and $h \in \mathcal{H}_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$?

The main purpose of this paper is to address this question above. Our main results can be formulated as follows:

Theorem 1.1. *Let* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *with satisfying* (1.1) *and* $h \in \mathcal{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$. *Then for* $\alpha \in \mathbb{R}$ *and* $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$, *there exists a constant* $C > 0$ *such that*

$$\|T_{h,\Omega,\mathcal{P}}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \le C\|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)},$$

where $C = C(n, d, p, q, h, \alpha, \deg(\mathcal{P}))$ *is independent of the coefficients of* P_j *for* $1 \le j \le d$.

Theorem 1.2. *Let* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *with satisfying* (1.1) *and* $h \in \mathcal{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$. *Then for* $\alpha \in \mathbb{R}$, $1 < q < \infty$ *and* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, *there exists a constant* $C > 0$ *such that*

$$\|T_{h,\Omega,\mathcal{P}}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} \le C\|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})}\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)},$$

where $C = C(n, d, p, q, h, \alpha, \deg(\mathcal{P}))$ *is independent of the coefficients of* P_j *for* $1 \le j \le d$.

Remark 1.3. Obviously, the range of q given in Theorem 1.1 is the full range $(1, \infty)$ when $\gamma \ge 2$. Thus Theorem 1.1 improves the results of Theorem B(i), even in the special case: $\mathcal{P}(y) = (y_1, y_2, \cdots, y_d)$ and $n = d$. We also remark that Theorems 1.1 and 1.2 are not true, if replacing $h \in \mathcal{H}_\gamma(\mathbb{R}^+)$ by $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$, because of that $L^\infty(\mathbb{R}^+) \subset \Delta_\gamma(\mathbb{R}^+)$, $L\log^+ L(S^{n-1}) \subsetneq L(\log^+ L)^\alpha(S^{n-1})$ for any $0 < \alpha < 1$, and Calderon-Zygmund's celebrated result in [3]. In addition, by (1.8), Theorems 1.1 and 1.2 are distinct from Theorem A.

Furthermore, by Theorems 1.1 and 1.2, and a switched method followed from [7], we can establish the corresponding results for the more general singular integral operators $T_{h,\Omega,\mathcal{P},\varphi}$ supported by the compound sub-manifolds as follows.

Theorem 1.4. *Let* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *with satisfying* (1.1) *and* $h \in \mathcal{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$. *Suppose that* φ *is a nonnegative* (*or non-positive*) *and monotonic* $\mathcal{C}^1$ *function on* $(0, \infty)$ *such that* $\Gamma(t) := \frac{\varphi(t)}{t\varphi'(t)}$ *with* $|\Gamma(t)| \le C$, *where* C

is a positive constant which depends only on φ*. Then for* $\alpha \in \mathbb{R}$ *and* $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$*, there exists a constant* $C > 0$ *such that*

$$\|T_{h,\Omega,\mathcal{P},\varphi}(f)\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^d)} \leq C\|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})}\|f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^d)},$$

where

$$T_{h,\Omega,\mathcal{P},\varphi}(f)(x) := p.v. \int_{\mathbb{R}^n} f(x - \mathcal{P}(\varphi(|y|)y'))\frac{\Omega(y)h(|y|)}{|y|^n}dy,$$

$y' = y/|y| \in S^{n-1}$ *and* $C = C(n, d, p, q, h, \alpha, \varphi, \deg(\mathcal{P}))$ *is independent of the coefficients of* P_j *for* $1 \leq j \leq d$*.*

Theorem 1.5. *Let* $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ *with satisfying* (1.1) *and* $h \in \mathcal{H}_{\gamma}(\mathbb{R}^+)$ *for some* $\gamma > 1$*. Suppose that* φ *is a nonnegative (or non-positive) and monotonic* $\mathcal{C}^1$ *function on* $(0, \infty)$ *such that* $\Gamma(t) := \frac{\varphi(t)}{t\varphi'(t)}$ *with* $|\Gamma(t)| \leq C$*, where* C *is a positive constant which depends only on* φ*. Then for* $\alpha \in \mathbb{R}$*,* $1 < q < \infty$ *and* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$*, there exists a constant* $C > 0$ *such that*

$$\|T_{h,\Omega,\mathcal{P},\varphi}(f)\|_{\dot{B}^{p,q}_{\alpha}(\mathbb{R}^d)} \leq C\|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})}\|f\|_{\dot{B}^{p,q}_{\alpha}(\mathbb{R}^d)},$$

where $C = C(n, d, p, q, h, \alpha, \varphi, \deg(\mathcal{P}))$ *is independent of the coefficients of* P_j *for* $1 \leq j \leq d$*.*

Remark 1.6. Under the assumptions on φ in Theorem 1.4, the following facts are obvious (see [7]):

(i) $\lim_{t\to 0} \varphi(t) = 0$ and $\lim_{t\to\infty} |\varphi(t)| = \infty$ if φ is nonnegative and increasing, or non-positive and decreasing;

(ii) $\lim_{t\to 0} |\varphi(t)| = \infty$ and $\lim_{t\to\infty} \varphi(t) = 0$ if φ is nonnegative and decreasing, or non-positive and increasing.

Moreover, the inhomogeneous versions of Triebel-Lizorkin space and Besov spaces, which are denoted by $F^{p,q}_{\alpha}(\mathbb{R}^d)$ and $B^{p,q}_{\alpha}(\mathbb{R}^d)$, respectively, are obtained by adding the term $\|\Phi * f\|_{L^p(\mathbb{R}^d)}$ to the right hand side of (1.3) or (1.4) with $\sum_{j\in\mathbb{Z}}$ replaced by $\sum_{j\geq 1}$, where $\Phi \in \mathcal{S}(\mathbb{R}^d)$, $\mathrm{supp}(\hat{\Phi}) \subset \{\xi : |\xi| \leq 2\}$, $\hat{\Phi}(x) > c > 0$ if $|x| \leq 5/3$. The following properties are well known (see [9, 13], for example):

$$F^{p,q}_{\alpha}(\mathbb{R}^d) \sim \dot{F}^{p,q}_{\alpha}(\mathbb{R}^d) \bigcap L^p(\mathbb{R}^d) \text{and} \\ \|f\|_{F^{p,q}_{\alpha}(\mathbb{R}^d)} \sim \|f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)} \ (\alpha > 0); \tag{1.9}$$

$$B^{p,q}_{\alpha}(\mathbb{R}^d) \sim \dot{B}^{p,q}_{\alpha}(\mathbb{R}^d) \bigcap L^p(\mathbb{R}^d) \text{and} \\ \|f\|_{B^{p,q}_{\alpha}(\mathbb{R}^d)} \sim \|f\|_{\dot{B}^{p,q}_{\alpha}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)} \ (\alpha > 0). \tag{1.10}$$

Hence, by (1.5), (1.9)-(1.10) and Theorems 1.4-1.5, we get the following conclusion immediately.

Corollary 1.7. *Under the same conditions of Theorems* 1.4 *and* 1.5 *with* $\alpha > 0$, *the operator* $T_{h,\Omega,\mathcal{P},\varphi}$ *defined as in Theorem* 1.4 *is bounded on* $F_\alpha^{p,q}(\mathbb{R}^d)$ *and* $B_\alpha^{p,q}(\mathbb{R}^d)$, *respectively.*

The paper is organized as follows. In Section 2, we will present some general vector-valued norm inequalities (see Propositions 2.2 and 2.3). In Section 3 we recall some notations and establish some necessary lemmas. Finally, the proofs of main results will be given in Section 4.

Throughout the paper, we let p' denote the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables.

2. Vector-valued Norm Inequalities

In this section we will recall and establish some important vector-valued norm inequalities, which will play the key roles in the proof of Theorem 1.1. The following result obtained by Chen, Ding and Liu in [4] is an extension of the famous result on the $L^p(\ell^q)$ boundedness of the Hardy-Littlewood maximal operator.

Lemma 2.1. (see [4, Theorem 1.4]). *Let* $\mathcal{P} = (P_1, \cdots, P_d)$ *with* P_j *being real-valued polynomials on* $\mathbb{R}^n$. *For* $1 < p,\ q < \infty$, *the operator* $\mathcal{M}_\mathcal{P}$ *given by*

$$\mathcal{M}_\mathcal{P}(f)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |f(x - \mathcal{P}(y))| dy$$

satisfies the following $L^p(\ell^q)$ *inequality*

$$\Big\| \Big(\sum_{i\in\mathbb{Z}} |\mathcal{M}_\mathcal{P}(f_i)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} \le C(p,q) \Big\| \Big(\sum_{i\in\mathbb{Z}} |f_i|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)},$$

where the positive constant $C(p,q)$ *is independent of the coefficients of* P_j *for* $1 \le j \le d$.

Proposition 2.2. *Let* $\Phi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\{a_k\}_{k\in\mathbb{Z}}$ *be a lacunary sequence of positive numbers satisfying* $\inf_{k\in\mathbb{Z}} a_{k+1}/a_k \ge a > 1$. *Define the Littlewood-Paley operator* Δ_k *associated with* Φ *by*

$$\Delta_k(f)(x) = \Phi_k * f(x)$$

for all $x \in \mathbb{R}^n$, *where* $\Phi_k(x) = a_k^{-n}\Phi(x/a_k)$. *Then for* $1 < p,\ q < \infty$ *and arbitrary functions* $\{f_j\} \in L^p(\ell^q, \mathbb{R}^n)$, *there exists a positive constant* $C(n,a)$ *such that*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |\Delta_k(f_j)|^2 \Big)^{q/2} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \le C(n,a) \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

Proof. The idea of proving this proposition comes from the proof of [13, Theorem 5.1.2]. First we introduce two Banach spaces $\mathcal{B}_1 = \mathbb{C}$ and $\mathcal{B}_2 = \ell^2$ and define an operator

$$\vec{T}(f) = \{\Delta_k(f)\}_{k\in\mathbb{Z}}.$$

It is clear that $\vec{T}(f)$ can be written by

$$\vec{T}(f)(x) = \int_{\mathbb{R}^n} \vec{K}(y)(f(x-y))dy, \tag{2.1}$$

where $\vec{K}$ is a bounded linear operator form $\mathcal{B}_1$ to $\mathcal{B}_2$ given by

$$\vec{K}(x)(g) = \{\Phi_k(x)g\}_{k\in\mathbb{Z}}.$$

It is easy to see that $\|\vec{K}(x)\|_{\mathcal{B}_1\to\mathcal{B}_2} = (\sum_{k\in\mathbb{Z}} |\Phi_k(x)|^2)^{1/2}$. In what follows, we will verify the following inequality

$$\int_{|x|\geq 2|y|} \|\vec{K}(x-y) - \vec{K}(x)\|_{\mathcal{B}_1\to\mathcal{B}_2} dx \leq C, \quad y \neq 0. \tag{2.2}$$

Since $\Phi \in \mathcal{S}(\mathbb{R}^n)$, there exists a constant $C > 0$, which depends only on n, such that

$$|\Phi(x)| + |\nabla\Phi(x)| \leq C(1+|x|)^{-n-1}. \tag{2.3}$$

This together with the mean value theorem of derivative implies

$$|\Phi_k(x-y) - \Phi_k(x)| \leq C\frac{1}{a_k^{n+1}}\Big(1+\frac{|x|}{2a_k}\Big)^{-n-1}|y|, \quad |x| \geq 2|y|. \tag{2.4}$$

In addition, it follows from (2.3) that

$$|\Phi_k(x-y) - \Phi_k(x)| \leq C\frac{1}{a_k^{n}}\Big(1+\frac{|x|}{2a_k}\Big)^{-n-1}, \quad |x| \geq 2|y|. \tag{2.5}$$

Thus by the geometric mean of (2.4) and (2.5), we get

$$|\Phi_k(x-y) - \Phi_k(x)| \leq C|y|^{1/2}\frac{1}{a_k^{n+1/2}}\Big(1+\frac{|x|}{2a_k}\Big)^{-n-1}. \tag{2.6}$$

This together with (2.4) yields

$$\begin{aligned}
\|\vec{K}(x-y) - \vec{K}(x)\|_{\mathcal{B}_1\to\mathcal{B}_2} &= \Big(\sum_{k\in\mathbb{Z}} |\Phi_k(x-y) - \Phi_k(x)|^2\Big)^{1/2} \\
&\leq \sum_{k\in\mathbb{Z}} |\Phi_k(x-y) - \Phi_k(x)| \\
&\leq C|y| \sum_{a_k > |x|/2} \frac{1}{a_k^{n+1}}\Big(1+\frac{|x|}{2a_k}\Big)^{-n-1} \\
&\quad + C|y|^{1/2} \sum_{a_k \leq |x|/2} \frac{1}{a_k^{n+1/2}}\Big(1+\frac{|x|}{2a_k}\Big)^{-n-1} \\
&\leq C(n,a)\Big(\frac{2^{n+1}|y|}{|x|^{n+1}} + \frac{|y|^{1/2}}{|x|^{n+1/2}}\Big),
\end{aligned}$$

which implies (2.2). Furthermore, $\vec{T}$ obviously maps $L^q(\mathcal{B}_1, \mathbb{R}^n)$ to $L^q(\mathcal{B}_2, \mathbb{R}^n)$. Applying [13, Proposition 4.6.4] yields Proposition 2.2. ■

Proposition 2.3. *Let* $0 < M \leq N$ *and* $H : \mathbb{R}^M \to \mathbb{R}^M$, $G : \mathbb{R}^N \to \mathbb{R}^N$ *be two nonsingular linear transformations. Let* $\{a_k\}_{k\in\mathbb{Z}}$ *be a lacunary sequence of positive numbers satisfying* $\inf_{k\in\mathbb{Z}} a_{k+1}/a_k \geq a > 1$. *Let* $\Phi(\xi) \in \mathcal{S}(\mathbb{R}^M)$ *and* $\Phi_k(\xi) = a_k^{-M}\Phi(\xi/a_k)$. *Define the transformations* J *and* X_k *by*

$$J(f)(x) = f(G^t(H^t \otimes id_{\mathbb{R}^{N-M}})x)$$

and

$$X_k(f)(x) = J^{-1}((\Phi_k \otimes \delta_{\mathbb{R}^{N-M}}) * J(f))(x).$$

Here we shall use $\delta_{\mathbb{R}^n}$ *to denote the Dirac delta function on* $\mathbb{R}^n$, J^{-1} *denote the inverse transform of* J *and* D^t *denote the transpose of the linear transformation* D. *Then there exists a positive constant* $C(M, a)$ *such that*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(f_j)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)} \leq C(M,a)\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)} \tag{2.7}$$

for arbitrary functions $\{f_j\} \in L^p(\ell^q, \mathbb{R}^N)$ *and* $1 < p, q < \infty$;

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(g_{k,j})|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)}\\ &\leq C(M,a)\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)}\end{aligned} \tag{2.8}$$

for arbitrary functions $\{g_{k,j}\} \in L^p(\ell^q(\ell^2), \mathbb{R}^N)$ *and* $1 < p, q < \infty$.

Proof. For convenience we denote $\xi = (\xi^1, \xi^2)$ with $\xi^1 = (\xi_1, \cdots, \xi_M)$ and $\xi^2 = (\xi_{M+1}, \cdots, \xi_N)$. Then using Proposition 2.2 and the change of the variables, we have

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(f_j)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)}^p\\ &= \int_{\mathbb{R}^N}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|J^{-1}((\Phi_k \otimes \delta_{\mathbb{R}^{N-M}}) * J(f_j))(\xi)|^2\Big)^{q/2}\Big)^{p/q}d\xi\\ &\leq C|J|\int_{\mathbb{R}^N}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|((\Phi_k \otimes \delta_{\mathbb{R}^{N-M}}) * J(f_j))(\xi)|^2\Big)^{q/2}\Big)^{p/q}d\xi\\ &\leq C|J|\int_{\mathbb{R}^{N-M}}\int_{\mathbb{R}^M}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|[\Phi_k * J(f_j)(\cdot, \xi^2)](\xi^1)|^2\Big)^{q/2}\Big)^{p/q}d\xi^1 d\xi^2\end{aligned}$$

$$\leq C(M,a)|J|\int_{\mathbb{R}^{N-M}}\int_{\mathbb{R}^M}\Big(\sum_{j\in\mathbb{Z}}|J(f_j)(\xi^1,\xi^2)|^q\Big)^{p/q}d\xi^1 d\xi^2$$
$$\leq C(M,a)\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^N)},$$

where $|J|$ denotes the Jacobian of the transformation J. Then (2.7) holds. Next we prove (2.8). Let $\mathcal{M}^M$ be the Hardy-Littlewood maximal function on $\mathbb{R}^M$. Note that

$$|X_k f(x)|\leq C(M,a)[J^{-1}\circ(\mathcal{M}^M\otimes\delta_{\mathbb{R}^{N-M}})\circ J](f)(x).$$

(2.8) follows from the following equality

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(g_{k,j})|^2\Big)^{q/2}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^N)}$$
$$\leq C(M,a)\int_{\mathbb{R}^N}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|(J^{-1}\circ(\mathcal{M}^M\otimes\delta_{\mathbb{R}^{N-M}})\circ J)g_{k,j}(\xi)|^2\Big)^{q/2}\Big)^{p/q}d\xi$$
$$\leq C(M,a)|J|\int_{\mathbb{R}^{N-M}}\int_{\mathbb{R}^M}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\mathcal{M}^M(J(g_{k,j})(\cdot,\xi^2))(\xi^1)|^2\Big)^{q/2}\Big)^{p/q}d\xi^1 d\xi^2$$
$$\leq C(M,a)|J|\int_{\mathbb{R}^{N-M}}\int_{\mathbb{R}^M}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|J(g_{k,j})(\xi^1,\xi^2)|^2\Big)^{q/2}\Big)^{p/q}d\xi^1 d\xi^2$$
$$\leq C(M,a)\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^N)}.$$

This proves Proposition 2.3. ∎

3. Auxiliary Lemmas

Following from [10], we first recall some notations. For $l,\ n\in\mathbb{Z}^+$, we denote $V_{n,l}$ as the space of real-valued homogeneous polynomials of degree l on $\mathbb{R}^n$ and $\mathcal{A}_n$ denote the class of polynomials of n variables with real coefficients. Let $\mathcal{P}(x)=(P_1(x),\cdots,P_d(x))$ with $P_j\in\mathcal{A}_n$ for $j=1,\cdots,d$. Then there are integers $0<l_1<l_2<\cdots<l_{\mathcal{N}}\leq\deg(\mathcal{P})$, and polynomials $Q_j^u\in V_{n,l_u}\subset\mathcal{A}_n,\ R_j\in\mathcal{A}_1$ with $\deg(R_j)\leq\deg(\mathcal{P})$ for $1\leq u\leq\mathcal{N},\ 1\leq j\leq d$ such that

$$\mathcal{P}(x)=\sum_{u=1}^{\mathcal{N}}\mathcal{Q}^u(x)+\mathcal{R}(|x|),$$

where $\mathcal{Q}^u(x)=(Q_1^u(x),Q_2^u(x),\cdots,Q_d^u(x))$ and $\mathcal{R}(t)=(R_1(t),R_2(t),\cdots,R_d(t))$;

$$Z_{l_u}(Q_j^u)=Q_j^u\ \text{ for }\ 1\leq u\leq\mathcal{N}\ \text{ and }\ 1\leq j\leq d.$$

For $j=1,\cdots,d$ and $1\leq u\leq\mathcal{N}$, write

$$Q_j^u(x) = \sum_{|\beta|=l_u} b_{uj\beta} x^\beta = \sum_{s=1}^{d(u)} b'_{ujs} x^{\beta(u,s)},$$

where $d(u) = \dim(V_{n,l_u})$. For $1 \le u \le \mathcal{N}$, we define the linear transformations $I_u : \mathbb{R}^d \to \mathbb{R}^{d(u)}$ by

$$I_u(\xi) = (\sum_{s=1}^{d} b'_{uj1}\xi_j, \cdots, \sum_{s=1}^{d} b'_{ujd(u)}\xi_j).$$

For $1 \le \eta \le \mathcal{N}$, we define

$$\Gamma_\eta(x) = \sum_{u=1}^{\eta} \mathcal{Q}^u(x) + \mathcal{R}(|x|) \ \text{ and } \ \Gamma_0(x) = \mathcal{R}(|x|).$$

Let $\Omega \in L(\log^+ L)^\alpha(S^{n-1})$ for $\alpha > 0$ and satisfy (1.1). Employing the notation in [2], let $E_m = \{y' \in S^{n-1} : \ 2^m < |\Omega(y')| \le 2^{m+1}\}$ for $m \in \mathbb{Z}$ and $E_0 = \{y' \in S^{n-1} : \ |\Omega(y')| < 2\}$. Set $N(\Omega) = \{m \in \mathbb{N} : \ \sigma(E_m) > 2^{-4m}\}$ and for $m \ge 1$,

$$\Omega_m(y') = \Omega(y')\chi_{E_m}(y') - \sigma(S^{n-1})^{-1} \int_{E_m} \Omega(y') d\sigma(y'),$$

and $\Omega_0(y') = \Omega(y') - \sum_{m \in N(\Omega)} \Omega_m(y')$. It is easy to check that

$$\int_{S^{n-1}} \Omega_m(y') d\sigma(y') = 0, \quad \text{for } m \in N(\Omega) \cup \{0\}; \tag{3.1}$$

$$\|\Omega_0\|_{L^2(S^{n-1})} \le C, \quad \|\Omega_0\|_{L^1(S^{n-1})} \le C; \tag{3.2}$$

$$\begin{aligned} &\|\Omega_m\|_{L^2(S^{n-1})} \le C 2^{2m} \|\Omega\|_{L^1(E_m)}, \quad \|\Omega_m\|_{L^1(S^{n-1})} \\ &\le C\|\Omega\|_{L^1(E_m)}, \quad \text{for } m \in N(\Omega); \end{aligned} \tag{3.3}$$

$$\Omega(y') = \sum_{m \in N(\Omega) \cup \{0\}} \Omega_m(y'); \tag{3.4}$$

$$\sum_{m \in N(\Omega) \cup \{0\}} (m+1)^\alpha \|\Omega\|_{L^1(E_m)} \le C \|\Omega\|_{L(\log^+ L)^\alpha(S^{n-1})}, \quad \text{for } \alpha > 0. \tag{3.5}$$

It is clear that

$$T_{h,\Omega,\mathcal{P}}(f)(x) = \sum_{m \in N(\Omega) \cup \{0\}} T_{h,\Omega_m,\mathcal{P}}(f)(x). \tag{3.6}$$

For $k \in \mathbb{Z}$ and $m \in N(\Omega) \cup \{0\}$, let $D_k = \{x \in \mathbb{R}^n : 2^{(m+1)k} \le |x| < 2^{(m+1)(k+1)}\}$. For $1 \le \eta \le \mathcal{N}$, we define the measures $\{\sigma_{k,\Gamma_\eta}\}_{k \in \mathbb{Z}}$ by

$$\int_{\mathbb{R}^d} f d\sigma_{k,\Gamma_\eta} = \int_{D_k} f(\Gamma_\eta(x)) \frac{h(|x|)\Omega_m(x)}{|x|^n} dx.$$

Obviously,

$$T_{h,\Omega_m,\mathcal{P}}(f)(x) = \sum_{k\in\mathbb{Z}} f * \sigma_{k,\Gamma_{\mathcal{N}}}(x). \tag{3.7}$$

For convenience, for $\gamma > 1$, we denote $\tilde{\gamma} = \max\{2, \gamma'\}$ and $A = (m+1)^{1/\gamma'} \|\Omega\|_{L^1(E_m)} \||h|\|_\gamma$, where

$$\||h|\|_\gamma = \sup_{k\in\mathbb{Z}} \Big(\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}} |h(t)|^\gamma \frac{dt}{t} \Big)^{1/\gamma}.$$

We have the following lemmas.

Lemma 3.1. *For $k \in \mathbb{Z}$, $m \in N(\Omega) \cup \{0\}$, $\xi \in \mathbb{R}^d$ and $1 \le \eta \le \mathcal{N}$, there exists a constant $C > 0$ such that*

$$|\widehat{\sigma_{k,\Gamma_\eta}}(\xi) - \widehat{\sigma_{k,\Gamma_{\eta-1}}}(\xi)| \le CA|2^{(m+1)(k+1)l_\eta} I_\eta(\xi)|^{1/(4(m+1)l_\eta\tilde{\gamma})}; \tag{3.8}$$

$$|\widehat{\sigma_{k,\Gamma_\eta}}(\xi)| \le CA \min\{1, |2^{(m+1)(k+1)l_\eta} I_\eta(\xi)|^{-1/(4(m+1)l_\eta\tilde{\gamma})}\}. \tag{3.9}$$

The constant C is independent of m and γ.

Proof. By the change of variables, we have

$$\begin{aligned} &|\widehat{\sigma_{k,\Gamma_\eta}}(\xi) - \widehat{\sigma_{k,\Gamma_{\eta-1}}}(\xi)| \\ &= \Big| \int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}} \int_{S^{n-1}} \Omega_m(y')(e^{-2\pi i\xi\cdot\Gamma_\eta(ry')} \\ &\quad - e^{-2\pi i\xi\cdot\Gamma_{\eta-1}(ry')}) d\sigma(y') h(r) \tfrac{dr}{r} \Big| \\ &\le C|2^{(m+1)(k+1)l_\eta} I_\eta(\xi)| \|\Omega_m\|_{L^1(S^{n-1})} \int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}} |h(r)| \frac{dr}{r} \\ &\le CA|2^{(m+1)(k+1)l_\eta} I_\eta(\xi)|. \end{aligned} \tag{3.10}$$

On the other hand,

$$|\widehat{\sigma_{k,\Gamma_\eta}}(\xi) - \widehat{\sigma_{k,\Gamma_{\eta-1}}}(\xi)| \le CA. \tag{3.11}$$

Interpolating between (3.10) and (3.11) implies (3.8). Below we prove (3.9). It is easy to see that

$$|\widehat{\sigma_{k,\Gamma_\eta}}(\xi)| \le CA. \tag{3.12}$$

Moreover, by Hölder's inequality we have

$$
\begin{aligned}
(3.13)\qquad |\widehat{\sigma_{k,\Gamma_\eta}}(\xi)| = &\ \Big|\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}}\int_{S^{n-1}}\Omega_m(y')e^{-2\pi i\xi\cdot\Gamma_\eta(ry')}d\sigma(y')h(r)\frac{dr}{r}\Big| \\
&\le |||h|||_\gamma H_{m,k}(\xi),
\end{aligned}
$$

where

$$
H_{m,k}(\xi) := \Big(\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}}\Big|\int_{S^{n-1}}\Omega_m(y')e^{-2\pi i\xi\cdot\Gamma_\eta(ry')}d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}.
$$

Applying [10, Corollary 4.3] with $\epsilon = 1/(8l_\eta)$ and $p = 2$, we have for any $r > 0$,

$$
\begin{aligned}
(3.14)\qquad &\Big(\int_r^{2r}\Big|\int_{S^{n-1}}\Omega_m(y')e^{-2\pi i\xi\cdot\Gamma_\eta(ty')}d\sigma(y')\Big|^2\frac{dt}{t}\Big)^{1/2} \\
&\le C\|\Omega_m\|_{L^2(S^{n-1})}|r^{l_\eta}I_\eta(\xi)|^{-1/(8l_\eta)}.
\end{aligned}
$$

Since $\gamma \ge 2$ implies $1 < \gamma' \le 2$, by (3.2)-(3.3), (3.14) and Hölder's inequality we have

$$
\begin{aligned}
&H_{m,k}(\xi) \\
&\le (m+1)^{1/\gamma'-1/2}\Big(\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}}\Big|\int_{S^{n-1}}\Omega_m(y')e^{-2\pi i\xi\cdot\Gamma_\eta(ry')}d\sigma(y')\Big|^2\frac{dr}{r}\Big)^{1/2} \\
&\le (m+1)^{1/\gamma'-1/2}\Big(\sum_{i=0}^{m}\int_{2^{(m+1)k}+i}^{2^{(m+1)(k+1)}+i+1}\Big|\int_{S^{n-1}}\Omega_m(y')e^{-2\pi i\xi\cdot\Gamma_\eta(ry')}d\sigma(y')\Big|^2\frac{dr}{r}\Big)^{1/2} \\
&\le C(m+1)^{1/\gamma'-1/2}(m+1)^{1/2}\|\Omega_m\|_{L^2(S^{n-1})}|2^{(m+1)kl_\eta}I_\eta(\xi)|^{-1/(8l_\eta)} \\
&\le C(m+1)^{1/\gamma'}2^{2m}\|\Omega\|_{L^1(E_m)}|2^{(m+1)kl_\eta}I_\eta(\xi)|^{-1/(8l_\eta)},
\end{aligned}
$$

which combining with (3.13) implies

$$
(3.15)\qquad |\widehat{\sigma_{k,\Gamma_\eta}}(\xi)| \le CA2^{2m}|2^{(m+1)kl_\eta}I_\eta(\xi)|^{-1/(8l_\eta)}, \quad \text{for } \gamma \ge 2.
$$

On the other hand, for $1 < \gamma < 2$, we have $\gamma' > 2$. Then

$$
\begin{aligned}
&H_{m,k}(\xi) \\
&\le C\|\Omega_m\|_{L^1(S^{n-1})}^{1-2/\gamma'}\Big(\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}}\Big|\int_{S^{n-1}}\Omega_m(y')e^{-2\pi i\xi\cdot\Gamma_\eta(ry')}d\sigma(y')\Big|^2\frac{dr}{r}\Big)^{1/\gamma'} \\
&\le C(m+1)^{1/\gamma'}\|\Omega\|_{L^1(E_m)}^{1-2/\gamma'}2^{4m/\gamma'}\|\Omega\|_{L^1(E_m)}^{2/\gamma'}|2^{(m+1)kl_\eta}I_\eta(\xi)|^{-1/(4l_\eta\gamma')} \\
&\le C(m+1)^{1/\gamma'}\|\Omega\|_{L^1(E_m)}2^{4m/\gamma'}|2^{(m+1)kl_\eta}I_\eta(\xi)|^{-1/(4l_\eta\gamma')}.
\end{aligned}
$$

Then for $1 < \gamma < 2$,

$$
(3.16)\qquad |\widehat{\sigma_{k,\Gamma_\eta}}(\xi)| \le CA2^{4m/\gamma'}|2^{(m+1)kl_\eta}I_\eta(\xi)|^{-1/(4l_\eta\gamma')}.
$$

Interpolating between (3.15)-(3.16) and (3.12) yields

$$|\widehat{\sigma_{k,\Gamma_\eta}}(\xi)| \le CA|2^{(m+1)kl_\eta} I_\eta(\xi)|^{-1/(4(m+1)l_\eta\tilde{\gamma})}. \tag{3.17}$$

(3.9) follows from (3.12) and (3.17). This completes the proof of Lemma 3.1. ■

Lemma 3.2. *Let A be as above and $m \in N(\Omega) \cup \{0\}$. For any $1 \le \eta \le \mathcal{N}$ and arbitrary functions $\{g_{k,j}\}_{k,j} \in L^p(\ell^q(\ell^2), \mathbb{R}^d)$, there exists a constant $C > 0$ which is independent of m and γ such that*

$$\begin{aligned} &\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\Gamma_\eta} * g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \\ &\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \end{aligned} \tag{3.18}$$

for $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$.

Proof. Since $|||h|||_2 \le (m+1)^{1/2-1/\gamma}|||h|||_\gamma$ when $\gamma \ge 2$, we may assume that $1 < \gamma \le 2$. By duality, it suffices to prove (3.18) for $2 < p, q < 2\gamma/(2-\gamma)$. Given functions $\{f_j\}$ with $\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'},\mathbb{R}^d)} \le 1$. By the similar arguments as in getting (7.7) in [10], we have

$$\begin{aligned} &\int_{\mathbb{R}^d}|\sigma_{k,\Gamma_\eta} * g_{k,j}(x)|^2 f_j(x)dx \\ &\le C\|\Omega\|_{L^1(E_m)}|||h|||_\gamma^\gamma\int_{\mathbb{R}^d}|g_{k,j}(x)|^2\mathcal{M}_{\Gamma_\eta}(f_j)(x)dx, \end{aligned} \tag{3.19}$$

where

$$\mathcal{M}_{\Gamma_\eta}(f)(x) = \int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}}\int_{S^{n-1}}|f(x+\Gamma_\eta(ty'))||\Omega_m(y')|d\sigma(y')|h(t)|^{2-\gamma}\frac{dt}{t}.$$

By Hölder's inequality we have

$$\begin{aligned} &\mathcal{M}_{\Gamma_\eta}(f)(x) \\ &\le |||h|||_\gamma^{2-\gamma}\int_{S^{n-1}}\Big(\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}}|f(x+\Gamma_\eta(ty'))|^{\gamma'/2}\frac{dt}{t}\Big)^{2/\gamma'}|\Omega_m(y')|d\sigma(y') \\ &\le |||h|||_\gamma^{2-\gamma}\int_{S^{n-1}}\Big(\sum_{i=0}^{m}\int_{2^{(m+1)k+i}}^{2^{(m+1)(k+1)+i+1}}|f(x+\Gamma_\eta(ty'))|^{\gamma'/2}\frac{dt}{t}\Big)^{2/\gamma'}|\Omega_m(y')|d\sigma(y') \\ &\le (m+1)^{2/\gamma'}|||h|||_\gamma^{2-\gamma}\int_{S^{n-1}}|\Omega_m(y')| \\ &\quad\times\Big(\sup_{r>0}\frac{1}{r}\int_{|t|<r}|f(x+\Gamma_\eta(ty'))|^{\gamma'/2}dt\Big)^{2/\gamma'}d\sigma(y'). \end{aligned}$$

By Lemma 2.1 and Minkowski's inequality, we have for $\gamma'/2 < u, v < \infty$,

$$\begin{aligned}(3.20)\qquad &\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{M}_{\Gamma_\eta}(f_j)|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^d)}\\ &\le (m+1)^{2/\gamma'}\||h|\|_\gamma^{2-\gamma}\|\Omega\|_{L^1(E_m)}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^d)}.\end{aligned}$$

Thus by (3.19)-(3.20), we get

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\Gamma_\eta}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^2\\ &=\sup_{\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'},\mathbb{R}^d)}\le 1}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\sigma_{k,\Gamma_\eta}*g_{k,j}(x)|^2 f_j(x)dx\\ &\le C\|\Omega\|_{L^1(E_m)}\||h|\|_\gamma^\gamma\sup_{\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'},\mathbb{R}^d)}\le 1}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|g_{k,j}(x)|^2\mathcal{M}_{\Gamma_\eta}(f_j)(x)dx\\ &\le C\|\Omega\|_{L^1(E_m)}\||h|\|_\gamma^\gamma\sup_{\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'},\mathbb{R}^d)}\le 1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathcal{M}_{\Gamma_\eta}(f_j)|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^d)}\\ &\quad\times\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^2\\ &\le C(m+1)^{2/\gamma'}\|\Omega\|_{L^1(E_m)}^2\||h|\|_\gamma^2\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where we take $u=(p/2)'$ and $v=(q/2)'$. Then we prove (3.18) for $1<\gamma\le 2$. When $\gamma>2$, since $(m+1)^{1/2}\||h|\|_2\le (m+1)^{1/\gamma'}\||h|\|_\gamma$, therefore (3.18) holds for $\gamma>2$. Lemma 3.2 is proved. ■

Lemma 3.3. *Let* Γ, φ *be as in Theorem* 1.4. *Suppose that* $h\in\mathcal{H}_\gamma(\mathbb{R}^+)$ *for some* $\gamma>1$, *then* $h(\varphi^{-1})\Gamma(\varphi^{-1})\in\mathcal{H}_\gamma(\mathbb{R}^+)$. *Precisely,*

$$\|h(\varphi^{-1})\Gamma(\varphi^{-1})\|_{\mathcal{H}_\gamma(\mathbb{R}^+)}\le C\|h\|_{\mathcal{H}_\gamma(\mathbb{R}^+)},$$

where the constant $C>0$ *depends only on* φ.

Proof. We only prove the lemma in the case where φ is positive and increasing, since in the other case one can prove similarly. By the change of variables $t=\varphi(r)$ and Remark 1.6 (i) we have

$$\int_0^\infty|h(\varphi^{-1}(t))\Gamma(\varphi^{-1}(t))|^\gamma\frac{dt}{t}=\int_0^\infty|h(r)\Gamma(r)|^\gamma\frac{\varphi'(r)}{\varphi(r)}dr\le C\|h\|_{\mathcal{H}_\gamma(\mathbb{R}^+)}^\gamma.$$

This completes the proof of Lemma 3.3. ■

Lemma 3.4. *Let* Γ *and* φ *be as in Theorem* 1.4*. Then*

(i) *if* φ *is nonnegative and increasing,* $T_{h,\Omega,\mathcal{P},\varphi}(f) = T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\Omega,\mathcal{P}}(f)$*;*

(ii) *if* φ *is nonnegative and decreasing,* $T_{h,\Omega,\mathcal{P},\varphi}(f) = -T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\Omega,\mathcal{P}}(f)$*;*

(iii) *if* φ *is non-positive and decreasing,* $T_{h,\Omega,\mathcal{P},\varphi}(f) = T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\tilde{\Omega},\mathcal{P}}(f)$*;*

(iv) *if* φ *is non-positive and increasing,* $T_{h,\Omega,\mathcal{P},\varphi}(f) = -T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\tilde{\Omega},\mathcal{P}}(f)$*, where* $\tilde{\Omega}(y) = \Omega(-y)$*.*

Proof. We can get this lemma by Remark 1.6 and the similar arguments as in [7, Lemma 2.3]. The details are omitted. ∎

4. Proofs of Main Results

For $\eta \in \{1, \cdots, \mathcal{N}\}$, we denote $s(\eta) = \operatorname{rank}(I_\eta)$. By [10, Lemma 6.1] (see in [10, (7.35)]), there are two nonsingular linear transformations $H_\eta : \mathbb{R}^{s(\eta)} \to \mathbb{R}^{s(\eta)}$ and $G_\eta : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$|H_\eta \pi^d_{s(\eta)} G_\eta \xi| \le |I_\eta(\xi)| \le \Lambda_\eta |H_\eta \pi^d_{s(\eta)} G_\eta \xi|. \tag{4.1}$$

For a function $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\phi(t) \equiv 1$ for $|t| \le 1/2$ and $\phi(t) \equiv 0$ for $|t| \ge 1$. Let $\psi(t) = \phi(t^2)$ and define the measures $\{\tau_{k,\eta}\}$ by

$$\begin{aligned}\widehat{\tau_{k,\eta}}(\xi) &= \widehat{\sigma_{k,\Gamma_\eta}}(\xi) \prod_{\eta<j\le\mathcal{N}} \psi\Big(|2^{(m+1)(k+1)l_j} H_j \pi^d_{s(j)} G_j \xi|\Big) \\ &\quad - \widehat{\sigma_{k,\Gamma_{\eta-1}}}(\xi) \prod_{\eta-1<j\le\mathcal{N}} \psi\Big(|2^{(m+1)(k+1)l_j} H_j \pi^d_{s(j)} G_j \xi|\Big)\end{aligned} \tag{4.2}$$

for $k \in \mathbb{Z}$ and $1 \le \eta \le \mathcal{N}$, where we use convention $\Pi_{j\in\emptyset} a_j = 1$. It is easy to check that

$$\sigma_{k,\Gamma_\mathcal{N}} = \sum_{\eta=1}^{\mathcal{N}} \tau_{k,\eta}. \tag{4.3}$$

In addition, we can obtain the following estimates by (3.8)-(3.9):

$$\begin{aligned}&|\widehat{\tau_{k,\eta}}(\xi)| \\ &\le CA\Big[\min\{2^{(m+1)(k+1)l_\eta}\Lambda_\eta^{-1}|I_\eta(\xi)|, (2^{(m+1)(k+1)l_\eta}\Lambda_\eta^{-1}|I_\eta(\xi)|)^{-1}\}\Big]^{1/(4(m+1)l_\eta\tilde{\gamma})}.\end{aligned} \tag{4.4}$$

Now we are in a position to prove our main results.

Proof of Theorem 1.1. Let A and $N(\Omega)$ be as in Section 3. By (3.6)-(3.7) and (4.3), we have

$$T_{h,\Omega,\mathcal{P}}(f)(x) = \sum_{\eta=1}^{\mathcal{N}} \sum_{m\in N(\Omega)\cup\{0\}} \sum_{k\in\mathbb{Z}} \tau_{k,\eta} * f(x) := \sum_{\eta=1}^{\mathcal{N}} \sum_{m\in N(\Omega)\cup\{0\}} B_\eta(f)(x). \tag{4.5}$$

By (3.5) and the fact that $\||h|\|_\gamma \leq C\|h\|_{\mathcal{H}_\gamma(\mathbb{R}^+)}$, to prove Theorem 1.1, it suffices to prove that for any $1 \leq \eta \leq \mathcal{N}$ and $\alpha \in \mathbb{R}$,

$$\|B_\eta(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq CA\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \tag{4.6}$$

for $\max\{|1/p-1/2|, |1/q-1/2|\} < \min\{1/2, 1/\gamma'\}$, where $C = C(n, d, h, p, q, \alpha, \varphi, \deg(\mathcal{P}))$ is independent of the coefficients of P_j for $1 \leq j \leq d$ and m.

Let $\lambda \in \mathcal{S}(\mathbb{R}^+)$ satisfying

$$0 \leq \lambda(t) \leq 1,\ \operatorname{supp}(\lambda) \subset [2^{-(m+1)l_\eta}\Lambda_\eta,\ 2^{(m+1)l_\eta}\Lambda_\eta],$$

and $\sum_{k\in\mathbb{Z}} \lambda_k^2(t) = 1$ with $\lambda_k(t) = \lambda(2^{(m+1)kl_\eta}t)$. Define the operator S_k by

$$\widehat{S_k f}(\xi) := \lambda_k(|\pi_{s(\eta)}^d \xi|)\hat{f}(\xi).$$

Observe that we can write

$$B_\eta(f) = \sum_{k\in\mathbb{Z}} \tau_{k,\eta} * \Big(\sum_{j\in\mathbb{Z}} S_{j+k}S_{j+k}f\Big) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} S_{j+k}(\tau_{k,\eta} * S_{j+k}f) := \sum_{j\in\mathbb{Z}} B_\eta^j(f). \tag{4.7}$$

When $I_\eta = \pi_{s(\eta)}^d$, invoking the Littlewood-Paley theory and Plancherel's theorem, we get

$$\|B_\eta^j(f)\|_{L^2(\mathbb{R}^d)}^2 \leq C\sum_{k\in\mathbb{Z}} \int_{E_{j+k}} |\widehat{\tau_{k,\eta}}(\xi)|^2|\hat{f}(\xi)|^2 d\xi,$$

where

$$E_{j+k} = \{\xi \in \mathbb{R}^d :\ 2^{-(j+k+1)(m+1)l_\eta}\Lambda_\eta \leq |\pi_{s(\eta)}^d \xi| \leq 2^{-(j+k-1)(m+1)l_\eta}\Lambda_\eta\}.$$

This together with (4.4) yields

$$\|B_\eta^j(f)\|_{L^2(\mathbb{R}^d)} \leq 2^{-|j|/(4\tilde{\gamma})}CA\|f\|_{L^2(\mathbb{R}^d)},$$

in other words (by (1.5)),

$$\|B_\eta^j(f)\|_{\dot{F}_0^{2,2}(\mathbb{R}^d)} \leq 2^{-|j|/(4\tilde{\gamma})}CA\|f\|_{\dot{F}_0^{2,2}(\mathbb{R}^d)}. \tag{4.8}$$

Next, it remains only to show that

$$\|B_\eta^j(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq CA\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \tag{4.9}$$

for $\max\{|1/p-1/2|, |1/q-1/2|\} < \min\{1/2, 1/\gamma'\}$, $\alpha \in \mathbb{R}$, $j \in \mathbb{Z}$ and $1 \leq \eta \leq \mathcal{N}$. To prove (4.9), it suffices to prove that

$$\Big\|\Big(\sum_{i\in\mathbb{Z}} |B_\eta^j(g_i)|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq CA\Big\|\Big(\sum_{i\in\mathbb{Z}} |g_i|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{4.10}$$

for $\max\{|1/p-1/2|,|1/q-1/2|\}<\min\{1/2,1/\gamma'\}$ and $\{g_i\}\in L^p(\ell^q,\mathbb{R}^d)$, where C is independent of j and m. In fact, (4.10) implies (4.9), that is,

$$\begin{aligned}\|B_\eta^j(f)\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)} &= \Big\|\Big(\sum_{i\in\mathbb{Z}}2^{-i\alpha q}|\Psi_i*B_\eta^j(f)|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\le \Big\|\Big(\sum_{i\in\mathbb{Z}}|B_\eta^j(2^{-i\alpha}\Psi_i*f)|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\le CA\Big\|\Big(\sum_{i\in\mathbb{Z}}2^{-i\alpha q}|\Psi_i*f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &= CA\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)}.\end{aligned}$$

In what follows, we show (4.10). Using Proposition 2.3, Lemma 3.2, the definition of $\tau_{k,\eta}$ and the similar argument in getting [4, Propostion 2.3], one can check that

$$\begin{aligned}&\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\tau_{k,\eta}*g_{k,i}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\le C\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,i}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned}\tag{4.11}$$

for $\max\{|1/p-1/2|,|1/q-1/2|\}<\min\{1/2,1/\gamma'\}$. Let $\widehat{\Psi_k}(\xi^1)=\hat\Psi(2^{(m+1)kl_\eta}\xi^1)=\lambda_k(|\pi_{s(\eta)}^d\xi|)$, where $\xi=(\xi^1,\xi^2)$ with $\xi^1=(\xi_1,\cdots,\xi_{s(\eta)})$ and $\xi^2=(\xi_{s(\eta)+1},\xi_{s(\eta)+2},\cdots,\xi_d)$. It is clear that $\Psi\in\mathcal{S}(\mathbb{R}^{s(\eta)})$. By the definition of S_k, we have

$$S_k(f)(x)=\Psi_k\otimes\delta_{d-s(\eta)}*f(x).$$

Using Proposition 2.3 again, for $1<p,q<\infty$ and arbitrary functions $\{g_i\}_{i\in\mathbb{Z}}\in L^p(\ell^q,\mathbb{R}^d)$, we have

$$\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_k(g_i)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\le C\Big\|\Big(\sum_{i\in\mathbb{Z}}|g_i|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.\tag{4.12}$$

By duality and using (4.11)-(4.12), we get

$$\begin{aligned}&\Big\|\Big(\sum_{i\in\mathbb{Z}}|B_\eta^j(g_i)|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &=\sup_{\|\{f_i\}\|_{L^{p'}(\ell^{q'},\mathbb{R}^d)}\le1}\Big|\int_{\mathbb{R}^d}\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}S_{j+k}(\tau_{k,\eta}*S_{j+k}(g_i))(x)f_i(x)dx\Big|\\ &\le C\sup_{\|\{f_i\}\|_{L^{p'}(\ell^{q'},\mathbb{R}^d)}\le1}\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_{j+k}^*(f_i)|^2\Big)^{q'/2}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^d)}\\ &\quad\times\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\tau_{k,\eta}*S_{j+k}(g_i)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

$$\leq CA \Big\| \Big(\sum_{i\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |S_{j+k}(g_i)|^2 \Big)^{q/2} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)}$$
$$\leq CA \Big\| \Big(\sum_{i\in\mathbb{Z}} |g_i|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)}.$$

This proves (4.10). Then by interpolation (see [8, 13]) between (4.8) and (4.9) implies that there exists $\epsilon > 0$ such that for $\max\{|1/p-1/2|, |1/q-1/2|\} < \min\{1/2, 1/\gamma'\}$, $\alpha \in \mathbb{R}$ and $1 \leq \eta \leq \mathcal{N}$.

$$\|B_\eta^j(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leq 2^{-|j|\epsilon/(4\tilde{\gamma})} CA \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}, \tag{4.13}$$

which together with (4.7) implies (4.6) and completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. The proof of Theorem 1.2 is to copy the arguments in proving [4, Theorem 1.2]. By Theorem 1.1 and (1.5), for $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$, there exists a constant $C > 0$ such that

$$\|T_{h,\Omega,\mathcal{P}}(f)\|_{L^p(\mathbb{R}^d)} \leq C \|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})} \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.14}$$

Then for $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$, $1 < q < \infty$ and $\alpha \in \mathbb{Z}$, we have

$$\begin{aligned}
\|T_{h,\Omega,\mathcal{P}}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} &= \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * T_{h,\Omega,\mathcal{P}}(f)\|_{L^p(\mathbb{R}^d)}^q \Big)^{1/q} \\
&= \Big(\sum_{i\in\mathbb{Z}} \|T_{h,\Omega,\mathcal{P}}(2^{-i\alpha}\Psi_i * f)\|_{L^p(\mathbb{R}^d)}^q \Big)^{1/q} \\
&\leq C \|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})} \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|_{L^p(\mathbb{R}^d)}^q \Big)^{1/q} \\
&= C \|\Omega\|_{L(\log^+ L)^{1/\gamma'}(S^{n-1})} \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

Theorem 1.2 is proved. ∎

Proofs of Theorems 1.4 and 1.5. Using Lemmas 3.3–3.4 and Theorem 1.1, we get Theorem 1.4. Also, Theorem 1.5 follows from Lemmas 3.3–3.4 and Theorem 1.2. ∎

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References

1. H. M. Al-Qassem, On the boundedness of maxiaml operators and singular operators with kernels in $L(\log^+ L)^\alpha(S^{n-1})$, *J. Inequal. Appl.*, **2006** (2006), 1-16.

2. A. Al-Salman and Y. Pan, Singular integrals with rough kernels in $L\log^{+}L(S^{n-1})$, *J. London Math. Soc.*, **66(2)** (2002), 153-174.

3. A. P. Calderón and A. Zygmund, On singular integrals, *Amer. J. Math. Soc.*, **78** (1956), 289-309.

4. Y. Chen, Y. Ding and H. Liu, Rough singular integrals supported on submanifolds, *J. Math. Anal. Appl.*, **368** (2010), 677-691.

5. L. Colzani, *Hardy Spaces on Spheres*, Ph.D thesis, Washington University, St. Louis, 1982.

6. J. Duoandikoetxea and J. L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, *Invent. Math.*, **84** (1986), 541-561.

7. Y. Ding, Q. Xue and Y. Yabuta, On singular interal operators with rough kernel along surfaces, *Integr. Equ. Oper. Theory*, **68** (2010), 151-161.

8. M. Frazier and B. Jawerth, A discrete transform an decompositions of distribution spaces, *J. Funct. Anal.*, **93** (1990), 34-170.

9. M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study of Function Spaces*, CBMS Reg. Conf. Ser., Vol. 79. Amer. Math. Soc., Providence, RI, 1991.

10. D. Fan and Y. Pan, Singular integral operators with rough kernels supported by subvarieties, *Amer. J. Math.*, **119** (1997), 799-839.

11. D. Fan and S. Sato, A note on the singular integrals associated with a variable surface of revolution, *Math. Ineq. Appl.*, **12(2)** (2009), 441-454.

12. D. Fan and H. Wu, Non-isotropic singular integrals and maximal operators along surfaces of revolution, *Math. Ineq. Appl.*, **16(2)** (2013), 461-476.

13. L. Grafakos, *Classical and Modern Fourier Analysis*, Prentice Hall, Upper Saddle River, NJ, 2003.

14. H. Viet Le, Singular integrals with dominating mixed smoothness in Triebel-Lizorkin spaces, Preprint, 2013.

15. F. Ricci and G. Weiss, *A Characterization of $H^1(\sum_{n-1})$*, *Harmonic Analysis in Euclidean Space*, Proc. Sympos. Pure Math., WilliamsColl., Williamstown, Mass., 1978, Part 1, pp. 163-165, Proc. Sympos. Pure Math. 35, Part, Amer. Math. Soc., Providence, R.I., 1979.

16. S. Sato, Estimates for singular integrals and extrapolation, *Studia Math.*, **192** (2009), 219-233.

17. E. M. Stein, *Problems in Harmonic Analysis Related to Curvature and Oscillatory Integrals*, Proc. Intern. Cong. Math., Berkeley, 1986, pp. 196-221.

18. E. M. Stein, *Harmonic Analysis: Real-variable methods, orthogonality and oscillatory integral*, Princeton University Press, 1993.

19. H. Triebel, *Theory of Function Spaces*, Monogr. Math., Vol. 78, Birkhäser Verlag, Basel, 1983.

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