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AN EXTENSION OPERATOR ASSOCIATED WITH CERTAIN G-LOEWNER CHAINS

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Abstract. In this paper we are concerned with an extension operator $\Phi_{n,\alpha,\beta}$ that provides a way of extending a locally univalent function f on the unit disc U to a locally biholomorphic mapping $F \in H(B^n)$. By using the method of Loewner chains, we prove that if f can be embedded as the first element of a g-Loewner chain on the unit disc, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $|\zeta| < 1$ and $\gamma \in (0,1)$, then $F = \Phi_{n,\alpha,\beta}(f)$ can also be embedded as the first element of a g-Loewner chain on B^n, whenever $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \le 1$. In particular, if f is starlike of order $\gamma \in (0,1)$ on U, then $F = \Phi_{n,\alpha,\beta}(f)$ is also starlike of order γ on B^n. Also, if f is spirallike of type δ and order γ on U, where $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0,1)$, then $F = \Phi_{n,\alpha,\beta}(f)$ is spirallike of type δ and order γ on B^n. We also obtain a subordination preserving result under the operator $\Phi_{n,\alpha,\beta}$ and we consider some radius problems associated with this operator.

1. Introduction and Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. For $n \ge 2$, let $\tilde{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ so that $z = (z_1, \tilde{z}) \in \mathbb{C}^n$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n is denoted by B^n. In the case of one complex variable, B^1 is denoted by U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, $\|A\| = \sup\{\|A(z)\| : \|z\| = 1\}$ and let I_n be the identity of $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, we denote by $H(\Omega)$ the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If $f \in H(B^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. We say that $f \in H(B^n)$ is locally biholomorphic

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on B^n if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in B^n$. Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n. A holomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B^n)$. It is well known that any univalent mapping on B^n (holomorphic and injective on B^n) is also biholomorphic. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n. We also denote by $S^*(B^n)$ (respectively $K(B^n)$) the subset of $S(B^n)$ consisting of starlike mappings with respect to zero (respectively convex mappings). In the case of one complex variable, we write $\mathcal{L}S_1 = \mathcal{L}S$, $S(B^1) = S$, $K(B^1) = K$ and $S^*(B^1) = S^*$.

We next consider some subclasses of $S(B^n)$ that will be useful in the next section.

The following notion of starlikeness of order γ was introduced in [8, 29].

Definition 1.1. Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping and let $\gamma \in [0, 1)$. The mapping f is said to be starlike of order γ if

$$\text{Re}\left[\frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle}\right] > \gamma,\ z \in B^n \setminus \{0\}.$$

Remark 1.1. (i) In the case of one complex variable, the above relation is equivalent to $\text{Re}\,[zf'(z)/f(z)] > \gamma$ for $z \in U$, which is the usual notion of starlikeness of order γ on the unit disc U.

(ii) It is obvious that f is starlike of order 0 on B^n if and only if f is starlike. Also, if $\gamma \in (0, 1)$, then f is starlike of order γ if and only if

$$\left|\frac{1}{\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma},\ z \in B^n \setminus \{0\}.$$

Let $S^*_\gamma(B^n)$ be the set of starlike mappings of order γ on B^n. In the case $n = 1$, $S^*_\gamma(B^1)$ is denoted by S^*_γ. Note that if $f \in S^*_\gamma(B^n)$, then

$$\text{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0, \quad z \in B^n \setminus \{0\},$$

and thus $f \in S^*(B^n)$ (see [40]).

Another notion that will occur in the next section is that of spirallikeness of type δ and order γ, where $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in [0, 1)$ ([31]; cf. [26]).

Definition 1.2. Let $f \in \mathcal{L}S_n$, $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in [0, 1)$. We say that f is spirallike of type δ and order γ if

$$\text{Re}\left[\frac{1}{(1 - i\tan\delta)\frac{1}{\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle + i\tan\delta}\right] > \gamma,\ z \in B^n \setminus \{0\}. \tag{1.1}$$

Remark 1.2.

(i) It is easy to see that f is spirallike of type δ and order 0 on B^n if and only if f is spirallike of type δ on B^n. Also, if $\gamma \in (0,1)$, then f is spirallike of type δ and order γ if and only if

$$\left| e^{-i\delta}\frac{1}{\|z\|^2}\langle [Df(z)]^{-1}f(z), z\rangle + i\sin\delta - \frac{\cos\delta}{2\gamma}\right| < \frac{\cos\delta}{2\gamma},\ z \in B^n \setminus \{0\}. \tag{1.2}$$

(ii) Note that any spirallike mapping f of type δ and order γ on B^n is also spirallike of type δ, since the relation (1.1) implies that

$$\mathrm{Re}[e^{-i\delta}\langle [Df(z)]^{-1}f(z), z\rangle] > 0, \quad z \in B^n \setminus \{0\}.$$

Hence f is biholomorphic on B^n, in view of [26]. The class of spirallike mappings of type δ on B^n is denoted by $\hat{S}_\delta(B^n)$. When $n = 1$, $\hat{S}_\delta(B^1)$ is denoted by $\hat{S}_\delta$.

The following class of holomorphic mappings on B^n was introduced by Pfaltzgraff [34]:

$$\mathcal{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I_n,\ \mathrm{Re}\ \langle h(z), z\rangle > 0, z \in B^n \setminus \{0\}\}.$$

This class is related to various subclasses of biholomorphic mappings on B^n, such as starlikeness, spirallikeness of type δ, mappings which have parametric representation, etc (see e.g. [15]).

Next, let $\gamma \in [0,1)$ and $g : U \to \mathbb{C}$ be given by $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$. Also, let $\mathcal{M}_g$ be the subclass of $H(B^n)$ given by (see [15])

$$\mathcal{M}_g = \Big\{h : B^n \to \mathbb{C}^n : h \in H(B^n),\ h(0) = 0,\ Dh(0) = I_n,$$
$$\Big\langle h(z), \frac{z}{\|z\|^2}\Big\rangle \in g(U),\ z \in B^n\Big\}.$$

Here $\langle h(z), \frac{z}{\|z\|^2}\rangle\big|_{z=0} = 1$, since h is normalized. It is clear that $\mathcal{M}_g \subseteq \mathcal{M}$. Obviously, if $\gamma = 0$, then $\mathcal{M}_g \equiv \mathcal{M}$. Also, if $\gamma \in (0,1)$, then g maps the unit disc U onto the open disc of center $1/(2\gamma)$ and radius $1/(2\gamma)$, and thus

$$\mathcal{M}_g = \Big\{h \in H(B^n) : h(0) = 0, Dh(0) = I_n,$$
$$\Big|\frac{1}{\|z\|^2}\langle h(z), z\rangle - \frac{1}{2\gamma}\Big| < \frac{1}{2\gamma},\ z \in B^n\setminus\{0\}\Big\}.$$

We remark that a more general class $\mathcal{M}_g$ was introduced in [15].

Next, we recall the definitions of subordination and Loewner chains. For various results related to Loewner chains in $\mathbb{C}^n$, the reader may consult [1, 2, 9, 15, 17, 20, 23, 24, 34, 41].

Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$.

Definition 1.3. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$ whenever $0 \leq s \leq t < \infty$.

The above subordination condition is equivalent to the fact that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that $f(z, s) = f(v(z, s, t), t)$ for $z \in B^n$, $t \geq s \geq 0$.

The following characterization of Loewner chains was obtained by Pfaltzgraff [34] (see also [15, 20, 23]).

Lemma 1.1. *Let $h = h(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:*

(i) $h(\cdot, t) \in \mathcal{M}$ for $t \geq 0$.

(ii) $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B^n$.

Let $f = f(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be a mapping such that $f(\cdot, t) \in H(B^n)$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Assume that

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t), \text{ a.e. } t \geq 0, \forall z \in B^n.$$

Further, assume that there exists an increasing sequence $\{t_m\}_{m \in \mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ and $\lim_{m \to \infty} e^{-t_m} f(z, t_m) = F(z)$ locally uniformly on B^n. Then $f(z, t)$ is a Loewner chain.

Remark 1.3. In the case of one complex variable, if $f(\zeta, t)$ is a Loewner chain, then it is well known that $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family on U, and there exists a function $p = p(\zeta, t)$ such that (see [15]) $p(\cdot, t) \in \mathcal{P}$ for $t \geq 0$, $p(\zeta, \cdot)$ is measurable on $[0, \infty)$ for $\zeta \in U$, and (see [35])

$$\frac{\partial f}{\partial t}(\zeta, t) = \zeta f'(\zeta, t)p(\zeta, t), \quad \text{a.e.} \quad t \geq 0, \quad \forall \zeta \in U. \tag{1.3}$$

Remark 1.4. (i) In higher dimensions, Graham, Kohr and Kohr [23] (see also [20]) proved that if $f(z, t)$ is a Loewner chain on B^n, then $f(z, \cdot)$ is locally Lipschitz on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Also, there exists a mapping

$h = h(z,t)$, which satisfies the conditions (i) and (ii) in Lemma 1.1, such that (see [15])

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geq 0, \quad \forall\, z \in B^n. \tag{1.4}$$

(ii) The mapping $h = h(z,t)$ which occurs in the Loewner differential equation (1.4) is unique up to a measurable set of measure zero which is independent of $z \in B^n$, i.e. if there is another mapping $q = q(z,t)$ such that $q(\cdot,t) \in \mathcal{M}$ for a.e. $t \geq 0$, $q(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B^n$, and such that the Loewner differential equation (1.4) holds for $q(z,t)$, then $h(\cdot,t) = q(\cdot,t)$, a.e. $t \geq 0$ (see e.g. [3]).

Now, we are able to recall the notions of a g-Loewner chain and g-parametric representation (cf. [15]; compare with [23] and [36] for $g(\zeta) \equiv \frac{1-\zeta}{1+\zeta}$). For our purpose, we consider these notions only for $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, where $\gamma \in [0,1)$.

Definition 1.4. A mapping $f = f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ is called a g-Loewner chain if $f(z,t)$ is a Loewner chain such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and the mapping $h = h(z,t)$ which occurs in the Loewner differential equation (1.4) satisfies the condition $h(\cdot,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$.

Definition 1.5. Let $f : B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping. We say that f has g-parametric representation if there exists a g-Loewner chain $f(z,t)$ such that $f = f(\cdot,0)$.

Let $S_g^0(B^n)$ be the set of mappings which have g-parametric representation, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, and $\gamma \in [0,1)$. If $g(\zeta) \equiv \frac{1-\zeta}{1+\zeta}$, then $S_g^0(B^n)$ reduces to the usual set $S^0(B^n)$ of mappings which have parametric representation (see [15]; cf. [36]). The notion of parametric representation was considered in [15, 20, 23, 25, 36].

Remark 1.5. In view of Remark 1.3, we conclude that in the case $n = 1$, a g-Loewner chain $f(\zeta,t)$ is a Loewner chain such that the function $p(\zeta,t)$ defined by (1.3) satisfies the condition $p(\cdot,t) \in g(U)$ for a.e. $t \geq 0$. In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, any Loewner chain on the unit disc is also a g-Loewner chain.

We close this section with some extension operators that preserve the notions of starlikeness, spirallikeness of type δ and parametric representation.

Let $\Phi_{n,\alpha,\beta}$ be the operator given by (see [18])

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z}\left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} \right), \ z = (z_1, \tilde{z}) \in B^n,$$

where $\alpha \geq 0$, $\beta \geq 0$ and f is a locally univalent function on U, normalized by $f(0) = f'(0) - 1 = 0$, and such that $f(z_1) \neq 0$ for $z_1 \in U \setminus \{0\}$. We choose the branches of the power functions such that

$$\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\Bigg|_{z_1=0} = 1 \text{ and } (f'(z_1))^{\beta}|_{z_1=0} = 1.$$

The operator $\Phi_{n,0,1/2}$ reduces to the well known Roper-Suffridge extension operator Φ_n (see [37])

$$\Phi_n(f)(z) = \Big(f(z_1), \tilde{z}(f'(z_1))^{1/2}\Big),\ z = (z_1, \tilde{z}) \in B^n.$$

We remark that $\Phi_n(K) \subset K(B^n)$ (see [37]), $\Phi_n(S^*) \subset S^*(B^n)$ (see [19]), and $\Phi_n(S) \subset S^0(B^n)$ (see [22]). On the other hand, the operator $\Phi_{n,\alpha,\beta}$ preserves the notions of starlikeness and parametric representation from dimension one into the n-dimensional case, whenever $\alpha \in [0,1]$, $\beta \in [0,1/2]$, and $\alpha + \beta \leq 1$ (see [18]). However, $\Phi_{n,\alpha,\beta}(K) \subset K(B^n)$ if and only if $(\alpha, \beta) = (0, 1/2)$ [18].

In this paper we consider g-Loewner chains associated with the extension operator $\Phi_{n,\alpha,\beta}$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, and $\gamma \in (0,1)$. We shall prove that if $f \in S$ can be embedded as the first element of a g-Loewner chain, then $F = \Phi_{n,\alpha,\beta}(f)$ can also be embedded as the first element of a g-Loewner chain on B^n, for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, and $\alpha + \beta \leq 1$. As a consequence, the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of starlikeness of order γ, for $\gamma \in (0,1)$. Also, the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of spirallikeness of type δ and order γ, where $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0,1)$. Finally, we prove a subordination preserving result under the operator $\Phi_{n,\alpha,\beta}$ and we consider some radius problems associated with the operator $\Phi_{n,\alpha,\beta}$.

Other extension operators that preserve some subclasses of biholomorphic mappings may be found in [5, 6, 10-13, 16, 18, 21, 28, 31-33, 42].

2. The Operator $\Phi_{n,\alpha,\beta}$ and g-Loewner Chains

The main result of this section yields that the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of g-Loewner chain for $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, where $\gamma \in (0,1)$. This result was obtained in [18], in the case $\gamma = 0$. In the case $\alpha = 0$ and $\gamma \in (0,1)$, Theorem 2.1 was recently obtained in [6].

Theorem 2.1. *Assume $f \in S$ can be embedded as the first element of a g-Loewner chain, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, and $\gamma \in (0,1)$. Then $F = \Phi_{n,\alpha,\beta}(f)$ can be embedded as the first element of a g-Loewner chain on B^n for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$.*

Proof. We may assume that $n = 2$, since the general case is then easily handled.

Let $f(z_1, t)$ be a g-Loewner chain such that $f(z_1) = f(z_1, 0)$ for $z_1 \in U$. Let $F_{\alpha,\beta}(z, t)$ be the map defined by

$$F_{\alpha,\beta}(z,t) = \left(f(z_1,t), e^{(1-\alpha-\beta)t} z_2 \left(\frac{f(z_1,t)}{z_1} \right)^{\alpha} (f'(z_1,t))^{\beta} \right) \tag{2.1}$$

for $z = (z_1, z_2) \in B^2$ and $t \geq 0$. We know that $F_{\alpha,\beta}(z, t)$ is a Loewner chain, since $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, and $\alpha + \beta \leq 1$ (see [18]).

Since $f(z_1, t)$ is a Loewner chain on U, there exists a function $p(z_1, t)$ that is holomorphic on U and measurable in $t \geq 0$, with $p(0, t) = 1$, Re $p(z_1, t) > 0$ for $z_1 \in U$, $0 \leq t < \infty$, and such that (see [35])

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t) p(z_1,t), \text{ a.e. } t \geq 0, \forall\, z_1 \in U.$$

The fact that $f(z_1, t)$ is a g-Loewner chain is equivalent to the condition

$$|2\gamma p(z_1,t) - 1| < 1, \text{ a.e. } t \geq 0, \ \forall z_1 \in U.$$

The mapping $h = h(z, t)$ which occurs in the Loewner differential equation

$$\frac{\partial F_{\alpha,\beta}}{\partial t}(z,t) = DF_{\alpha,\beta}(z,t) h(z,t), \text{ a.e. } t \geq 0, \ \forall z \in B^2$$

is given by [18]

$$h(z,t) = (z_1 p(z_1,t), z_2(1 - \alpha - \beta + (\alpha+\beta) p(z_1,t) + \beta z_1 p'(z_1,t))),$$

for $z = (z_1, z_2) \in B^2$ and $t \geq 0$.

We have to prove that $h(\cdot, t) \in \mathcal{M}_g$ for a.e. $t \geq 0$, which is equivalent to

$$\left| \frac{1}{\|z\|^2} \langle h(z,t), z \rangle - \frac{1}{2\gamma} \right| < \frac{1}{2\gamma}, \text{ a.e. } t \geq 0, \ \forall\, z \in B^2 \backslash \{0\}.$$

If $z = (z_1, 0)$ then

$$\left| \frac{1}{\|z\|^2} \langle h(z,t), z \rangle - \frac{1}{2\gamma} \right| = \left| p(z_1,t) - \frac{1}{2\gamma} \right| < \frac{1}{2\gamma}, \text{ a.e. } t \geq 0,$$

in view of the fact that $f(z_1, t)$ is a g-Loewner chain. Hence it suffices to assume that $z = (z_1, z_2) \in B^2 \setminus \{0\}$ with $z_2 \neq 0$.

Taking into account the maximum principle for holomorphic functions, it is enough to prove that

$$|2\gamma \langle h(z,t), z \rangle - 1| \leq 1, \text{ a.e. } t \geq 0, \ \forall\, z = (z_1, z_2) \in \mathbb{C}^2, \ |z_1|^2 + |z_2|^2 = 1, \ z_2 \neq 0.$$

By elementary computations, we obtain that

$$\begin{aligned}
&|2\gamma\langle h(z,t),z\rangle - 1|\\
&= |2\gamma p(z_1,t)[|z_1|^2(1-\alpha-\beta)+(\alpha+\beta)]\\
&\quad +2\gamma(1-|z_1|^2)\beta z_1 p'(z_1,t)+2\gamma(1-|z_1|^2)(1-\alpha-\beta)-1|.
\end{aligned}$$

Therefore, we need to prove that

$$\begin{aligned}
&|(2\gamma p(z_1,t)-1)[|z_1|^2(1-\alpha-\beta)+(\alpha+\beta)]\\
&+2\gamma(1-|z_1|^2)\beta z_1 p'(z_1,t)+(1-\alpha-\beta)(1-|z_1|^2)(2\gamma-1)|\le 1.
\end{aligned}$$

Since $p(\cdot,t)$ is a holomorphic function on the unit disc U and

$$|2\gamma p(z_1,t)-1|<1,\ |z_1|<1,$$

we deduce in view of the Schwarz-Pick lemma that

$$2\gamma|p'(z_1,t)|\le \frac{1-|2\gamma p(z_1,t)-1|^2}{1-|z_1|^2},\ |z_1|<1.$$

Hence we obtain that

$$\begin{aligned}
&|(2\gamma p(z_1,t)-1)[|z_1|^2(1-\alpha-\beta)+(\alpha+\beta)]\\
&+2\gamma(1-|z_1|^2)\beta z_1 p'(z_1,t)+(1-\alpha-\beta)(1-|z_1|^2)(2\gamma-1)|\\
\le\ & |2\gamma p(z_1,t)-1|[(1-\alpha-\beta)|z_1|^2+(\alpha+\beta)]\\
&+(1-\alpha-\beta)(1-|z_1|^2)|2\gamma-1|+\beta|z_1|(1-|2\gamma p(z_1,t)-1|^2).
\end{aligned}$$

Denote by $q(z_1)=2\gamma p(z_1,t)-1$. Then $|q(z_1)|\in[0,1)$. Using the fact that $|2\gamma-1|<1$, for $\gamma\in(0,1)$, we obtain that

$$\begin{aligned}
&|2\gamma\langle h(z,t),z\rangle-1|\\
&\le |q(z_1)|[(1-\alpha-\beta)|z_1|^2+(\alpha+\beta)]+(1-\alpha-\beta)(1-|z_1|^2)\\
&\quad +\beta|z_1|(1-|q(z_1)|^2)-1+1\\
&=\beta|z_1|(1-|q(z_1)|^2)+1+(\alpha+\beta)(1-|z_1|^2)(|q(z_1)|-1)+|z_1|^2(|q(z_1)|-1)\\
&=(1-|q(z_1)|)[\beta|z_1|(1+|q(z_1)|)-|z_1|^2-(\alpha+\beta)(1-|z_1|^2)]+1\\
&\le (1-|q(z_1)|)(2\beta|z_1|-|z_1|^2-(\alpha+\beta)(1-|z_1|^2))+1.
\end{aligned}$$

We may consider the following two cases (cf. [30]):

Case 1. If $|z_1| \le \sqrt{2}-1$, then

$$\begin{aligned}&(1-|q(z_1)|)(2\beta|z_1|-|z_1|^2-\alpha(1-|z_1|^2)-\beta(1-|z_1|^2))+1\\ &\le (1-|q(z_1)|)(\beta(2|z_1|-1+|z_1|^2)-|z_1|^2)+1.\end{aligned}$$

Therefore, to prove the inequality $|2\gamma\langle h(z,t),z\rangle-1|\le 1$, a.e. $t\ge 0$, $z=(z_1,z_2)\in\mathbb{C}^2$, $|z_1|^2+|z_2|^2=1$, $z_2\ne 0$, it suffices to prove that

$$\beta(|z_1|^2+2|z_1|-1)-|z_1|^2\le 0.$$

The roots of the quadratic equation $x^2+2x-1=0$ are $x_1=-1-\sqrt{2}$ and $x_2=\sqrt{2}-1$, therefore $|z_1|^2+2|z_1|-1\le 0$, for $|z_1|\le\sqrt{2}-1$. Hence the above relation is proven.

Case 2. If $\sqrt{2}-1\le|z_1|<1$, using the fact that $\beta\in[0,1/2]$, we obtain that

$$\begin{aligned}&(1-|q(z_1)|)(2\beta|z_1|-|z_1|^2-\alpha(1-|z_1|^2)-\beta(1-|z_1|^2))+1\\ &\le (1-|q(z_1)|)(\beta(2|z_1|-1+|z_1|^2)-|z_1|^2)+1\\ &\le \frac{1-|q(z_1)|}{2}(-|z_1|^2+2|z_1|-1)+1=-\frac{1-|q(z_1)|}{2}(|z_1|-1)^2+1\le 1.\end{aligned}$$

Finally, it remains to prove that $\{e^{-t}F_{\alpha,\beta}(\cdot,t)\}_{t\ge0}$ is a normal family on B^n. Indeed, since $\{e^{-t}f(\cdot,t)\}_{t\ge0}$ is a normal family on U, there exists a sequence (t_m) such that $0<t_m\to\infty$ and $e^{-t_m}f(z_1,t_m)\to r(z_1)$ locally uniformly on U as $m\to\infty$. It is clear that $r\in S$, in view of Hurwitz's theorem. Then it is easy to see that $e^{-t_m}F_{\alpha,\beta}(z,t_m)\to R(z)$ locally uniformly on B^n as $m\to\infty$, where $R=\Phi_{n,\alpha,\beta}(r)$, and thus $\{e^{-t}F_{\alpha,\beta}(\cdot,t)\}_{t\ge0}$ is also a normal family on B^n.

Combining the above arguments, we deduce that $F_{\alpha,\beta}(z,t)$ is a g-Loewner chain, as desired. This completes the proof. ■

In view of Theorem 2.1, we obtain the following particular cases. Corollary 2.1 was obtained in [18], in the case $\gamma=0$. Also, Corollary 2.1 was recently obtained in [6], in the case $\alpha=0$.

Corollary 2.1. *If $f:U\to\mathbb{C}$ has g-parametric representation and $\alpha\in[0,1]$, $\beta\in[0,1/2]$, $\alpha+\beta\le1$, then $F=\Phi_{n,\alpha,\beta}(f)\in S_g^0(B^n)$, where $g(\zeta)=\frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta\in U$, and $\gamma\in(0,1)$.*

Proof. Since f has g-parametric representation, there exists a g-Loewner chain $f(\zeta,t)$ such that $f=f(\cdot,0)$. In view of the proof of Theorem 2.1, we deduce that the mapping $F_{\alpha,\beta}(z,t)$ given by (2.1) is also a g-Loewner chain. Since $F=F_{\alpha,\beta}(\cdot,0)$, we deduce that $F\in S_g^0(B^n)$, as desired. This completes the proof. ■

The following result was obtained by Hamada, Kohr and Kohr [27], in the case $\alpha=0$, $\beta=\gamma=1/2$, and by Liu [30], in the case $\gamma\in(0,1)$ and $\alpha\in[0,1]$, $\beta\in[0,1/2]$, $\alpha+\beta\le1$. If $\gamma=0$, the result below was obtained in [18].

Corollary 2.2. *If $f \in S^*_\gamma$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \le 1$, then $F = \Phi_{n,\alpha,\beta}(f) \in S^*_\gamma(B^n)$, where $\gamma \in (0,1)$. In particular, the Roper-Suffridge extension operator preserves the notion of starlikeness of order γ.*

Proof. Since f is starlike of order γ, it follows that $f(\zeta,t) = e^t f(\zeta)$ is a g-Loewner chain (see e.g. [40]), where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$. In view of the proof of Theorem 2.1, we deduce that the mapping $F_{\alpha,\beta}(z,t)$ given by (2.1) is a g-Loewner chain. Since $f(\zeta,t) = e^t f(\zeta)$ it follows that $F_{\alpha,\beta}(z,t) = e^t F(z)$, and thus $F = \Phi_{n,\alpha,\beta}(f)$ is starlike of order γ, as desired. ∎

Remark 2.6. Since $K \subset S^*_{1/2}$ (see e.g. [20] and [35]), it follows in view of Corollary 2.2 that $\Phi_{n,\alpha,\beta}(K) \subset S^*_{1/2}(B^n)$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \le 1$. However, $\Phi_{n,\alpha,\beta}(K) \nsubseteq K(B^n)$ for $(\alpha,\beta) \neq (0,1/2)$ (see [18]).

The following result is due to Liu and Liu [31] (see also [30] and [42]).

Corollary 2.3. *Let $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \le 1$, $\delta \in (-\pi/2,\pi/2)$ and $\gamma \in (0,1)$. Also, let $f : U \to \mathbb{C}$ be a spirallike function of type δ and order γ on U, and let $F = \Phi_{n,\alpha,\beta}(f)$. Then F is also spirallike of type δ and order γ on B^n.*

Proof. First, we prove that $f(z_1,t) = e^{(1-ia)t} f(e^{iat}z_1)$ is a g-Loewner chain, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$ and $a = \tan\delta$ (see also [7]). Indeed, since f is spirallike of type δ and order γ, it is also spirallike of type δ on U. Hence $f(z_1,t)$ is a Loewner chain (see [26]). The mapping $p = p(z_1,t)$ which occurs in the Loewner differential equation

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t)p(z_1,t), \text{ a.e. } t \ge 0, \ \forall z_1 \in U$$

is given by

$$p(z_1,t) = ia + (1-ia)\frac{f(e^{iat}z_1)}{e^{iat}z_1 f'(e^{iat}z_1)}, \ z_1 \in U, \ t \ge 0.$$

From the relation (1.2) we obtain that $p(z_1,t) \in g(U)$ a.e. $t \ge 0$ and $z_1 \in U$.

It remains to prove that $\{e^{-t}f(\cdot,t)\}_{t\ge 0}$ is a normal family on U. Indeed, since f is bounded on each closed disc $\overline{U}_r$, $r \in (0,1)$, it follows that for each $r \in (0,1)$ there exists $M = M(r) \ge 0$ such that

$$|e^{-t}f(z_1,t)| = |e^{-iat}f(e^{iat}z_1)| = |f(e^{iat}z_1)| \le M(r), \ |z_1| \le r, \ t \ge 0.$$

Consequently, $\{e^{-t}f(\cdot,t)\}_{t\ge 0}$ is a locally uniformly bounded family on U and thus is normal. Hence $f(z_1,t) = e^{(1-ia)t}f(e^{iat}z_1)$ is a g-Loewner chain.

In view of the proof of Theorem 2.1, we deduce that the mapping $F_{\alpha,\beta}(z,t)$ given by (2.1) is a g-Loewner chain. It is easily seen that $F_{\alpha,\beta}(z,t) = e^{(1-ia)t}F(e^{iat}z)$.

Thus we know that $\frac{1}{\|z\|^2}\langle h(z,t),z\rangle \in g(U)$, a.e. $t\geq 0$, $z\in B^n\setminus\{0\}$, where $h(z,t)$ is obtained from the Loewner differential equation

$$\frac{\partial F_{\alpha,\beta}}{\partial t}(z,t)=DF_{\alpha,\beta}(z,t)h(z,t),\ \text{a.e. } t\geq 0,\ \forall z\in B^n.$$

The mapping $h(z,t)$ is given by

$$h(z,t)=iaz+(1-ia)e^{-iat}[DF(e^{iat}z)]^{-1}F(e^{iat}z),\ z\in B^n,\ t\geq 0.$$

It is easily seen that the relation $\frac{1}{\|z\|^2}\langle h(z,t),z\rangle\in g(U)$ implies relation (1.2), therefore F is spirallike of type δ and order γ, as desired. This completes the proof. ■

3. Subordination Associated with the Operator $\Phi_{n,\alpha,\beta}$

We next obtain a subordination preserving result under the operator $\Phi_{n,\alpha,\beta}$. More precisely, we prove the following (see [27], in the case $\alpha=0$ and $\beta=1/2$):

Theorem 3.1. *Let $f,g:U\to\mathbb{C}$ be two locally univalent functions such that $f(0)=g(0)=0$, $f'(0)=a$ and $g'(0)=b$, where $0<a\leq b$. Assume that $f(z_1)\neq 0$ and $g(z_1)\neq 0$ for $0<|z_1|<1$. If $\alpha\geq 0$, $\beta\in[0,1/2]$ and $f\prec g$, then $\Phi_{n,\alpha,\beta}(f)\prec\Phi_{n,\alpha,\beta}(g)$. We choose the branches of the power functions such that*

$$[f'(z_1)]^\beta|_{z_1=0}=a^\beta,\ \left.\left[\frac{f(z_1)}{z_1}\right]^\alpha\right|_{z_1=0}=a^\alpha,$$

$$[g'(z_1)]^\beta|_{z_1=0}=b^\beta,\ \left.\left[\frac{g(z_1)}{z_1}\right]^\alpha\right|_{z_1=0}=b^\alpha.$$

Proof. Let $F=\Phi_{n,\alpha,\beta}(f)$ and $G=\Phi_{n,\alpha,\beta}(g)$. Since $f\prec g$ it follows that there exists a Schwarz function $v=v(z_1)$ such that $f(z_1)=g(v(z_1))$, $z_1\in U$. It is clear that $v'(0)=\frac{a}{b}$ and since f and g are locally univalent on U, v is locally univalent on U too. Let $V:B^n\to\mathbb{C}^n$ be given by

$$V(z)=\left(v(z_1),\tilde{z}\left[\frac{v(z_1)}{z_1}\right]^\alpha[v'(z_1)]^\beta\right),\ z=(z_1,\tilde{z})\in B^n.$$

We choose the branches of the power functions such that $[v'(z_1)]^\beta|_{z_1=0}=\left(\frac{a}{b}\right)^\beta$ and $\left.\left[\frac{v(z_1)}{z_1}\right]^\alpha\right|_{z_1=0}=\left(\frac{a}{b}\right)^\alpha$. Then V is a locally biholomorphic mapping on B^n, $V(0)=0$ and it is easy to deduce that $V(z)\in B^n$, $z\in B^n$. Indeed, fix $z\in B^n$ and let $w=V(z)$. Applying the Schwarz-Pick lemma, we deduce that

$$\begin{aligned}|w_1|^2 + \|\tilde{w}\|^2 &= |v(z_1)|^2 + \|\tilde{z}\|^2 \left|\frac{v(z_1)}{z_1}\right|^{2\alpha} |v'(z_1)|^{2\beta} \\ &\leq |v(z_1)|^2 + \|\tilde{z}\|^2 \left|\frac{v(z_1)}{z_1}\right|^{2\alpha} \left[\frac{1-|v(z_1)|^2}{1-|z_1|^2}\right]^{2\beta} \\ &\leq |v(z_1)|^2 + \|\tilde{z}\|^2 \frac{1-|v(z_1)|^2}{1-|z_1|^2} < |v(z_1)|^2 + 1 - |v(z_1)|^2 = 1.\end{aligned}$$

Here we have used that $|v(z_1)| \leq |z_1|$, $z_1 \in U$ and $\alpha \geq 0$, $\beta \in [0, 1/2]$. Hence $w \in B^n$, as desired. Moreover, we can easily deduce that $F(z) = G(V(z))$, $z \in B^n$. Indeed, since $v(z_1) \neq 0$ for $z_1 \neq 0$,

$$\begin{aligned}G(V(z)) &= (g(v(z_1)), \tilde{z} \left[\frac{v(z_1)}{z_1}\right]^{\alpha} [v'(z_1)]^{\beta} \left[\frac{g(v(z_1))}{v(z_1)}\right]^{\alpha} [g'(v(z_1))]^{\beta}) \\ &= (g(v(z_1)), \tilde{z} \left[\frac{g(v(z_1))}{z_1}\right]^{\alpha} [(g \circ v)'(z_1)]^{\beta}) \\ &= (f(z_1), \tilde{z} \left[\frac{f(z_1)}{z_1}\right]^{\alpha} [f'(z_1)]^{\beta}) = F(z), \ z \in B^n.\end{aligned}$$

Therefore $F \prec G$. This completes the proof. ■

We next obtain certain consequences of the above result. These results were obtained in [27], for $\alpha = 0$ and $\beta = 1/2$.

Corollary 3.1. *Let $f \in \mathcal{L}S$ and $M \geq 1$ be such that $|f(z_1)| \leq M$, $z_1 \in U$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq M$, $z \in B^n$, whenever $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$.*

Proof. Let $g(z_1) = Mz_1$, for $z_1 \in U$. Then $f \prec g$ and hence $\Phi_{n,\alpha,\beta}(f) \prec \Phi_{n,\alpha,\beta}(g)$, for $\alpha \geq 0$, $\beta \in [0, 1/2]$, by Theorem 3.1. Since $\Phi_{n,\alpha,\beta}(g)(z) = (Mz_1, \tilde{z}M^{\alpha+\beta})$, $z = (z_1, \tilde{z}) \in B^n$ and $\alpha + \beta \leq 1$, it is easy to see that $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq M$, $z \in B^n$. ■

Corollary 3.2. *Let $f \in \mathcal{L}S$ and $M \geq 1$ be such that $|f(z_1)| \leq M$, $z_1 \in U$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Then $\Phi_{n,\alpha,\beta}(f) \in S^0(B^n_r)$, where $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$, and $r = 1/(M + \sqrt{M^2 - 1})$.*

Proof. Assume first that $M = 1$. Then $|f(z_1)| \leq 1$, $z_1 \in U$. Taking into account the Schwarz lemma and the fact that f is normalized by $f(0) = 0$ and $f'(0) = 1$, we deduce that $f(z_1) = z_1$ for $z_1 \in U$. Hence, in this case the conclusion is obvious.

Assume next that $M > 1$. Since $|f(z_1)| \leq M$, $z_1 \in U$, it follows in view of a well-known result of Landau (see e.g. [4, Theorem 1]) that f is univalent on the disc U_r, where $r = 1/(M + \sqrt{M^2 - 1})$. Now, let $f_r(z_1) = f(rz_1)/r$, for $z_1 \in U$. Then

$f_r \in S$ and hence $\Phi_{n,\alpha,\beta}(f_r) \in S^0(B^n)$, since $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$ (see [18]). On the other hand, it is easy to see that

$$\Phi_{n,\alpha,\beta}(f_r)(z) = \frac{1}{r}\Phi_{n,\alpha,\beta}(f)(rz),\ z \in B^n.$$

The conclusion now follows. ■

Corollary 3.3. *Let $f : U \to \mathbb{C}$ be a locally univalent function on U such that $f(0) = 0$ and $f'(0) = a$, where $a \in (0,1]$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Also let $g \in S$ and assume that $f \prec g$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq \|z\|/(1-\|z\|)^2$, $z \in B^n$, whenever $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$.*

Proof. Since $g \in S$ it follows that $\Phi_{n,\alpha,\beta}(g) \in S^0(B^n)$, for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$ (see [18]). Hence $\|\Phi_{n,\alpha,\beta}(g)(z)\| \leq \|z\|/(1-\|z\|)^2$, $z \in B^n$, by [15, Corollary 2.4]. Next, it suffices to apply Theorem 3.1. ■

Corollary 3.4. *Let f be a locally univalent function on the unit disc U with $f(0) = 0$ and $f'(0) = a \in (0,1]$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Also let $g \in S^*_\gamma$, $\gamma \in (0,1)$, and assume that $f \prec g$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq \|z\|/(1-\|z\|)^{2(1-\gamma)}$, $z \in B^n$, whenever $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$.*

Proof. Since $g \in S^*_\gamma$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$, it follows that $\Phi_{n,\alpha,\beta}(g) \in S^*_\gamma(B^n)$, by Corollary 2.2. Hence $\|\Phi_{n,\alpha,\beta}(g)(z)\| \leq \|z\|/(1-\|z\|)^{2(1-\gamma)}$, $z \in B^n$ (see e.g. [8]) . Next, it suffices to apply Theorem 3.1. ■

In view of Corollary 3.4, we obtain the following consequence.

Corollary 3.5. *Let f be a locally univalent function on the unit disc U with $f(0) = 0$ and $f'(0) = a \in (0,1]$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Also let $g \in K$ and assume that $f \prec g$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq \|z\|/(1-\|z\|)$, $z \in B^n$, whenever $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$.*

Proof. Since $g \in K$, it follows that $g \in S^*_{1/2}$. The result follows in view of Corollary 3.4. ■

We now present another consequence of Theorem 3.1 (see [27] for $\alpha = 0$ and $\beta = 1/2$).

Corollary 3.6. *Let $F = \Phi_{n,\alpha,\beta}(f)$ and $G = \Phi_{n,\alpha,\beta}(g)$ where f is a locally univalent function on the unit disc such that $f(0) = 0$, $f'(0) = a \in (0,1]$, $f(z_1) \neq 0$ for $0 < |z_1| < 1$, $g \in K$, $\alpha \geq 0$, $\beta \in [0,1/2]$. Assume $DF(z)(z) \prec DG(z)(z)$, $z \in B^n$. Then $F(z) \prec G(z)$, $z \in B^n$.*

Proof. We may assume that $n = 2$, since the general case is easily handled. A short computation yields that

$$DF(z)(z) = \left(z_1 f'(z_1), z_1 z_2 \alpha [f'(z_1)]^\beta \left[\frac{f(z_1)}{z_1}\right]^{\alpha-1} \left[\frac{f(z_1)}{z_1}\right]' \right.$$
$$\left. + z_1 z_2 \beta \left[\frac{f(z_1)}{z_1}\right]^\alpha [f'(z_1)]^{\beta-1} f''(z_1) + z_2 \left[\frac{f(z_1)}{z_1}\right]^\alpha [f'(z_1)]^\beta \right)$$

and

$$DG(z)(z) = \left(z_1 g'(z_1), z_1 z_2 \alpha [g'(z_1)]^\beta \left[\frac{g(z_1)}{z_1}\right]^{\alpha-1} \left[\frac{g(z_1)}{z_1}\right]' \right.$$
$$\left. + z_1 z_2 \beta \left[\frac{g(z_1)}{z_1}\right]^\alpha [g'(z_1)]^{\beta-1} g''(z_1) + z_2 \left[\frac{g(z_1)}{z_1}\right]^\alpha [g'(z_1)]^\beta \right),$$

for all $z = (z_1, z_2) \in B^2$. Let $S(z) = DF(z)(z)$ and $T(z) = DG(z)(z)$. Since $S \prec T$, there exists a Schwarz mapping ω such that $S(z) = T(\omega(z))$, $z \in B^2$. Therefore $z_1 f'(z_1) = \omega_1(z) g'(\omega_1(z))$, where $\omega(z) = (\omega_1(z), \omega_2(z))$, $z = (z_1, z_2) \in B^2$. Taking $z = (z_1, 0) \in B^2$, we obtain that

$$z_1 f'(z_1) = \omega_1(z_1, 0) g'(\omega_1(z_1, 0)). \tag{3.1}$$

Let $w(\zeta) = \omega_1(\zeta, 0)$, $|\zeta| < 1$. Then w is holomorphic on U, $w(0) = 0$ and

$$|w(\zeta)| = |\omega_1(\zeta, 0)| \le \|\omega(\zeta, 0)\| \le \|(\zeta, 0)\| = |\zeta|, \ |\zeta| < 1.$$

Here we have used the fact that ω is a Schwarz mapping. We have obtained that $|w(\zeta)| \le |\zeta| < 1$, $\zeta \in U$, hence w is a Schwarz function on U.

Relation (3.1) can be written as $z_1 f'(z_1) = w(z_1) g'(w(z_1))$, where w is the above Schwarz function on the unit disc. Hence $z_1 f'(z_1) \prec z_1 g'(z_1)$, $z_1 \in U$. Since $g \in K$, we may apply a well known result (see [39]), to deduce that $f(z_1) \prec g(z_1)$. Finally, in view of Theorem 3.1, the conclusion follows, as desired. ■

4. RADIUS PROBLEMS AND THE OPERATOR $\Phi_{n,\alpha,\beta}$

We next consider some radius problems associated with the operator $\Phi_{n,\alpha,\beta}$. First, we recall the concept of the radius for a certain property in a certain set (see e.g. [14] and [20]).

Definition 4.1. Given $\mathcal{F}$ a nonempty subset of $S(B^n)$ and a property $\mathcal{P}$ which the mappings in $\mathcal{F}$ may or may not have in a ball B_r^n, the radius for the property $\mathcal{P}$ in the set $\mathcal{F}$ is denoted by $R_{\mathcal{P}}(\mathcal{F})$ and is the largest R such that every mapping in the set $\mathcal{F}$ has the property $\mathcal{P}$ in each ball B_r^n for every $r < R$.

We let $R_{S^*}(\mathcal{F})$ be the radius of starlikeness of $\mathcal{F}$, $R_K(\mathcal{F})$ the radius of convexity, $R_{S^*_\gamma}(\mathcal{F})$ the radius of starlikeness of order γ and $R_{\hat{S}_\delta}(\mathcal{F})$ the radius of spirallikeness of type δ of $\mathcal{F}$.

It is well known that $R_K(S) = R_K(S^*) = 2 - \sqrt{3}$ and $R_{S^*}(S) = \tanh(\pi/4)$ (see e.g. [35]). Graham, Kohr and Kohr [22] obtained the radius of starlikeness and convexity associated with $\Phi_n(S)$. Also, Graham, Hamada, Kohr and Suffridge [18] obtained the radius of starlikeness associated with $\Phi_{n,\alpha,\beta}(S)$. In this section, we shall obtain other radius problems for some subsets of $S(B^n)$ associated with the operator $\Phi_{n,\alpha,\beta}$. We begin with the following remark (cf. [18]):

Remark 4.1. If $\Phi_{n,\alpha,\beta}(f) \in S(B^n_r)$, then $f \in S(U_r)$, for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ such that $\alpha + \beta \leq 1$ and $r \in (0,1)$. On the other hand, if $\Phi_{n,\alpha,\beta}(f) \in S^*(B^n_r)$ (respectively $K(B^n_r)$, $S^*_\gamma(B^n_r)$, $\hat{S}_\delta(B^n_r)$), then $f \in S^*(U_r)$ ($K(U_r)$, $S^*_\gamma(U_r)$, $\hat{S}_\delta(U_r)$, respectively). Also, if $f \in S(U_r)$ then $\Phi_{n,\alpha,\beta}(f) \in S^0(B^n_r)$, for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha + \beta \leq 1$, since the equality

$$\Phi_{n,\alpha,\beta}(f_r)(z) = \frac{1}{r}\Phi_{n,\alpha,\beta}(f)(rz)$$

holds on B^n, where $f_r(\zeta) = \frac{1}{r}f(r\zeta)$, $\zeta \in U$.

Now, we obtain the following result regarding the radius of spirallikeness of type δ for the set $\Phi_{n,\alpha,\beta}(S)$.

Theorem 4.1. $R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(S)) = \tanh\left[\frac{\pi}{4} - \frac{|\delta|}{2}\right]$, *for* $\alpha \in [0,1]$, $\beta \in [0,1/2]$ *such that* $\alpha + \beta \leq 1$ *and* $\delta \in (-\pi/2, \pi/2)$.

Proof. It is known that if $f \in S$, then f is spirallike of type δ in U_r, where $r = \tanh\left[\frac{\pi}{4} - \frac{|\delta|}{2}\right]$ and this number is the radius of spirallikeness of type δ for the class S (see [38, Theorem 4] for $\beta = 0$). Hence

$$\operatorname{Re} \frac{e^{i\delta} z_1 f'(z_1)}{f(z_1)} > 0, \ |z_1| < r,$$

and the left hand side of the above inequality can be negative if $|z_1| > r$.

Let $F_{\alpha,\beta} = \Phi_{n,\alpha,\beta}(f)$. Using Remark 4.1 and the fact that the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of spirallikeness of type δ, for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ such that $\alpha + \beta \leq 1$ (see e.g. [30]), we deduce that $F_{\alpha,\beta} \in \hat{S}_\delta(B^n_r)$. Moreover, $F_{\alpha,\beta}$ may fail to be spirallike of type δ in any ball $B^n_{r_1}$ with $r_1 > r$. Therefore $r = \tanh\left[\frac{\pi}{4} - \frac{|\delta|}{2}\right]$ is the biggest radius for which each $F_{\alpha,\beta} \in \Phi_{n,\alpha,\beta}(S)$ is spirallike of type δ in B^n_r. This completes the proof. ∎

Remark 4.2. If we take $\delta = 0$, $\alpha \in [0,1]$, $\beta \in [0,1/2]$ with $\alpha + \beta \leq 1$, then from Theorem 4.1 we obtain that $R_{S^*}(\Phi_{n,\alpha,\beta}(S)) = \tanh(\pi/4)$. This result was proven in [18].

With arguments similar to those in the proof of Theorem 4.1, we may obtain the following result regarding the radius of starlikeness of order γ for the class $\Phi_{n,\alpha,\beta}(S)$.

Theorem 4.2. *$R_{S^*_\gamma}(\Phi_{n,\alpha,\beta}(S)) = r$, where r is the unique root of the equation*

$$\left(\frac{1-r}{1+r}\right)^{\cos x} \cos x - \gamma = 0, \tag{4.1}$$

for $\gamma \in (0, 1/e)$, in which $x = x(r)$, $0 < x < \pi$ is uniquely determined by the equation

$$\sin x \ln\left(\frac{1+r}{1-r}\right) - x = 0$$

and $r = \frac{1-\gamma}{1+\gamma}$, for $\gamma \in [1/e, 1)$.

Proof. Let $F_{\alpha,\beta} \in \Phi_{n,\alpha,\beta}(S)$. Then $F_{\alpha,\beta} = \Phi_{n,\alpha,\beta}(f)$, where $f \in S$. It is known that f is starlike of order γ in U_r, where r is defined as above. This number is the radius of starlikeness of order γ for S (see [38, Theorem 3] for $\alpha = 0$ and [38, Theorem 4] for $\gamma = 0$). Hence

$$\text{Re}\, \frac{z_1 f'(z_1)}{f(z_1)} > \gamma, \ |z_1| < r$$

and the left hand side of the above inequality can be negative if $|z_1| > r$.

From Remark 4.1 and Corollary 2.2, we obtain that $F_{\alpha,\beta} \in S^*_\gamma(B^n_r)$ and $F_{\alpha,\beta}$ may not be starlike of order γ in any ball $B^n_{r_1}$ with $r_1 > r$. Therefore $R_{S^*_\gamma}(\Phi_{n,\alpha,\beta}(S)) = r$. This completes the proof. ■

Using the fact that $R_K(S^*_{1/2}) = \sqrt{2\sqrt{3}-3}$ (see e.g. [14, II p. 87]), Remark 4.1 and the fact that the Roper-Suffridge extension operator preserves convexity (see [19] and [37]), with reasoning similar to those in Theorems 4.1 and 4.2, we may obtain the following result.

Theorem 4.3. $R_K(\Phi_n(S^*_{1/2})) = \sqrt{2\sqrt{3}-3}$.

Similarly, using the results regarding radii of univalence in [14, Chapter 13] and the fact that the operator $\Phi_{n,\alpha,\beta}$ preserves the notions of starlikeness ([18]), starlikeness of order $\gamma \in (0, 1)$ (Corollary 2.2) and spirallikeness of type $\delta \in (-\pi/2, \pi/2)$ (see e.g. [30]), we may obtain the following results.

Theorem 4.4. *If $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$ such that $\alpha + \beta \le 1$, then the following relations hold:*

(*i*) *$R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(S^*_\gamma))$ is the smallest positive root of*

$$((1-2\gamma)\cos\delta)x^2 - 2(1-\gamma)x + \cos\delta = 0, \ \delta \in (-\pi/2, \pi/2), \ \gamma \in (0, 1).$$

(ii) $R_{S^*_\gamma}(\Phi_{n,\alpha,\beta}(K)) = \sin(\gamma\pi/2)$, $\gamma \in (0,1)$.

(iii) $R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(K)) = \cos\delta$, $0 \leq \delta < 1$.

(iv) $R_{S^*}(\Phi_{n,\alpha,\beta}(\hat{S}_\delta)) = 1/(\cos\delta + |\sin\delta|)$, $\delta \in (-\pi/2, \pi/2)$.

Remark 4.3. It would be interesting to see if the results contained in this paper remain true in the case of g-Loewner chains for other univalent functions g.

Remark 4.4. It would be interesting to see whether the results in this paper may be generalized to the case of complex Hilbert spaces.

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