TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 5, pp. 1627-1650, October 2013
DOI: 10.11650/tjm.17.2013.2798
This paper is available online at http://journal.taiwanmathsoc.org.tw

MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS SCHRÖDINGER-POISSON SYSTEMS WITH THE ASYMPTOTICAL NONLINEARITY IN $\mathbb{R}^3$

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Abstract. In this paper, we study nonhomogeneous Schrödinger-Poisson systems

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)f(u) + h(x), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $f(t)$ is either asymptotically linear or asymptotically 3-linear with respect to t at infinity. Under appropriate assumptions on $K,\ a,\ f$ and h, the existence of two positive solutions of the above system is obtained by using the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory.

1. Introduction and Main Results

In this paper, we are concerned with the existence of two positive solutions for the following nonhomogeneous Schrödinger-Poisson system

$$(1)\qquad \begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)f(u) + h(x), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $K,\ h \in L^2(\mathbb{R}^3),\ K,\ h, \geq\ (\not\equiv)0$, a is a nonnegative function, the function $f \in C(\mathbb{R}, \mathbb{R}^+)$ and F is the primitive function of f with $F(t) = \int_0^t f(s)ds$. This

Received December 28, 2012, accepted March 21, 2013.
Communicated by Biagio Ricceri.
2010 *Mathematics Subject Classification*: Primary 45M20; Secondary 35J20, 35J60.
Key words and phrases: Nonhomogeneous Schrödinger-Poisson system, Variational methods, Positive solutions, Asymptotically linear.
The first author was supported by National Natural Science Foundation of China (No. 11101347), Post-doctor Foundation of China (No. 2012M510363) and the Key Project in Science and Technology Research Plan of the Education Department of Hubei Province (No. D20112605, D20122501), while the second one by the National Natural Science Foundation of China (No. 11201323) and the Fundamental Research Funds for the Central Universities (No. XDJK2013D007).

system arises in an interesting physical model which describes the interaction of a charged particle with electrostatic field (we refer the reader to [4] and the references therein for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve system (1).

When $K \equiv 0$, system (1) becomes into a single equation

$$-\Delta u + u = a(x)f(u) + h(x). \tag{2}$$

Problem (2) with $h(x) \equiv 0$ (homogeneous) has been studied extensively in the last decade, see [13, 17, 18, 19, 24] and so on. In these mentioned papers, the condition: $f(t)/t$ is nondecreasing in $t \geq 0$ is usually assumed to prove that the (PS) sequence is bounded. In the case of $h(x) \not\equiv 0$ (nonhomogeneous), Zhu in [28] proved that problem (2) has at least two positive solutions in $\mathbb{R}^N$ with $a(x) = 1$ and $f(t) = t^p(p \in (1, 2^* - 1)$ if $h(x)$ is small in some sense. After [28], there has been quite a lot of interesting existence results of positive solutions to problem (2) in $\mathbb{R}^N$, see [7, 9, 14, 27] and the references herein, the results in these papers are on the base of assuming that $f(t)$ satisfies usual Ambrosetti-Rabinowitz(AR) condition in [2]:

$$(AR)\ \ 0 < F(t) = \int_0^t f(s)ds \leq \theta t f(t), \quad \text{for } t > 0 \tag{3}$$

and some $\theta \in \left(0, \frac{1}{2}\right)$. Wang and Zhou in [26] obtained the existence of two positive solutions for problem (2) in $\mathbb{R}^N(N \geq 3)$, for suitable a and h under the conditions (f1)-(f3) (seen in Theorem 1.1). From all above papers, we find that methods used in the homogenous case are difficult to apply to the nonhomogenous case of $h(x) \not\equiv 0$. However, in this paper, we shall obtain solutions for nonhomogeneous Schrödinger-Poisson systems. Moreover, these systems have the asymptotical nonlinearity: the asymptotically linear or the asymptotically 3-linear at infinity. Clearly, the nonlinearity assumed in the following main results satisfies the (AR) condition as in (3) with $\theta = \frac{1}{4}$.

When $K \not\equiv 0$, Cerami and Vaira in [8] studied system (1) with $f(t) = |t|^{p-1}u(p \in (3, 5))$ and $h(x) \equiv 0$ (homogeneous) and obtained the existence of positive ground state solutions by minimizing I restricted to the Nehari manifold when K and a satisfy different assumptions, respectively. Sun, Chen and Nieto in [23] also studied system (1) with general f which is asymptotically linear at infinity($\lim_{t\to+\infty} \frac{f(t)}{t} = l < +\infty$) and obtain the existence of positive ground state solution under suitable K and a by Mountain Pass Theorem. Wang and Zhou in [25] studied Schrödinger-Poisson systems with external potential, parameter λ and $f(x, t)$ which is asymptotically linear with respect to t at infinity

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi(x)u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim_{|x|\to+\infty} \phi(x) = 0 & x \in \mathbb{R}^3, \end{cases} \tag{4}$$

and obtained a positive solution for small λ and not obtained any nontrivial solution for λ large. Zhu in [29] generalized system (4) with autonomous nonlinearity $f(t)$ to system (4) with non-autonomous nonlinearity $K(x)f(t)$ and obtained the same results as in [25] with the vanishing potential at infinity. Later, Zhu in [30] studied system (4) with $V(x) = \beta$ and asymptotically linear nonlinearity $f(x,t)t$ where $f(x,t)$ tends to $p(x)$ and $q(x) \in L^\infty(\mathbb{R}^3)$, respectively, as $t \to 0$ and $t \to +\infty$ and obtained existence and nonexistence results depending on the parameters β and λ. Furthermore, there are abundant results with respect to Schrödinger-Poisson systems, see [1, 3, 6, 11, 12, 21, 22] and so on. But there are few results for system (1) with $K \not\equiv 0$, $h(x) \not\equiv 0$ (nonhomogeneous) and asymptotically linear or 3-linear at infinity. So I think it is worth to study. To our best knowledge, this is the first paper which consider this type of problem.

Now, we firstly give some notations. For any $1 \leq s \leq +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$. $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard product and norm

$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)dx, \quad \|u\| := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx\right)^{\frac{1}{2}}.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Here, we state our main results as follows.

Theorem 1.1. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K,\ h \geq (\not\equiv)0$, and the following conditions hold:*

(f1) $f \in C(\mathbb{R},\mathbb{R}^+)$, $f(0) = 0$, and $f(t) \equiv 0$ for $t < 0$.

(f2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$.

(f3) $\lim_{t\to +\infty} \frac{f(t)}{t} = l$ with $0 < l < +\infty$.

(A1) $a(x)$ is a positive continuous function and there exists $R_0 > 0$ such that

$$\sup\{f(t)/t:\ t > 0\} < \inf\{1/a(x):\ |x| \geq R_0\}.$$

(A2) There exists a constant $\beta \in (0,1)$ such that

$$(1-\beta)l > \mu^* := \inf\left\{\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx : u \in H^1(\mathbb{R}^3,\mathbb{R}^+),\ \int_{\mathbb{R}^3} a(x)F(u)dx \geq \frac{l}{2},\right.$$
$$\left.\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2dx < 2\beta l\right\}.$$

Then there exists $m > 0$ such that system (1) has at least two positive solutions $u_0,\ u_1 \in H^1(\mathbb{R}^3)$ satisfying $I(u_0) < 0$ and $I(u_1) > 0$ if $\|h\|_2 < m$.

Remark 1.1. In this paper, $K \not\equiv 0$ and $h(x) \not\equiv 0$, system (1) is nonhomogeneous and the existence of two positive solutions for system (1) has been proved in our Theorem 1.1. Note that the first local minimum solution exists due to the homogeneous term which is looked a small perturbation because $\|h\|_2 < m$. Moreover, the second solution u_1 is the mountain pass solution with the positive energy. Furthermore, functions $K,\ a,\ f$ which satisfy the above conditions of Theorem 1.1 exist. For example, for any $R_0 > 0$ and $r > 0$, taking $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\psi(x) = 1$ if $|x| \le r$, $\psi(x) = 0$ if $|x| \ge 2r$ and $|\nabla\psi(x)| \le \frac{C}{r}$ for all $x \in \mathbb{R}^3$, where $C > 0$ is an arbitrary constant independent of x, and $K \in L^2(\mathbb{R}^3)$ such that $K(x) \ge 0$ for all $x \in \mathbb{R}^3$, $K(x) \not\equiv 0$ and $\|K\|_2^2 \le \frac{9}{2\times 32^2\pi^2} S^{-2}\overline{S}^{-4} R_0^{-2}\left(C^2 + R_0^2\right)^{-2}$, where $S,\ \overline{S}$ are also seen in Section 2. Let

$$a(x) = \begin{cases} 1000/(1+|x|), & \text{if } |x| \le \frac{R_0}{2}, \\ 1/(1+R_0), & \text{if } |x| \ge R_0, \end{cases}$$

and

$$f(t) = \begin{cases} R_0 t^2/(1+t), & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases}$$

These functions are also seen in Remark 1.1 of [23].

Theorem 1.2. *Suppose that* $K,\ h \in L^2(\mathbb{R}^3)$, $a \in L^3(\mathbb{R}^3)$. *Let* $K,\ h$ *and* $a \ge (\not\equiv) 0$. *Assume* (*f1*), (*f2*) *and the following conditions hold:*

(*f4*) $\lim_{t\to+\infty} \frac{f(t)}{t^3} = l$ *with* $0 < l < +\infty$.

(*f5*) $\frac{F(t)}{t^4}$ *is nondecreasing for* $t > 0$.

(A3) There exists a constant $\beta \in (0,1)$ such that

$$(1-\beta)l > \mu^* := \inf\left\{\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx :\ u \in H^1(\mathbb{R}^3, \mathbb{R}^+),\ \int_{\mathbb{R}^3} a(x)u^4 dx \ge 1, \right.$$
$$\left. \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 dx < \beta l\right\}.$$

Then there exists $m > 0$ such that system (1) has at least two positive solutions $u_0,\ u_1 \in H^1(\mathbb{R}^3)$ satisfying $I(u_0) < 0$ and $I(u_1) > 0$ if $\|h\|_2 < m$.

Remark 1.2. In Theorem 1.2, f is superlinear at zero and asymptotical 3-linear at infinity, of course, is also superlinear at infinity. We usual need the (AR) condition as in (3) with $\theta \in \left(0, \frac{1}{4}\right)$, to deal with this superlinear case(seen [10] and the references herein). But here, (f5) only satisfies condition (3) with $\theta = \frac{1}{4} \notin \left(0, \frac{1}{4}\right)$. Furthermore, since f is asymptotical 3-linear at infinity, (A1) is not meaning. We try to replaced (A1) by (A4) as follows:

(A4) $a(x)$ is a positive continuous function and there exists $R_0 > 0$ such that

$$\sup\{f(t)/t^3 :\ t > 0\} < \inf\{1/a(x) :\ |x| \geq R_0\}.$$

We find that other conditions are needed to prove our result. So, I can't consider (A4). In order to obtain the compact result: $\int_{|x|\geq R}(|\nabla u_n|^2 + |u_n|^2)dx \leq \varepsilon$, we assume that $a \in L^3(\mathbb{R}^3)$ and $a \geq (\not\equiv)0$.

Remark 1.3. It is not difficult to find some functions $K,\ a,\ f$ satisfying conditions of Theorem 1.2. For example, for any $r > 0$, taking $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\psi(x) = 1$ if $|x| \leq r$, $\psi(x) = 0$ if $|x| \geq 2r$ and $|\nabla\psi(x)| \leq \frac{C}{r}$ for all $x \in \mathbb{R}^3$, where $C > 0$ is an arbitrary constant independent of x, and $K \in L^2(\mathbb{R}^3)$ such that $K(x) \geq 0$ for all $x \in \mathbb{R}^3$, $K(x) \not\equiv 0$ and $\|K\|_2^2 \leq \frac{9}{2\times 32^2\pi^2}S^{-2}\overline{S}^{-4}R_0^{-2}\left(C^2 + R_0^2\right)^{-2}$. Setting $a(x) = \frac{3\sqrt[3]{1+R_0}}{4\pi R_0^3}\frac{1}{\sqrt[3]{1+|x|}}$ if $|x| \leq r$, $a(x) = 0$ if $|x| \geq r$. Let $f(t) = t^3$ if $t \geq 0$ and $f(t) = 0$ if $t < 0$. Clearly, f satisfies (f1), (f2), (f4) and (f5) and $l = 1 \in (0, +\infty)$. Taking $\beta = \frac{1}{2}$. Furthermore, for any $r < R_0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} a(x)\psi(x)^4 dx &\geq \frac{3\sqrt[3]{1+R_0}}{4\pi R_0^3}\int_{|x|\leq r}\frac{1}{\sqrt[3]{1+|x|}}dx \\
&\geq \frac{3\sqrt[3]{1+R_0}}{4\pi R_0^3}\frac{1}{\sqrt[3]{1+R_0}}\int_{|x|\leq R_0}dx \\
&= \frac{3\sqrt[3]{1+R_0}}{4\pi R_0^3}\frac{1}{\sqrt[3]{1+R_0}}\frac{4\pi}{3}R_0^3 = 1, \\
\int_{\mathbb{R}^3}(|\nabla\psi|^2 + |\psi|^2)dx &\leq \int_{|x|\leq 2r}\frac{C^2}{r^2}dx + \int_{|x|\leq 2r}dx \\
&\leq \left(1 + \frac{C^2}{r^2}\right)\frac{32\pi}{3}r^3 \\
&= \frac{32\pi}{3}r\left(C^2 + r^2\right).
\end{aligned} \tag{5}$$

This and (11) in section 2 yield

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\phi_\psi\psi^2 dx &\leq S^2\overline{S}^4\|K\|_2^2\|\psi\|^4 \\
&\leq S^2\overline{S}^4\|K\|_2^2\frac{32^2\pi^2}{9}r^2\left(C^2 + r^2\right)^2 \\
&\leq S^2\overline{S}^4\|K\|_2^2\frac{32^2\pi^2}{9}R_0^2\left(C^2 + R_0^2\right)^2 \\
&\leq \beta l = \frac{1}{2}.
\end{aligned}$$

Taking $R_0 = 1$, $r = \frac{1}{8}R_0 = \frac{1}{8}$ and $C = \frac{r}{4} = \frac{1}{32}$. Moreover, in view of the definition of μ^* and (5), one has

$$\mu^* \leq \int_{\mathbb{R}^3} (|\nabla\psi|^2 + |\psi|^2)dx \leq \frac{32\pi}{3} r \left(C^2 + r^2\right) < \frac{R_0}{2} = (1-\beta)l.$$

So, condition (A3) holds.

As we know, in order to obtain two different solutions, for the asymptotically linear case, the method is standard. Precisely, similar to [27], by the Ekeland's variational principle [15], it is not difficult to get a weak solution u_0 for $\|h\|_2$ suitably small. Moreover, u_0 is the local minimizer of I and $I(u_0) < 0$. However, under our assumptions, it seems difficult to get the Mountain Pass solution (different from the local minimum solution) of (1) by applying the Mountain Pass Theorem as the mentioned references because $h(x) \geq (\not\equiv) 0$, the nonlinearity is asymptotical and the working space is $H^1(\mathbb{R}^3)$. We have to find new ways to show that a Cerami sequence is bounded in $H^1(\mathbb{R}^3)$. Once a Cerami sequence is bounded in $H^1(\mathbb{R}^3)$, the usual strategy is try to show this sequence converges to a solution different from u_0, but this seems not so easy because the imbedding of $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(p \in (2,6))$ is not compact. In fact, firstly, this difficulty can be avoided by restricting I to the subspace of $H^1(\mathbb{R}^3)$ such as radially functions subspace usually denoted by $H_r^1(\mathbb{R}^3)$, see [1, 3, 8, 11, 21, 22]. Especially, many authors avoid the lack of the compactness by the external potential $V(x)$, some conditions are assumed on $V(x)$ to make the working space which is a subspace of $H^1(\mathbb{R}^3)$ have compactness imbeddings, see [10, 21, 23, 29]. However, for the asymptotically case, we have to find another method to verify Cerami condition. Motivated by [26, 23], we consider system (1) with the following two asymptotical cases at infinity: asymptotically linear and asymptotically 3-linear in this paper, respectively. Secondly, in order to recover the compactness, we establish the equi-absolutely-continuity at infinity: $\int_{|x|\geq R} (|\nabla u_n|^2 + |u_n|^2)dx \leq \varepsilon$ which is also called the compactness result in this paper.

This paper is organized as follows. In section 2, some important preliminaries are listed out. In sections 3 and 4, we manage to give proofs of Theorems 1.1 and 1.2. In the following discussion, we denote various positive constants as C or $C_i (i = 0, 1, 2, 3, ...)$ for convenience.

2. Preliminaries

System (1) has a variational structure. Indeed we consider the functional

$$\mathcal{J} : \ H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$$

defined by

$$\mathcal{J}(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi u^2 dx - \int_{\mathbb{R}^3} a(x)F(u)dx - \int_{\mathbb{R}^3} h(x)u dx.$$

Evidently, the action functional $\mathcal{J}$ belongs to $C^1(H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3),\mathbb{R})$ and the partial derivatives in (u,ϕ) are given, for $\xi\in H^1(\mathbb{R}^3)$ and $\eta\in D^{1,2}(\mathbb{R}^3)$, by

$$\begin{aligned}\left\langle \tfrac{\partial\mathcal{J}}{\partial u}(u,\phi),\xi\right\rangle &= \int_{\mathbb{R}^3}(\nabla u\cdot\nabla\xi+u\xi+K(x)\phi u\xi-a(x)f(u)\xi-h(x)\xi)dx,\\ \left\langle \tfrac{\partial\mathcal{J}}{\partial \phi}(u,\phi),\eta\right\rangle &= \frac{1}{2}\int_{\mathbb{R}^3}(-\nabla\phi\cdot\nabla\eta+K(x)u^2\eta)dx.\end{aligned}$$

Thus, the pair (u,ϕ) is a weak solution of system (1) if and only if it is a critical point of $\mathcal{J}$ in $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$. Clearly, the action functional $\mathcal{J}$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [5], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

For all u in $H^1(\mathbb{R}^3)$, the Lax-Milgram theorem (see [16]) implies that there exists a unique $\phi_u\in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u=K(x)u^2$ in a weak sense. Then, insert ϕ_u into the first equation of (1), we have

$$-\Delta u+u+K(x)\phi_u(x)u=a(x)f(u)+h(x). \tag{6}$$

That is, system (1) can be easily transformed to a nonlinear Schrödinger equation (6) with a non-local term. Moreover, we can write an integral expression for ϕ_u in the explicit form:

$$\phi_u(x)=\int_{\mathbb{R}^3}\frac{K(y)u(y)^2}{|x-y|}dy \tag{7}$$

for any $u\in H^1(\mathbb{R}^3)$. So, we can consider the functional $I:H^1(\mathbb{R}^3)\to\mathbb{R}$ defined by $I(u)=\mathcal{J}(u,\phi_u)$. After multiplying $-\Delta\phi_u=K(x)u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx=\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2dx. \tag{8}$$

Therefore, the reduced functional takes the form

$$\begin{aligned}&\widetilde{I}(u)\\ &=\frac{1}{2}\|u\|^2+\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2dx-\int_{\mathbb{R}^3}a(x)F(u)dx-\int_{\mathbb{R}^3}h(x)udx,\quad u\in H^1(\mathbb{R}^3).\end{aligned}$$

Using the reduction method, this indefiniteness for $\mathcal{J}$ can be removed and we are led to study a one variable functional $\widetilde{I}$ that does not present such a strongly indefinite nature structure.

Recall the Sobolev 's inequalities with the best constant S and $\overline{S}$

$$\|v\|_6 \leq S\|v\|_{D^{1,2}(\mathbb{R}^3)}, \quad \|v\|_6 \leq \overline{S}\|v\|, \tag{9}$$

together with (8) and the Hölder's inequality, we have

$$\begin{aligned}
\|\phi_u\|^2_{D^{1,2}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 dx \\
&\leq \|K\|_2\|u^2\|_3\|\phi_u\|_6 \\
&= \|K\|_2\|u\|_6^2\|\phi_u\|_6 \\
&\leq S\overline{S}^2\|K\|_2\|u\|^2\|\phi_u\|_{D^{1,2}(R^3)}.
\end{aligned}$$

This yields

$$\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \leq S\overline{S}^2\|K\|_2\|u\|^2, \quad \|\phi_u\|_6 \leq S\|\phi_u\|_{D^{1,2}(R^3)} \leq S^2\overline{S}^2\|K\|_2\|u\|^2. \tag{10}$$

Therefore, by the Hölder's inequality, (10) and (9) we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)\phi_u(x)u(x)^2 dx \\
\leq\ & \|K\|_2\|u\|_6^2\|\phi_u\|_6 \\
\leq\ & S^2\overline{S}^4\|K\|_2^2\|u\|^4 \\
:=\ & C_0\|u\|^4.
\end{aligned} \tag{11}$$

In this paper, we shall look for the positive solution of problem (1). By assumption (f1), we know that to seek a nonnegative weak solution of problem (1) is equivalent to finding a nonzero critical point of the following functional I on $H^1(\mathbb{R}^3)$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)(u^+)^2 dx - \int_{\mathbb{R}^3} a(x)F(u^+)dx - \int_{\mathbb{R}^3} h(x)u dx, \tag{12}$$

where $u^+ = \max\{u, 0\}$. Combining (10), (11), (f1)-(f3), and Lemma 3.3 in [23], I is well defined. Furthermore, I is C^1 and we have

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv + K(x)\phi_u(x)u^+v - a(x)f(u^+)v - h(x)v)dx.$$

Hence, if $u \in H^1(\mathbb{R}^3)$ is a nonzero critical point of I, then (u, ϕ_u) with ϕ_u as in (7), is a nonnegative solution of (1). In fact, by (f1) and $h \geq 0$, we have $\langle I'(u), u^-\rangle = -\|u^-\|^2 - \int_{\mathbb{R}^3} h(x)u^- dx = 0$, where $u^- = \max\{-u, 0\}$. This yields that $u^- = 0$, then $u = u^+ - u^- = u^+ \geq 0$. By the strong maximum principle, the nonzero critical point of I is the positive solution for problem (1).

In the following sections, we shall discuss system (1) with the two cases: asymptotically linear case and asymptotically cubic case at infinity, respectively.

3. The Asymptotically Linear Case

In this section, we prove that system (1) has a mountain pass type solution and a local minimum solution. For this purpose, we use a variant version of Mountain Pass Theorem [15], which allows us to find a so-called Cerami type (PS) sequence (Cerami sequence, in short). The properties of this kind of Cerami sequence are very helpful in showing its boundedness in the asymptotically case. The following lemmas will show that I defined in (12) has the so-called mountain pass geometry.

Lemma 3.1. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K \geq (\not\equiv)0$, (f1)-(f3) and $(A1)$ hold. Then there exist $\rho,\ \alpha,\ m > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$ for $\|h\|_2 < m$.*

Proof. For any $\varepsilon > 0$, it follows from (f1)-(f3) that there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^5 \text{ for all } t \in \mathbb{R}. \tag{13}$$

Therefore, we have

$$|F(t)| \leq \frac{1}{2}\varepsilon|t|^2 + \frac{C_\varepsilon}{6}|t|^6 \text{ for all } t \in \mathbb{R}. \tag{14}$$

Furthermore, by (f1)-(f3) and (A1), there exists $C_1 > 0$ such that

$$a(x) \leq C_1 \text{ for all } x \in \mathbb{R}^3. \tag{15}$$

According to (14), (15) and (9), we deduce

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} a(x)F(u^+)dx\right| &\leq \frac{\varepsilon C_1}{2}\int_{\mathbb{R}^3}|u^+|^2dx + \frac{C_1C_\varepsilon}{6}\int_{\mathbb{R}^3}|u^+|^6dx \\
&\leq \frac{\varepsilon C_1}{2}\|u^+\|^2 + C_2\|u^+\|^6 \\
&\leq \frac{\varepsilon C_1}{2}\|u\|^2 + C_2\|u\|^6,
\end{aligned}$$

where $C_2 = \frac{C_1C_\varepsilon\overline{S}^6}{6}$. Together with (7), $K \geq (\not\equiv)0$, $h \in L^2(\mathbb{R}^3)$ and the Hölder inequality, one has

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon C_1}{2}\|u\|^2 - C_2\|u\|^6 - \|h\|_2\|u\|_2 \\
&\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon C_1}{2}\|u\|^2 - C_2\|u\|^6 - \|h\|_2\|u\| \\
&\geq \|u\|\left(\frac{1-\varepsilon C_1}{2}\|u\| - C_2\|u\|^5 - \|h\|_2\right).
\end{aligned} \tag{16}$$

Taking $\varepsilon = \frac{1}{2C_1}$ and setting $g(t) = \frac{1}{4}t - C_2t^5$ for $t \geq 0$, we see that there exists $\rho = \left(\frac{1}{20C_2}\right)^{\frac{1}{4}}$ such that $\max_{t\geq 0} g(t) = g(\rho) := m > 0$. Then it follows from (16) that there exists $\alpha > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$ for $\|h\|_2 < m$. Of course, ρ can be chosen small enough, we can obtain the same result: there exist $\alpha > 0$, $m > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$ for $\|h\|_2 < m$. ■

Lemma 3.2. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K,\ h \geq (\not\equiv)0$, (f1)-(f3) and (A1)-(A2) hold. Then there exists $v \in H^1(\mathbb{R}^3)$ with $\|v\| > \rho$, ρ is given by Lemma 3.1, such that $I(v) < 0$.*

Proof. By (A2) and $h \geq (\not\equiv)0$, in view of the definition of μ^* and $(1-\beta)l > \mu^*$, there is a nonnegative function $v \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} a(x)F(v)dx > \frac{l}{2},\ \int_{\mathbb{R}^3} K(x)\phi_v v^2 dx < 2\beta l,\ \int_{\mathbb{R}^3} h(x)vdx \geq 0,$$

and $\mu^* \leq \|v\|^2 < (1-\beta)l$. Then, we have

$$\begin{aligned}
I(v) &= \frac{1}{2}\|v\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_v(x)v^2dx - \int_{\mathbb{R}^3} a(x)F(v)dx - \int_{\mathbb{R}^3} h(x)vdx \\
&\leq \frac{1}{2}\|v\|^2 + \frac{1}{4} \times 2\beta l - \frac{l}{2} \\
&= \frac{1}{2}(\|v\|^2 - (1-\beta)l) < 0.
\end{aligned}$$

Choosing $\rho > 0$ small enough in Lemma 3.1 such that $\|v\| > \rho$, then this Lemma is proved. ■

From Lemmas 3.1, 3.2 and Mountain Pass Lemma in [15], taking α as in Lemma 3.1 and v as in Lemma 3.2, there is a Cerami sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$\|I'(u_n)\|_{H^{-1}}(1 + \|u_n\|) \to 0 \quad \text{and} \quad I(u_n) \to c \geq \alpha > 0 \text{ as } n \to \infty, \tag{17}$$

here, H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$ and c denotes by

$$c = \inf_{\gamma\in\tau} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\tau = \{\gamma \in ([0,1], H^1(\mathbb{R}^3)) | \gamma(0) = 0, \gamma(1) = v\}.$$

In the following, we shall prove that I satisfies the Cerami condition, that is, the Cerami sequence $\{u_n\}$ has a convergence subsequence.

Lemma 3.3. Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K,\ h \geq (\not\equiv)0$, (f1)-(f3) and (A1) hold. Then $\{u_n\}$ defined in (17) is bounded in $H^1(\mathbb{R}^3)$.

Proof. By contradiction, let $\|u_n\| \to \infty$. Define $w_n = u_n\|u_n\|^{-1}$. Clearly, $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and there is a $w \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{cases} w_n \rightharpoonup w \ \text{ weakly in } H^1(\mathbb{R}^3), \\ w_n \to w \ \text{ a.e. in } \mathbb{R}^3, \\ w_n \to w \text{ strongly in } \ L^2_{loc}(\mathbb{R}^3) \end{cases}$$

as $n \to \infty$. Therefore, we obtain that $w_n^{\pm} = u_n^{\pm}\|u_n\|^{-1}$ and

$$\begin{cases} w_n^{\pm} \rightharpoonup w^{\pm} \ \text{ weakly in } H^1(\mathbb{R}^3), \\ w_n^{\pm} \to w^{\pm} \ \text{ a.e. in } \mathbb{R}^3, \\ w_n^{\pm} \to w^{\pm} \text{ strongly in } \ L^2_{loc}(\mathbb{R}^3) \end{cases}$$

as $n \to \infty$.

Firstly, we claim that w is nontrivial, that is $w \not\equiv 0$. Otherwise, if $w \equiv 0$, the Sobolev embedding implies that $w_n \to 0$ strongly in $L^2(B_{R_0})$, R_0 is given by (A1). By (f1)-(f3), there exists $C_3 > 0$ such that

$$\frac{f(t)}{t} \leq C_3 \text{ for all } t \in R. \tag{18}$$

Then, by (15) and (18), for all $n \in N$, we have

$$0 \leq \int_{|x|<R_0} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx \leq C_1C_3\int_{|x|<R_0}(w_n^+)^2 dx \leq C_1C_3\int_{|x|<R_0} w_n^2 dx \to 0.$$

This yields

$$\lim_{n\to\infty}\int_{|x|<R_0} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx = 0. \tag{19}$$

Furthermore, by $(A1)$, there exists a constant $\theta \in (0,1)$ such that

$$\sup\{f(t)/t:\ t>0\} \leq \theta \inf\{1/a(x):\ |x| \geq R_0\}. \tag{20}$$

Then, for all $n \in N$, we have

$$\int_{|x|\geq R_0} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx \leq \theta\int_{|x|\geq R_0}(w_n^+)^2 dx \leq \theta\|w_n\|^2 = \theta < 1. \tag{21}$$

Combining (19) and (21), we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx < 1. \tag{22}$$

By (17), we get

$$0 \le |\langle I'(u_n), u_n\rangle| \le \|I'(u_n)\|_{H^{-1}}\|u_n\| \le \|I'(u_n)\|_{H^{-1}}(1+\|u_n\|) \to 0 \tag{23}$$

as $n \to \infty$. Together with $\|u_n\| \to \infty$ as $n \to \infty$, it follows that

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^2} = o(1),$$

that is,

$$\begin{aligned} o(1) &= \|w_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{w_n}(u_n^+)^2 dx - \int_{\mathbb{R}^3} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx \\ &\ge 1 - \int_{\mathbb{R}^3} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx, \end{aligned}$$

where, and in what follows, $o(1)$ denotes a quantity which goes to zero as $n \to \infty$. Therefore, we deduce that

$$\int_{\mathbb{R}^3} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx + o(1) \ge 1,$$

which contradicts (22). So, $w \not\equiv 0$.

Furthermore, because $\|u_n\| \to \infty$ as $n \to \infty$, it follows from (23) that

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^4} = o(1),$$

that is,

$$o(1) = \frac{1}{\|u_n\|^2} + \int_{\mathbb{R}^3} K(x)\phi_{w_n}(w_n^+)^2 dx - \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^3} a(x)\frac{f(u_n^+)}{u_n^+}(w_n^+)^2 dx.$$

Together with (15) and (18), one has

$$\int_{\mathbb{R}^3} K(x)\phi_{w_n}(w_n^+)^2 dx = o(1). \tag{24}$$

By the same method of Lemma 3.3 in [23], we can prove

$$\int_{\mathbb{R}^3} K(x)\phi_{w_n}(w_n^+)^2 dx = \int_{\mathbb{R}^3} K(x)\phi_w(w^+)^2 dx + o(1). \tag{25}$$

Here we omit its proof. (24) and (25) show that

$$\int_{\mathbb{R}^3} K(x)\phi_w(w^+)^2 dx = 0,$$

which implies $w^+ \equiv 0$. By (23), (f1), $h \in L^2(\mathbb{R}^3)$ and $\|u_n\| \to \infty$ as $n \to \infty$, we obtain

$$0 = \lim_{n\to\infty} \frac{\langle I'(u_n), u_n^- \rangle}{\|u_n\|^2} = - \lim_{n\to\infty} \|w_n^-\|^2.$$

This and $w_n^- \rightharpoonup w^-$ weakly in $H^1(\mathbb{R}^3)$ imply that $w^- = 0$. Thus $w = w^+ - w^- = 0$. That is a contradiction. Therefore, $\{u_n\}$ is a bounded in $H^1(\mathbb{R}^3)$. ∎

From the idea of Lemma 3.4 in [23] or Lemma 2.1 in [26], we have the following Lemma. The proof of this Lemma follows from Lemma 3.4 in [23](also seen Lemma 2.1 in [26]). Here we write it for the completeness because this Lemma plays a key role to prove our Theorem 1.1.

Lemma 3.4. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K,\ h \geq (\not\equiv)0$, (f1)-(f3), and (A1) hold. Then for any $\varepsilon > 0$, there exist $R(\varepsilon) > R_0$ and $n(\varepsilon) > 0$ such that $\{u_n\}$ defined in* (17) *satisfies $\int_{|x|\geq R}(|\nabla u_n|^2 + |u_n|^2)dx \leq \varepsilon$ for $n > n(\varepsilon)$ and $R \geq R(\varepsilon)$.*

Proof. Let $\xi_R :\ \mathbb{R}^3 \to [0,1]$ be a smooth function such that

$$\xi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq R/2, \\ 1, & |x| \geq R. \end{cases} \tag{26}$$

Moreover, there exists a constant C_4 independent of R such that

$$|\nabla \xi_R(x)| \leq \frac{C_4}{R} \text{ for all } x \in \mathbb{R}^3. \tag{27}$$

Then, for all $n \in N$ and $R \geq R_0$, by (26), (27) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla(u_n\xi_R)|^2 dx \\
\leq& \int_{\mathbb{R}^3} |\nabla u_n|^2|\xi_R|^2 dx + \int_{\mathbb{R}^3} |u_n|^2|\nabla \xi_R|^2 dx + 2\int_{\mathbb{R}^3} |u_n||\xi_R||\nabla u_n||\nabla\xi_R| dx \\
\leq& \int_{R/2<|x|<R} |\nabla u_n|^2 dx + \int_{|x|>R} |\nabla u_n|^2 dx + \frac{C_4^2}{R^2}\int_{\mathbb{R}^3} |u_n|^2 dx \\
&+2\left(\int_{\mathbb{R}^3} |\nabla u_n|^2|\xi_R^2| dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u_n|^2|\nabla\xi_R|^2 dx\right)^{\frac{1}{2}} \\
\leq& \int_{R/2<|x|<R} |\nabla u_n|^2 dx + \int_{|x|>R} |\nabla u_n|^2 dx + \frac{C_4^2}{R^2}\int_{\mathbb{R}^3} |u_n|^2 dx \\
&+2\left(\int_{R/2<|x|<R} |\nabla u_n|^2 dx + \int_{|x|>R} |\nabla u_n|^2 dx\right)^{\frac{1}{2}} \left(\frac{C_4^2}{R^2}\int_{\mathbb{R}^3} |u_n|^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \left(2+\frac{C_4^2}{R^2}+\frac{2\sqrt{2}C_4}{R}\right)\|u_n\|^2$$
$$\leq \left(2+\frac{C_4^2}{R_0^2}+\frac{2\sqrt{2}C_4}{R_0}\right)\|u_n\|^2.$$

This implies that

$$\|u_n\xi_R\| \leq C_5\|u_n\| \tag{28}$$

for all $n \in N$ and $R \geq R_0$, where $C_5 = \left(3+\frac{C_4^2}{R_0^2}+\frac{2\sqrt{2}C_4}{R_0}\right)^{\frac{1}{2}}$. From Lemma 3.3, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Together with (17), we obtain that $I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$. Moreover, for $\varepsilon > 0$, there exists $n(\varepsilon) > 0$ such that

$$\langle I'(u_n), \xi_R u_n\rangle \leq C_5\|I'(u_n)\|_{H^{-1}(R^3)}\|u_n\| \leq \frac{\varepsilon}{4}$$

for $n > n(\varepsilon)$ and $R > R_0$. Note that

$$\begin{aligned}\langle I'(u_n), \xi_R u_n\rangle &= \int_{\mathbb{R}^3}(|\nabla u_n|^2+|u_n|^2)\xi_R dx + \int_{\mathbb{R}^3} u_n\nabla u_n\cdot\nabla\xi_R dx\\ &\quad + \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)(u_n^+)^2\xi_R dx\\ &\quad - \int_{\mathbb{R}^3} a(x)f(u_n^+)u_n^+\xi_R dx - \int_{\mathbb{R}^3} h(x)u_n\xi_R dx\\ &\leq \frac{\varepsilon}{4}.\end{aligned}$$

This yields

$$\begin{aligned}&\int_{\mathbb{R}^3}[(|\nabla u_n|^2+|u_n|^2)\xi_R + u_n\nabla u_n\cdot\nabla\xi_R]dx\\ &\leq \int_{\mathbb{R}^3} a(x)f(u_n^+)u_n^+\xi_R dx + \int_{\mathbb{R}^3} h(x)u_n\xi_R dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)(u_n^+)^2\xi_R dx + \frac{\varepsilon}{4}\\ &\leq \int_{\mathbb{R}^3} a(x)f(u_n^+)u_n^+\xi_R dx + \int_{\mathbb{R}^3} h(x)u_n\xi_R dx + \frac{\varepsilon}{4}.\end{aligned} \tag{29}$$

By (20), we have

$$a(x)f(u_n^+)u_n^+ \leq \theta(u_n^+)^2 \quad \text{for } \theta\in(0,1) \text{ and } |x|\geq R_0.$$

This yields

$$\int_{\mathbb{R}^3} a(x)f(u_n^+)u_n^+\xi_R dx \leq \theta\int_{\mathbb{R}^3}(u_n^+)^2\xi_R dx \leq \theta\int_{\mathbb{R}^3} u_n^2\xi_R dx \tag{30}$$

for all $n \in N$ and $|x| \geq R_0$. For any $\varepsilon > 0$, there exists $R(\varepsilon) \geq R_0$ such that

$$\frac{1}{R^2} \leq \frac{4\varepsilon^2}{C_4^2} \quad \text{for all} \quad R > R(\varepsilon). \tag{31}$$

Because $h \in L^2(\mathbb{R}^3)$, $h \geq 0$, there exists $\overline{\rho} = \overline{\rho}(\varepsilon)$ such that

$$\|h\|_{2,\mathbb{R}^3 \setminus B_\rho(0)} < \varepsilon, \quad \forall \rho \geq \overline{\rho}. \tag{32}$$

By the Hölder inequality, (32), (26) and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} h(x) u_n \xi_R dx \leq \|h(x)\xi_R\|_2 \|u_n\|_2 \\ \leq\ & \|h(x)\|_{2,|x|>R/2} \|u_n\|_2 \leq \frac{\varepsilon}{4} \quad \text{for all} \quad R > R(\varepsilon). \end{aligned} \tag{33}$$

By the Young inequality, (27) and (31), for all $n \in N$ and $R > R(\varepsilon)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} |u_n \nabla u_n \cdot \nabla \xi_R| dx \\ =\ & \int_{\mathbb{R}^3} \sqrt{2\varepsilon} |\nabla u_n| \frac{1}{\sqrt{2\varepsilon}} |u_n| |\nabla \xi_R| dx \\ \leq\ & \varepsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4\varepsilon} \int_{|x| \leq R} |u_n|^2 \frac{C_4^2}{R^2} dx \\ \leq\ & \varepsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \varepsilon \int_{|x| \leq R} |u_n|^2 dx \\ \leq\ & \varepsilon \|u_n\|^2. \end{aligned} \tag{34}$$

Combining (29), (30), (33) and (34), there exists $C_6 > 0$ such that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + (1-\theta)|u_n|^2) \xi_R dx \leq \frac{\varepsilon}{2} + \varepsilon \|u_n\|^2 \leq C_6 \varepsilon \quad \text{for all} \quad R > R(\varepsilon).$$

Noting that C_6 is independent of ε. So, for any $\varepsilon > 0$, we can choose $R(\varepsilon) > R_0$ and $n(\varepsilon) > 0$ such that $\int_{|x| \geq R} (|\nabla u_n|^2 + |u_n|^2) dx \leq \varepsilon$ holds. ■

Lemma 3.5. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K,\ h \geq (\not\equiv) 0$, (f1)-(f3), and (A1)-(A2) hold. Then the sequence $\{u_n\}$ in* (17) *has a convergent subsequence. Moreover, u is a positive solution of problem* (1) *and $I(u) > 0$.*

Proof. By Lemma 3.3, the sequence $\{u_n\}$ in (17) is bounded in $H^1(\mathbb{R}^3)$. We may assume that, up to a subsequence, such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, $u_n \to u$ a.e.

in $\mathbb{R}^3$ and $u_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^3)$ for some $u \in H^1(\mathbb{R}^3)$. Now, we shall show that $\|u_n\| \to \|u\|$ as $n \to \infty$.

By (17), we have

$$\begin{aligned}&\langle I'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2 + K(x)\phi_{u_n}(x)(u_n^+)^2 - a(x)f(u_n^+)u_n^+ - h(x)u_n)dx = o(1),\end{aligned} \tag{35}$$

and

$$\begin{aligned}&\langle I'(u_n), u\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla u + u_n u + K(x)\phi_{u_n}(x)u_n^+ u - a(x)f(u_n^+)u^+ - h(x)u)dx = o(1).\end{aligned} \tag{36}$$

Since $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla u + u_n u)dx = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx + o(1). \tag{37}$$

By the same argument of proof of Theorem 3.1 in [23], we have the following equalities:

$$\int_{\mathbb{R}^3} a(x)f(u_n^+)u_n^+ dx = \int_{\mathbb{R}^3} a(x)f(u_n^+)u^+ dx + o(1), \tag{38}$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)(u_n^+)^2 dx = \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^+ u dx + o(1). \tag{39}$$

Moreover, $h \in L^2(\mathbb{R}^3)$ imply that for any $\varepsilon > 0$ there exists $\overline{\rho} = \overline{\rho}(\varepsilon)$ such that

$$\|h\|_{2,\mathbb{R}^3\setminus B_\rho(0)} < \varepsilon, \quad \forall \rho \geq \overline{\rho}. \tag{40}$$

Since $h \in L^2(\mathbb{R}^3)$, the Hölder inequaltiy, $u_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^3)$ and (40), we have

$$\begin{aligned}&\int_{\mathbb{R}^3} h(x)u_n dx - \int_{\mathbb{R}^3} h(x)u dx \\ &\leq \int_{\mathbb{R}^3\setminus B_\rho(0)} |h(x)(u_n - u)|dx + \int_{B_\rho(0)} |h(x)(u_n - u)|dx \\ &\leq \|h(x)\|_{2,\mathbb{R}^3\setminus B_\rho(0)}\|u_n - u\|_{2,\mathbb{R}^3\setminus B_\rho(0)} + \|h(x)\|_{2,B_\rho(0)}\|u_n - u\|_{2,B_\rho(0)} \\ &\leq \varepsilon\|u_n - u\|_{2,\mathbb{R}^3\setminus B_\rho(0)} + \varepsilon\|u_n - u\|_{2,B_\rho(0)} \\ &\leq C_6\varepsilon.\end{aligned}$$

This yields

$$\int_{\mathbb{R}^3} h(x)u_n dx = \int_{\mathbb{R}^3} h(x)u dx + o(1). \tag{41}$$

By (35)-(39) and (41), we obtain

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2 + u_n^2)dx - \int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)dx = o(1).$$

This yields that $\|u_n\| \to \|u\|$ as $n \to \infty$ and u is a nonzero critical point of I in $H^1(\mathbb{R}^3)$ and $I(u) = c > 0$ by Mountain Pass Theorem in [15]. Therefore, u is a positive solution of problem (1). ∎

Now, we give local properties of the variational functional I, which is required by using Ekeland's variational principle.

Lemma 3.6. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $K,\ a,\ h \geq (\not\equiv)0$, (f1)-(f3) and (A1) hold. If $\|h\|_2 < m$, then there exists $u_0 \in H^1(\mathbb{R}^3)$ such that*

$$I(u_0) = \inf\{I(u):\ u \in \overline{B}_\rho\} < 0, \quad \textit{where } B_\rho = \{u \in H^1(\mathbb{R}^3):\ \|u\| < \rho\},$$

$m,\ \rho$ are given by Lemma 3.1 and u_0 is a positive solution of system (1).

Proof. Because $h \in L^2(\mathbb{R}^3)$, $h \geq (\not\equiv)0$, we can choose a nonnegative function $\varphi \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} h(x)\varphi dx > 0. \tag{42}$$

Together with (11), (f1), $a \geq (\not\equiv)0$ and (42), for $t > 0$, we have

$$\begin{aligned} I(t\varphi) &= \frac{t^2}{2}\|\varphi\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{t\varphi}(x)(t\varphi)^2 dx - \int_{\mathbb{R}^3} a(x)F(t\varphi)dx - \int_{\mathbb{R}^3} h(x)t\varphi dx \\ &\leq \frac{t^2}{2}\|\varphi\|^2 + \frac{t^4}{4}C_0\|\varphi\|^4 - t\int_{\mathbb{R}^3} h(x)\varphi dx \\ &\leq 0 \end{aligned}$$

for $t > 0$ small enough. Thus there exists u small enough such that $I(u) < 0$. By Lemma 3.1, we deduce that

$$c_0 := \inf_{u \in \overline{B}_\rho} I(u) < 0 < \inf_{u \in \partial \overline{B}_\rho} I(u).$$

By applying Ekeland's variational principle (Theorem 4.1 in [20]) in $\overline{B}_\rho$, there is a minimizing sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$\text{(i) } c_0 \le I(u_n) < c_0 + \frac{1}{n}, \quad \text{(ii) } I(w) \ge I(u_n) - \frac{1}{n}\|w - u_n\| \text{ for all } w \in \overline{B}_\rho.$$

Clearly, $\{u_n\}$ is a bounded (PS) sequence of I. Then, by a standard procedure, Lemmas 3.4 and 3.5 imply that there exists $u_0 \in H^1(\mathbb{R}^3)$ such that $I'(u_0) = 0$, $I(u_0) = c_0 < 0$. Moreover, $I(u_0) = c_0 < 0$ implies that $u_0 \neq 0$. Therefore, u_0 is a nonzero critical point of I, thus u_0 is a positive solution of problem (1). So this Lemma is proved. ■

Proof of Theorem 1.1. By Lemmas 3.1-3.6, we know that system (1) has two different positive solutions u_0 and u. Moreover, $I(u_0) = c_0 < 0$ and $I(u) > 0$. ■

4. Asymptotically 3-Linear Case

To obtain two positive solutions of system (1) with asymptotically 3-linear at infinity, we also use the same method as Theorem 1.1. I can obtain corresponding results by suitably modifying the proofs of Lemmas 3.1-3.6 as follows. Here, some proofs of the following Lemmas which are the same as ones of Lemmas 3.1-3.6 are omitted.

Lemma 4.1. *Suppose that* $K,\ h \in L^2(\mathbb{R}^3)$, $a \in L^3(\mathbb{R}^3)$, $K,\ a \ge (\not\equiv)0,\ h \ge 0$. *Assume that (f1), (f2) and (f4) hold. Then there exist* $\widetilde{\rho},\ \widetilde{\alpha},\ \widetilde{m} > 0$ *such that* $I(u)|_{\|u\|=\widetilde{\rho}} \ge \widetilde{\alpha} > 0$ *for* $\|h\|_2 < \widetilde{m}$.

Proof. For any $\varepsilon > 0$, it follows from (f1), (f2) and (f4) that there exists $C'_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C'_\varepsilon|t|^3 \text{ for all } t \in \mathbb{R}. \tag{43}$$

Therefore, we have

$$|F(t)| \le \frac{1}{2}\varepsilon|t|^2 + \frac{C'_\varepsilon}{4}|t|^4 \text{ for all } t \in \mathbb{R}. \tag{44}$$

According to (44), $a \in L^3(\mathbb{R}^3)$, $a(x) \ge (\not\equiv)0$, the Hölder inequality, and Sobolev imbedding theorem, we deduce

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} a(x)F(u^+)dx\right| &\le \frac{\varepsilon}{2}\int_{\mathbb{R}^3} a(x)|u^+|^2dx + \frac{C'_\varepsilon}{4}\int_{\mathbb{R}^3} a(x)|u^+|^4dx \\
&\le \frac{\varepsilon}{2}\int_{\mathbb{R}^3} a(x)|u|^2dx + \frac{C'_\varepsilon}{4}\int_{\mathbb{R}^3} a(x)|u|^4dx \\
&\le \frac{\varepsilon}{2}\|a(x)\|_3\|u\|_3^2 + \frac{C'_\varepsilon}{4}\|a(x)\|_3\|u\|_6^4 \\
&\le \frac{\varepsilon C_7}{2}\|u\|^2 + C_8\|u\|^4
\end{aligned}$$

for some $C_7,\ C_8 > 0$. Together with (7), $h \in L^2(\mathbb{R}^3)$ and the Hölder inequality, one has

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2+\frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)(u^+)^2dx-\int_{\mathbb{R}^3} a(x)F(u^+)dx-\int_{\mathbb{R}^3} h(x)udx \\
&\geq \frac{1}{2}\|u\|^2-\frac{\varepsilon C_7}{2}\|u\|^2 - C_8\|u\|^4 - \|h\|_2\|u\| \qquad (45)\\
&\geq \|u\|\left(\frac{1-\varepsilon C_7}{2}\|u\| - C_8\|u\|^3 - \|h\|_2\right).
\end{aligned}
$$

Taking $\varepsilon = \frac{1}{2C_7}$ and setting $\widetilde{g}(t) = \frac{1}{4}t - C_8 t^3$ for $t \geq 0$, we see there exists $\rho = \left(\frac{1}{12C_8}\right)^{\frac{1}{2}}$ such that $\max_{t\geq 0}\widetilde{g}(t) = \widetilde{g}(\rho) := \widetilde{m}$. Then it follows from (45) that there exists $\widetilde{\alpha} > 0$ such that $I(u)|_{\|u\|=\widetilde{\rho}} \geq \widetilde{\alpha} > 0$ for $\|h\|_2 < \widetilde{m}$. We also choose $\widetilde{\rho}$ small enough to obtain the same result. ∎

Lemma 4.2. *Suppose that $K,\ h \in L^2(\mathbb{R}^3)$, $a \in L^3(\mathbb{R}^3)$, $K,\ a \geq (\not\equiv)0$, $h \geq 0$, (f1), (f2), (f4) and (A3) hold. Then there exists $\widetilde{v} \in H^1(\mathbb{R}^3)$ with $\|\widetilde{v}\| > \widetilde{\rho}$, $\widetilde{\rho}$ is given by Lemma 4.1, such that $I(\widetilde{v}) < 0$.*

Proof. By (A3), in view of the definition of μ^* and $(1-\beta)l > \mu^*$, there is a nonnegative function $\widetilde{v} \in H^1(\mathbb{R}^3)$ such that

$$
\int_{\mathbb{R}^3} a(x)\widetilde{v}^4dx \geq 1,\ \int_{\mathbb{R}^3} K(x)\phi_{\widetilde{v}}\widetilde{v}^2dx < \beta l,\ \int_{\mathbb{R}^3} h(x)\widetilde{v}dx > 0,
$$

and $\mu^* \leq \|\widetilde{v}\|^2 < (1-\beta)l$. Together with (f4), we have

$$
\begin{aligned}
&\lim_{t\to+\infty}\frac{I(t\widetilde{v})}{t^4} \\
&= \lim_{t\to+\infty}\left(\frac{1}{2t^2}\|\widetilde{v}\|^2+\frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\widetilde{v}}(x)\widetilde{v}^2dx-\int_{\mathbb{R}^3} a(x)\widetilde{v}^4\frac{F(t\widetilde{v})}{(t\widetilde{v})^4}dx-\frac{1}{t^3}\int_{\mathbb{R}^3} h(x)\widetilde{v}dx\right) \\
&= \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\widetilde{v}}(x)\widetilde{v}^2dx - \frac{l}{4} \\
&\leq \frac{\beta l - l}{4} \\
&< 0.
\end{aligned}
$$

Choosing $\widetilde{\rho} > 0$ small enough in Lemma 4.1 such that $\|v\| > \widetilde{\rho}$, then this Lemma is proved. ∎

From Lemmas 4.1, 4.2 and Mountain Pass Lemma in [15], there is a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$
\|I'(u_n)\|_{H^{-1}}(1+\|u_n\|) \to 0 \quad \text{and} \quad I(u_n) \to \widetilde{c} \geq \widetilde{\alpha} > 0 \text{ as } n \to \infty, \tag{46}
$$

where $\widetilde{c}$ denotes by

$$\widetilde{c} = \inf_{\gamma \in \tau} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\tau = \{\gamma \in ([0,1], H^1(\mathbb{R}^3)) | \gamma(0) = 0, \gamma(1) = \widetilde{v}\}.$$

In the following, we shall prove that sequence $\{u_n\}$ has a convergence subsequence.

Lemma 4.3. *Suppose that* $K,\ h \in L^2(\mathbb{R}^3)$*,* $a \in L^3(\mathbb{R}^3)$*,* $K,\ a \geq (\not\equiv) 0$*,* $h \geq 0$*, (f1), (f2) and (f5) hold. Then* $\{u_n\}$ *defined in* (46) *is bounded in* $H^1(\mathbb{R}^3)$*.*

Proof. By (46), we have

$$|\langle I'(u_n), u_n\rangle| \leq \|I'(u_n)\| \|u_n\| \leq (1 + \|u_n\|)\|I'(u_n)\|_{H^{-1}} \to 0$$

as $n \to \infty$. From (f1) and (f5), we obtain

$$f(t)t - 4F(t) \geq 0 \quad \text{for all} \ \ t \in \mathbb{R}.$$

Thus, we deduce

$$\begin{aligned} 1 + \widetilde{c} &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} a(x)\left[\frac{1}{4} f(u_n^+)u_n^+ - F(u_n^+)\right] dx - \frac{3}{4}\int_{\mathbb{R}^3} h(x) u_n dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{3}{4}\|h(x)\|_2 \|u_n\|_2 \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{3}{4}\|h(x)\|_2 \|u_n\| \end{aligned} \tag{47}$$

for n large enough. This yields that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, since $\|h\|_2 < \widetilde{m}$.

Lemma 4.4. *Suppose that* $K,\ h \in L^2(\mathbb{R}^3)$*,* $a \in L^3(\mathbb{R}^3)$*,* $K,\ a \geq (\not\equiv) 0$*,* $h \geq 0$*, (f1), (f2), (f4) and (f5) hold. Then for any* $\varepsilon > 0$*, there exist* $R(\varepsilon) > R_0$ *and* $n(\varepsilon) > 0$ *such that* $\{u_n\}$ *defined in* (46) *satisfies* $\int_{|x| \geq R} (|\nabla u_n|^2 + |u_n|^2) dx \leq \varepsilon$ *for* $n > \widetilde{n}(\varepsilon)$ *and* $R \geq \widetilde{R}(\varepsilon)$*.*

Proof. Since $a \in L^3(\mathbb{R}^3)$ and $a(x) \geq (\not\equiv) 0$, there exists $\overline{r} = \overline{r}(\varepsilon) > 0$ such that

$$\|a(x)\|_{3, \mathbb{R}^3 \setminus B_r(0)} < \varepsilon \text{ for all } \ \forall r > \overline{r}. \tag{48}$$

Let $\xi_R:\ \mathbb{R}^3 \to [0,1]$ be a smooth function defined by (26) and (27). By the same method of Lemma 3.4, we also obtain

$$\|u_n \xi_R\| \leq C_9 \|u_n\|$$

for all $n \in N$ and $R \geq \widetilde{R}_0(\varepsilon) > 2\overline{r}$. Moreover, for $\varepsilon > 0$, there exists $\widetilde{n}(\varepsilon) > 0$ such that

$$\langle I'(u_n), \xi_R u_n \rangle \leq C_9 \|I'(u_n)\|_{H^{-1}(R^3)} \|u_n\| \leq \frac{\varepsilon}{4}$$

for $n > \widetilde{n}(\varepsilon)$ and $R > \widetilde{R}_0(\varepsilon) > 2\overline{r}$. By (43), the Hölder inequality, Sobolev imbedding inequalities, (48) and the boundedness of u_n, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} a(x) f(u_n^+) u_n^+ \xi_R dx \\
&\leq \varepsilon \int_{\mathbb{R}^3} a(x)(u_n^+)^2 \xi_R dx + C'_\varepsilon \int_{\mathbb{R}^3} a(x)(u_n^+)^4 \xi_R dx \\
&\leq \varepsilon \int_{|x|>R/2} a(x) u_n^2 dx + C'_\varepsilon \int_{|x|>R/2} a(x) u_n^4 dx \\
&\leq \varepsilon \|a(x)\|_{3,|x|>R/2} \|u_n\|_3^2 + C'_\varepsilon \|a(x)\|_{3,|x|>R/2} \|u_n\|_6^4 \\
&\leq \varepsilon \|a(x)\|_{3,\mathbb{R}^3 \setminus B_r(0)} \|u_n\|^2 + C'_\varepsilon \|a(x)\|_{3,\mathbb{R}^3 \setminus B_r(0)} \|u_n\|^4 \\
&\leq \varepsilon
\end{aligned} \tag{49}$$

for all $n \in N$ and $|x| \geq \widetilde{R}_0(\varepsilon) > \overline{r}$.

Combining (29), (49), (33) and (34), there exists $C_{14} > 0$ such that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2) \xi_R dx \leq \frac{3\varepsilon}{4} + \varepsilon \|u_n\|^2 \leq C_{14}\varepsilon \quad \text{for all} \quad R > R(\varepsilon).$$

Noting that C_{14} is independent of ε. So, for any $\varepsilon > 0$, we can choose $R(\varepsilon) > \widetilde{R}_0$ and $\widetilde{n}(\varepsilon) > 0$ such that $\int_{|x|\geq R} (|\nabla u_n|^2 + |u_n|^2) dx \leq \varepsilon$ holds. ■

Lemma 4.5. *Suppose that K, $h \in L^2(\mathbb{R}^3)$, $a(x) \in L^3(\mathbb{R}^3)$, K, $a \geq (\not\equiv) 0$, $h \geq 0$, (f1), (f2), (f4), (f5) and (A3) hold. Then the sequence $\{u_n\}$ in (46) has a convergent subsequence. Moreover, I possesses a nonzero critical point $\widetilde{u}$ in $H^1(\mathbb{R}^3)$, $I(\widetilde{u}) > 0$ and $\widetilde{u}$ is a positive solution of problem (1).*

Proof. By Lemma 3.3, the sequence $\{u_n\}$ in (46) is bounded in $H^1(\mathbb{R}^3)$. We may assume that, up to a subsequence $u_n \rightharpoonup \widetilde{u}$ weakly in $H^1(\mathbb{R}^3)$ for some $\widetilde{u} \in H^1(\mathbb{R}^3)$. Now, we shall show that $\|u_n\| \to \|\widetilde{u}\|$ as $n \to \infty$. Under the conditions of this Lemma, (37), (39) and (41) of Lemma 3.5 still hold. Now, we only need to prove that (38) still holds under conditions of Lemma 4.5. By Lemma 4.3, we know that u_n is bounded and weakly converge to $\widetilde{u}$ in $H^1(\mathbb{R}^3)$. Together with Hölder inequality, (43) and Sobolev inequalities, we obtain

$$
\begin{aligned}
&\int_{|x|\geq \widetilde{R}(\varepsilon)} a(x)f(u_n^+)u_n^+dx - \int_{|x|\geq \widetilde{R}(\varepsilon)} a(x)f(u_n^+)\widetilde{u}^+dx \\
= &\int_{|x|\geq \widetilde{R}(\varepsilon)} a(x)f(u_n^+)(u_n^+ - \widetilde{u}^+)dx \\
\leq &\int_{|x|\geq \widetilde{R}(\varepsilon)} |a(x)f(u_n^+)||u_n - \widetilde{u}|dx \\
\leq &\left(\int_{|x|\geq R(\varepsilon)} |f(u_n^+)|^2dx\right)^{\frac{1}{2}} \left(\int_{|x|\geq R(\varepsilon)} a(x)^2|u_n - \widetilde{u}|^2dx\right)^{\frac{1}{2}} \\
\leq &\left(\int_{|x|\geq R(\varepsilon)} (\varepsilon|u_n|^2 + 2\varepsilon C'_\varepsilon|u_n|^4 + {C'_\varepsilon}^2|u_n|^6)dx\right)^{\frac{1}{2}} \\
&\left(\int_{|x|\geq R(\varepsilon)} |u_n - \widetilde{u}|^6dx\right) \|a(x)\|_{3,\mathbb{R}^3\backslash B_r(0)} \\
\leq &\ C_{15}\varepsilon.
\end{aligned}
$$

This and the compactness of embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2_{loc}(\mathbb{R}^3)$ imply (38). Combining (35), (36), (37), (38), (39) and (41), we have

$$\int_{\mathbb{R}^3}(|\nabla u_n|^2 + u_n^2)dx - \int_{\mathbb{R}^3}(|\nabla \widetilde{u}|^2 + |\widetilde{u}|^2)dx = o(1).$$

This yields that $\|u_n\| \to \|\widetilde{u}\|$ as $n \to \infty$ and $\widetilde{u}$ is a nonzero critical point of I in $H^1(\mathbb{R}^3)$ and $I(\widetilde{u}) = \widetilde{c} > 0$ by Mountain Pass Theorem in [15]. Therefore, $\widetilde{u}$ is a positive solution of problem (1). ■

Proof of Theorem 1.2. By the same method of Lemma 3.6, we can obtain system (1) has a local minimum positive solution $\widetilde{u}_0$ and $I(\widetilde{u}_0) < 0$. By Lemmas 4.1-4.5, we know that system (1) has a mountain pass solution and $I(\widetilde{u}) > 0$. Thus this Theorem is proved. ■

Acknowledgments

The authors express their thanks to Professor Chun-Lei Tang for his valuable help and suggestion during they stay in Southwest University. The authors would like to thank the referee for valuable suggestions.

References

1. T. D' Aprile and D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **134(5)** (2004), 893-906.

2. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis*, **14** (1973), 349-381.
3. A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, *Commun. Contemp. Math.*, **10(3)** (2008), 391-404.
4. V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, **11(2)** (1998), 283-293.
5. V. Benci, D. Fortunato, A. Masiello and L. Pisani, Solitons and the electromagnetic field, *Math. Z.*, **232(1)** (1999), 73-102.
6. A. M. Candela and A. Salvatore, Multiple solitary waves for non-homogeneous Schrödinger-Maxwell equations, *Mediterr. J. Math.*, **3(3-4)** (2006), 483-493.
7. D. M. Cao and H. S. Zhou, Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, **126(2)** (1996), 443-463.
8. G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differential Equations*, **248(3)** (2010), 521-543.
9. K. J. Chen, Exactly two entire positive solutions for a class of nonhomogeneous elliptic equations, *Differential Integral Equations*, **17(1-2)** (2004), 1-16.
10. S. J. Chen and C. L. Tang, High energy solutions for the superlinear Schrödinger-Maxwell equations, *Nonlinear Anal.*, **71(10)** (2009), 4927-4934.
11. G. M. Coclite, A multiplicity result for the nonlinear Schrödinger-Maxwell equations, *Commun. Appl. Anal.*, **7(2-3)** (2003), 417-423.
12. G. M. Coclite, A multiplicity result for the Schrödinger-Maxwell equations with negative potential, *Ann. Polon. Math.*, **79(1)** (2002), 21-30.
13. D. G. Costa and H. Tehrani, On a class of asymptotically linear elliptic problems in $\mathbb{R}^N$, *J. Differential Equations*, **173(2)** (2001), 470-494.
14. Y. B. Deng, Existence of multiple positive solutions for a semilinear equation with critical exponent, *Proc. Roy. Soc. Edinburgh Sect. A*, **122(1-2)** (1992), 161-175.
15. I. Ekeland, *Convexity Methods in Hamiltonian Mechanics*, Springer-Verlag, Berlin, 1990.
16. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition. Springer-Verlag, Berlin, 2001.
17. L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, **129(4)** (1999), 787-809.
18. G. B. Li and H. S. Zhou, The existence of a positive solution to asymptotically linear scalar field equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **130(1)** (2000), 81-105.
19. Z. L. Liu and Z. Q. Wang, Existence of a positive solution of an elliptic equation on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, **134(1)** (2004), 191-200.
20. J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York, 1989.

21. D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.*, **237(2)** (2006), 655-674.

22. A. Salvatore, Multiple solitary waves for a non-homogeneous Schrödinger-Maxwell system in $\mathbb{R}^3$, *Advanced Nonlinear Studies*, **6(2)** (2006), 157-169.

23. J. T. Sun, H. B. Chen and J. J. Nieto, On ground state solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differential Equations*, **252(5)** (2012), 3365-3380.

24. C. A. Stuart and H. S. Zhou, Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, **24(9-10)** (1999), 1731-1758.

25. Z. P. Wang and H. S. Zhou, Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$, *Discrete Contin. Dyn. Syst.*, **18(4)** (2007), 809-816.

26. Z. P. Wang and H. S. Zhou, Positive solutions for a nonhomogeneous elliptic equation on R^N without (AR) condition, *J. Math. Anal. Appl.*, **353(1)** (2009), 470-479.

27. X. P. Zhu and H. S. Zhou, Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains, *Proc. Roy. Soc. Edinburgh Sect. A*, **115(3-4)** (1990), 301-318.

28. X. P. Zhu, A perturbation result on positive entire solutions of a semilinear elliptic equation, *J. Differential Equations*, **92(2)** (1991), 163-178.

29. H. B. Zhu, Asymptotically linear Schrödinger-Poisson systems with potential vanishing at infinity, *J. Math. Appl.*, **380(2)** (2011), 501-510.

30. H. B. Zhu, An asymptotically linear Schrödinger-Poisson systems on $\mathbb{R}^3$, *Nonlinear Anal.*, **75(13)** (2012), 5261-5269.

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