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WEAK E-OPTIMAL SOLUTION IN VECTOR OPTIMIZATION

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Abstract. In a real locally convex Hausdorff topological vector space, we first introduce the concept of nearly E-subconvexlikeness of set-valued maps via improvement set and obtain an alternative theorem. Furthermore, under the assumption of nearly subconvexlikeness, we establish scalarization theorem, Lagrange multiplier theorem, weak E-saddle point criteria and weak E-duality for weak E-optimal solution in vector optimization with set-valued maps. We also propose some examples to illustrate the main results.

1. INTRODUCTION

In recent years, there has been growing interest in vector optimization problems and there are many related literatures, see [1-3] and the references therein. In particular, it is meaningful to study the scalarization theorems, Lagrange multiplier theorems, saddle point theorems and duality of various kinds of solutions in vector optimization with set-valued maps under the suitable assumptions of generalized convexity, see [4-8] and the references therein. The concepts of approximate solutions have been playing an important role when there are no exact solutions in vector optimization problems. Base on the classic ideas of efficiency and weak efficiency, various kinds of approximate solutions such as (weak) ε-efficiency [9-14], $C(\varepsilon)$-efficiency [15-19] and E-optimal solution [20] via improvement set are proposed and some properties are studied. Especially, under the cone-subconvexlikeness, Rong and Wu established scalarization theorems and Lagrange multiplier theorems of weak ε-minimal solutions of vector optimization problems with set-valued maps in [14]. Gutierrez et al studied optimality conditions of $C(\varepsilon)$-efficiency via scalarization in [17]. Chicco et al. proposed the concept of E-optimal solution based on improvement set and discussed the

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existence of E-optimal solution for vector optimization problems in finite dimensional space in [20]. Gutierrez et al generalize the concept of improvement set and E-optimal solution to a real locally convex Hausdorff topological vector space and studied some properties of improvement set and E-optimal solution in [21].

We notice that the concept of E-optimal solution via improvement set unifies the concepts of optimal points, approximate optimal solution in scalar optimization, efficient solution, weak efficient solution and approximate efficient solution in vector optimization, pareto equilibria and approximate equilibria, see [20-21]. Thus, in a unified framework and under the assumptions of suitable generalized convexity, it will be meaningful to establish scalarization theorems, Lagrange multiplier theorems, saddle theorems and duality theorems of weak E-optimal solution in vector optimization problems with set-valued maps.

Motivated by the works of [5-6, 14, 20-21] we first propose the concept of nearly E-subconvexlikeness of set-valued maps via improvement set and obtain an alternative theorem of nearly E-subconvexlikeness in terms of support function in a real locally convex Hausdorff topological vector space. Furthermore, under the assumption of nearly subconvexlikeness, we establish scalarization theorem, Lagrange multiplier theorem, weak E-saddle point criteria and weak E-duality for weak E-optimal solution in vector optimization with set-valued maps. We also propose some examples to illustrate the Lagrange multiplier theorem and weak E-saddle point criteria.

2. Preliminaries

Let X be a linear space, let Y and Z be real locally convex Hausdorff topological vector space with positive cones K and P, respectively. For a subset $A \subset Y$, int$A$ and cl$A$ denote its topological interior and closure, respectively. $\langle x, y\rangle$ indicates the inner product of x and y. A cone $K \subset Y$ is called pointed if $K \bigcap(-K) = \{0\}$. Let K and P be closed convex pointed cones in Y and Z with nonempty interiors int$K$ and int$P$, respectively. For any $x, y \in Y$, we consider the following notation:

$$x \leqq_K y \Leftrightarrow y - x \in K.$$

The generated cone of a set $A \subset Y$ is cone$(A) = \{\lambda a | \ \lambda \geq 0, a \in A\}$. We say that A is solid if int$A \neq \varnothing$ and free disposal with respect to K (free disposal for short) if $A + K = A$ (see [22]). Let Y^* denote the topological dual space of Y. The positive dual cone of $A \subset Y$ is defined as

$$A^+ = \{\mu \in Y^* | \langle y, \mu\rangle \geq 0, \forall y \in A\},$$

the strict positive dual cone of A is defined as

$$A^{+s} = \{\mu \in Y^* | \langle y, \mu\rangle > 0, \forall y \in A \backslash \{0\}\}.$$

Definition 2.1. [21]. A nonempty set $E \subset Y$ is said to be an improvement set (or an improvement set with respect to K) if $0 \notin E$ and E is free disposal. We denote the family of the improvement sets in Y by $\mathfrak{T}_Y$.

Throughout this paper, we suppose that $E \in \mathfrak{T}_Y$ and $\text{int}E \neq \varnothing$. Consider the following vector optimization problem with set-valued maps:

$$\begin{aligned}&\text{(VP)} \ \min F(x)\\&\text{s.t. } x \in C = \{x \in S | G(x) \bigcap (-P) \neq \varnothing\},\end{aligned}$$

where $S \subset X$, $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are set-valued maps with nonempty value. We assume that the feasible set C of (VP) is nonempty and let

$$\langle F(x), \mu\rangle = \{\langle y, \mu\rangle | y \in F(x)\}, F(A) = \bigcup_{x\in A} F(x), \langle F(A), \mu\rangle = \bigcup_{x\in A} \langle F(x), \mu\rangle,$$

where $\mu \in Y^*$ and $A \subset X$. Let $L(Z, Y)$ be the set of all continuous linear operators from Z to Y. A subset $L^+(Z, Y)$ of $L(Z, Y)$ is defined as

$$L^+ = L^+(Z, Y) = \{T \in L(Z, Y) | T(P) \subset K\}.$$

Let (F, G) be the map from S to $Y \times Z$ defined by $(F, G)(x) = F(x) \times G(x)$. If $T \in L(Z, Y)$, we also define $F + TG : S \rightrightarrows Y$ by $(F + TG)(x) = F(x) + T(G(x))$. If there exists $\hat{x} \in S$ such that $G(\hat{x}) \bigcap (-\text{int}P) \neq \varnothing$, we say that (VP) satisfies the generalized Slater constraint qualification (see [23]).

Gutirrez et al. proposed the concepts of E-optimal solution and weak E-optimal solution for vector optimization with vector-valued maps in [21]. Similarly, we introduce the corresponding concepts for (VP).

Definition 2.2. A point $x_0 \in C$ is called an E-optimal solution of (VP) if there exists $y_0 \in F(x_0)$ such that

$$(y_0 - E) \bigcap F(C) = \varnothing.$$

Definition 2.3. A point $x_0 \in C$ is called a weak E-optimal solution of (VP) if there exists $y_0 \in F(x_0)$ such that

$$(y_0 - \text{int}E) \bigcap F(C) = \varnothing.$$

Remark 2.1. It is clear that E-optimality of (VP) implies weak E-optimality of (VP), but the converse is not necessarily true. The following example illustrates this point.

Example 2.1. Let $X = Y = Z = \mathbb{R}^2$, $K = P = \mathbb{R}^2_+$, $E = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \geq 1\}$, $S = [-1, 1] \times [-1, 1]$. The set-valued maps $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are respectively defined as

$$F(x) = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in [0, |x_1|] \times [0, |x_2|]\},\ \forall (x_1, x_2) \in S,$$

$$G(x) = \{(z_1, z_2) \in \mathbb{R}^2 | (z_1, z_2) \in [-1, 0] \times [-|x_1| - |x_2|, 0]\},\ \forall (x_1, x_2) \in S.$$

Clearly, the feasible set $C = S$. Let $x_0 = (1, 0)$ and $y_0 = (1, 0) \in F(x_0)$. We can verify that x_0 is a weak E-optimal solution of (VP). However, $(y_0 - E) \bigcap F(C) = \{(0, 0)\} \neq \varnothing$, which implies that x_0 is not a E-optimal solution of (VP).

Let $O^{\text{int}E}(F(C))$ be the set of weak E-optimal solutions of (VP) and

$$\overline{O^{\text{int}E}}(F(C)) = \{y_0 \in F(x_0) | (y_0 + \text{int}E) \bigcap F(C) = \varnothing, x_0 \in C\}.$$

Definition 2.4. A point pair (x_0, y_0) is called a weak E-optimal point of (VP) if $x_0 \in C$, $y_0 \in F(x_0)$ and

$$(y_0 - \text{int}E) \bigcap F(C) = \varnothing.$$

Lemma 2.1. *intE* $= E +$ *intK*.

Proof. The proof is similar with Lemma 3.4 in [24] and is omitted. ∎

Lemma 2.2. *clcone*$(A + E) =$ *clcone*$(A +$ *intE*$)$.

Proof. From $E \in \mathfrak{T}_Y$, Proposition 3.3 in [5] and Lemma 2.1, we have

$$\text{clcone}(A+E) = \text{clcone}(A+E+K) = \text{clcone}(A+E+\text{int}K) = \text{clcone}(A+\text{int}E). \quad ∎$$

3. Nearly E-Subconvexlikeness

In this section, we introduce the concept of nearly E-subconvexlikeness of set-valued maps and establish an alternative theorem in terms of improvement set.

Definition 3.1. Let $F : S \rightrightarrows Y$ be a set-valued map. F is said to be nearly E-subconvexlike on S if clcone$(F(S) + E)$ is a convex set.

Remark 3.1. If E such that int$K \subset E \subset K \backslash \{0\}$, then from Lemma 2.2 and Proposition 3.3 in [5], it follows that clcone$(F(S) + E) =$ clcone$(F(S) + \text{int}K) =$ cone$(F(S) + K)$. This implies that nearly E-subconvexlikeness coincides with nearly K-subconvexlikeness.

Remark 3.2. Nearly E-subconvexlikeness of set-valued map does not necessarily imply nearly K-subconvexlikeness, the following example illustrates it.

Example 3.1. Let $X = Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $E = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \geq 1\}$, $S = [-1, 1] \times [-1, 1]$. The set-valued map $F : S \rightrightarrows Y$ is defined as

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in [-1, x_1] \times [0, |x_2|]\}, \ \forall (x_1, x_2) \in [-1, 0) \times [-1, 1], \\ \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in [0, x_1] \times [-1, x_2]\}, \ \forall (x_1, x_2) \in [0, 1] \times [-1, 1]. \end{cases}$$

Clearly, clcone$(F(S) + E)$ is a convex set and clcone$(F(S) + \mathbb{R}^2_+)$ is not a convex set.

Let the support functional of Q at y be defined as $\sigma_Q(y^*) = \sup\limits_{y \in Q}\{y^*(y)\}, \forall y^* \in Y^*$.

Theorem 3.1. *Let the set-valued map F be nearly E-subconvexlike on S. Then, one and only one of the following statements is true:*

(i) $\exists\ x \in S$, $F(x) \bigcap (-intE) \neq \varnothing$;

(ii) $\exists\ \mu \in K^+ \backslash \{0_{Y^*}\}$, $\langle y, \mu \rangle - \sigma_{-E}(\mu) \geq 0$, $\forall y \in F(S)$.

Proof. Assume that both (i) and (ii) hold. Then, there exists $x \in S$ such that

$$F(x) \bigcap (-\text{int}E) \neq \varnothing.$$

From Lemma 2.1, this implies that there exist $y \in F(x)$ and $e \in E$ such that $y + e \in -\text{int}K$. Hence, $\langle y + e, \mu \rangle < 0$, i.e.

$$\langle y, \mu \rangle < \langle -e, \mu \rangle \leq \sup_{\tilde{e} \in -E} \langle \tilde{e}, \mu \rangle = \sigma_{-E}(\mu).$$

Then$\langle y, \mu \rangle - \sigma_{-E}(\mu) < 0$, which contradicts (ii).

Assume that (i) does not hold. Clearly, $(F(S) + E) \bigcap (-\text{int}K) = \varnothing$. Thus,

$$\text{clcone}(F(S) + E) \bigcap (-\text{int}K) = \varnothing.$$

From the nearly E-subconvexlikeness of F, clcone$(F(S) + E)$ is a convex set. Hence, there exists $\mu \in Y^* \backslash \{0_{Y^*}\}$ such that

$$(1) \qquad \langle y + e + \varepsilon r, \mu \rangle \geq 0, \forall y \in F(S), \forall e \in E, \forall r \in \text{int}K, \forall \varepsilon > 0.$$

Letting $\varepsilon \to +\infty$ in (1), we have $\langle r, \mu \rangle \geq 0, \forall r \in \text{int}K$. Consequently, $\langle r, \mu \rangle \geq 0, \forall r \in K$. Therefore, $\mu \in K^+ \backslash \{0_{Y^*}\}$. Letting $\varepsilon \to 0$ in (1), we obtain

$$\langle y, \mu \rangle \geq \langle -e, \mu \rangle, \forall y \in F(S), \forall e \in E.$$

Then

$$\langle y, \mu \rangle \geq \sup_{\tilde{e} \in -E} \langle \tilde{e}, \mu \rangle = \sigma_{-E}(\mu), \forall y \in F(S).$$

This implies that (ii) does hold. ■

4. Scalarization Theorem

In this section, we establish scalarization theorem of weak E-optimal solution for (VP) by making use of the alternative theorem for nearly E-subconvexlike set-valued map. Consider the following scalar optimization problem:

$$(\text{VP})_\mu \ \min_{x\in C}\langle F(x),\mu\rangle,\ \mu\in Y^*\backslash\{0_{Y^*}\}.$$

Definition 4.1 A point $x_0 \in C$ is called an optimal solution of $(\text{VP})_\mu$ with respect to E if there exists $y_0 \in F(x_0)$ such that

$$\langle y-y_0,\mu\rangle \geq \sigma_{-E}(\mu), \forall x\in C, \forall y \in F(x). \tag{2}$$

The point pair (x_0, y_0) is called an optimal point of $(\text{VP})_\mu$ with respect to E.

Remark 4.1. If $\varepsilon \in K\backslash\{0\}$, $E = \varepsilon + K$ and $\mu \in K^+\backslash\{0_{Y^*}\}$, then $E \in \mathfrak{T}_Y$ and (2) reduces to

$$\langle y,\mu\rangle \geq \langle y_0,\mu\rangle - \langle \varepsilon,\mu\rangle, \forall x\in C, \forall y\in F(x).$$

Theorem 4.1. *Assume that $x_0 \in C, y_0 \in F(x_0)$ and $F - y_0$ is nearly E-subconvexlike on C. Then (x_0, y_0) is a weak E-optimal point of (VP) if and only if there exists $\mu \in K^+\backslash\{0_{Y^*}\}$ such that (x_0, y_0) is an optimal point of $(\text{VP})_\mu$ with respect to E.*

Proof. If (x_0, y_0) is a weak E-optimal point of (VP), then we have $(y_0 - \text{int}E)\bigcap F(C) = \varnothing$, i.e.,

$$(F(C) - y_0)\bigcap(-\text{int}E) = \varnothing.$$

From Theorem 3.1, there exists $\mu \in K^+\backslash\{0_{Y^*}\}$ such that

$$\langle y-y_0,\mu\rangle - \sigma_{-E}(\mu) \geq 0, \forall x\in C, \forall y\in F(C).$$

Then, for any $x \in C, y \in F(x)$,

$$\langle y-y_0,\mu\rangle \geq \sigma_{-E}(\mu),$$

which implies that (x_0, y_0) is an optimal point of $(\text{VP})_\mu$ with respect to E.

Conversely, assume that there exists $\mu \in K^+\backslash\{0_{Y^*}\}$ such that (x_0, y_0) is an optimal point of $(\text{VP})_\mu$ with respect to E. If (x_0, y_0) is not a weak E-optimal point of (VP), then from Lemma 2.1, we have

$$(y_0 - E - \text{int}K)\bigcap F(C) \neq \varnothing.$$

Hence, there exists $\widehat{x} \in C, \widehat{y} \in F(\widehat{x})$ and $\widehat{e} \in E$ such that

$$\widehat{y} - y_0 + \widehat{e} \in -\text{int}K.$$

Since $\mu \in K^+ \backslash \{0_{Y^*}\}$, the above relation implies that

$$\langle \widehat{y} - y_0, \mu \rangle - \sigma_E(\mu) \leq \langle \widehat{y} - y_0 + \widehat{e}, \mu \rangle < 0,$$

which contradicts the assumption that (x_0, y_0) is an E-optimal point of (VP)$_\mu$. ∎

Remark 4.2 Theorem 4.1 is a generalization of Theorem 6.1 in [5], Theorem 2.1 in [14] and Theorem 5.1 and Theorem 5.3 in [21].

5. Lagrange Multiplier Theorem

In this section, Lagrange multiplier theorem which implies that a weak E-optimal point of (VP) is a weak E-optimal point of an appropriate unconstrained vector optimization problem with set-valued maps are presented under the assumption of nearly E-subconvexlikeness.
The Lagrangian function of (VP) $L : S \times L^+(Z, Y) \to 2^Y$ is defined by

$$L(x, T) := F(x) + T(G(x)), \ (x, T) \in S \times L^+(Z, Y).$$

Theorem 5.1. *Let $(F - y_0, G)$ be nearly $(E \times P)$-subconvexlike on S and (VP) satisfy the generalized Slater constraint qualification. If (x_0, y_0) is a weak E-optimal point of (VP) and $0 \in G(x_0)$, then there exists $T \in L^+$ such that (x_0, y_0) is a weak E-optimal point of the following unconstrained vector optimization problem:*

$$\begin{aligned} &\textit{(UVP)} \ \min L(x, T), \\ &\textit{s.t.} \ (x, T) \in S \times L^+(Z, Y) \end{aligned}$$

and $-T(G(x_0) \bigcap (-P)) \subset (\mathit{int}K \bigcup \{0\}) \backslash \mathit{int}E$.

Proof. Since (x_0, y_0) is a weak E-optimal point of (VP), then $x_0 \in C, y_0 \in F(x_0)$ and

$$(F(S) - y_0) \bigcap (-\text{int}E) = \varnothing.$$

Hence,

$$(F(S) - y_0, G(S)) \bigcap (-\text{int}E, -P) = \varnothing. \tag{3}$$

From the nearly $(E \times P)$-subconvexlikeness on S of $(F - y_0, G)$ and by Theorem 4.1 and (3), there exists $(\mu, \varphi) \in K^+ \times P^+ \backslash \{(0_{Y^*}, 0_{Z^*})\}$ such that

$$\langle y - y_0, \mu \rangle - \sigma_{-E}(\mu) + \langle z, \varphi \rangle - \sigma_{-P}(\varphi) \geq 0, \forall x \in S, \forall y \in F(x), \forall z \in G(x).$$

That is, for any $x \in S, y \in F(x), z \in G(x), e \in E, z' \in P$,

$$\begin{aligned} \langle y - y_0, \mu \rangle + \langle z, \varphi \rangle &\geq \sigma_{-E}(\mu) + \sigma_{-P}(\varphi) \\ &= \sup_{\tilde{e} \in -E} \langle \tilde{e}, \mu \rangle + \sup_{\tilde{z} \in -P} \langle \tilde{z}, \varphi \rangle \\ &\geq \langle -e, \mu \rangle + \langle -z', \varphi \rangle. \end{aligned} \tag{4}$$

In particular, letting $z' = 0$ in (4), we obtain that

$$\langle y - y_0 + e, \mu \rangle + \langle z, \varphi \rangle \geq 0, \forall x \in S, \forall y \in F(x), \forall z \in G(x), \forall e \in E. \tag{5}$$

Since G satisfies the generalized Slater constraint qualification and from (5), we have $\mu \in K^* \backslash \{0_{Y^*}\}$. Taking $k_0 \in \text{int}K$ satisfying $\langle k_0, \mu \rangle = 1$ and we define $T : Z \to Y$ by

$$T(z) = \langle z, \varphi \rangle k_0, z \in Z. \tag{6}$$

Clearly, $T \in L^+(Z, Y)$. Taking $x = x_0$, $y = y_0$ and $z = z_0 \in G(x_0) \bigcap (-P)$ in (5) and from $\varphi \in P^+$, we obtain that

$$-\langle e, \mu \rangle \leq \langle z_0, \varphi \rangle \leq 0. \tag{7}$$

From the right inequality of (7), we have that

$$-T(z_0) = -\langle z_0, \varphi \rangle k_0 \in \text{int}K \bigcup \{0\}.$$

From the left inequality of (7), we have that $-T(z_0) \notin \text{int}E$. Otherwise, from Lemma 2.1, there exists $\overline{e} \in E$ such that $-T(z_0) - \overline{e} \in \text{int}K$. Consequently, $\langle T(z_0) + \overline{e}, \mu \rangle < 0$, i.e., $\langle z_0, \varphi \rangle < -\langle \overline{e}, \mu \rangle$, which contradicts to the left inequality of (7). Noticing that z_0 is arbitrary in the set $G(x_0) \bigcap (-P)$, we obtain that

$$-T(G(x_0) \bigcap (-P)) \subset (\text{int}K \bigcup \{0\}) \backslash \text{int}E.$$

Furthermore, from $T \in L^+(Z, Y)$ and $0 \in G(x_0)$, it follows that $0 \in T(G(x_0))$. Thus, we get that $y_0 \in F(x_0) \subset F(x_0) + T(G(x_0)) = L(x_0, T)$. Hence, from (5) and (6), it follows that

$$\begin{aligned} \langle y + T(z), \mu \rangle &= \langle y, \mu \rangle + \langle z, \varphi \rangle \langle k_0, \mu \rangle \\ &= \langle y, \mu \rangle + \langle z, \varphi \rangle \\ &\geq \langle y_0 - e, \mu \rangle, \forall x \in S, \forall y \in F(x), \forall z \in G(x), \forall e \in E. \end{aligned}$$

This implies that

$$\langle y + T(z) - y_0, \mu \rangle \geq \sigma_E(\mu), \forall x \in S, \forall y \in F(x), \forall z \in G(x). \tag{8}$$

Hence, (x_0, y_0) is an optimal point with respect to E for the problem $(\text{UVP})_\mu$ given by

$$(\text{UVP})_\mu \min_{(x,T)\in S\times L^+(Z,Y)} \langle L(x,T), \mu\rangle.$$

Therefore, (x_0, y_0) is a weak E-optimal point of (UVP). Otherwise, from Lemma 2.1 and $\mu \in K^+\backslash\{0_{Y^*}\}$, it follows that there exist $\widehat{x} \in S, \widehat{y} \in F(\widehat{x}), \widehat{z} \in G(\widehat{x})$ and $\widehat{e} \in E$ such that

$$\langle \widehat{y} + T(\widehat{z}) - y_0, \mu\rangle < \sigma_E(\mu),$$

which contradicts (8). This completes the proof. ∎

Remark 5.1. Theorem 5.1 is a generalization of Theorem 5.1 in [5] and Theorem 3.1 and Theorem 3.2 in [14].

The following example illustrates the above Theorem 5.1.

Example 5.1. Let $X = Y = Z = \mathbb{R}^2$, $K = P = \mathbb{R}^2_+$, $E = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \geq 1\}$, $S = [-1, 1] \times \{0\}$. The set-valued maps $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are, respectively, defined as

$$F(x_1, x_2) = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in \{1\} \times [x_1 + 1, x_1 + 2]\}, \forall (x_1, x_2) \in S,$$

$$G(x_1, x_2) = \{(z_1, z_2) \in \mathbb{R}^2 | (z_1, z_2) \in [-|x_1|, 0] \times [-|x_1|, 0]\}, \forall (x_1, x_2) \in S.$$

It is clear that the feasible set $C = [-1, 1] \times \{0\}$. Let $x_0 = (-1, 0), y_0 = (1, 0) \in F(x_0)$. We can verify that all conditions of Theorem 5.1 are satisfied and (x_0, y_0) is a weak E-optimal point of (VP). Then there exists $\bar{T}(z_1, z_2) = (0.1z_1, 0.1z_1) \in L^+(Z, Y)$ such that (x_0, y_0) is a weak E-optimal point of (UVP).

6. Weak E-Saddle Points

In this section, we give the concept of weak E-saddle point for a set-valued Lagrangian map and establish a weak E-saddle point theorem.

Definition 6.1. An ordered pair $(\bar{x}, \bar{T}) \in S \times L^+(Z, Y)$ is called a weak E-saddle point of the set-valued Lagrangian map $L(x, T)$ if

$$L(\bar{x}, \bar{T}) \bigcap O^{\text{int}E}(L(S, \bar{T})) \bigcap \overline{O^{\text{int}E}}(L(\bar{x}, L^+)) \neq \varnothing.$$

Theorem 6.1. *An ordered pair $(\bar{x}, \bar{T}) \in S \times L^+(Z, Y)$ is a weak E-saddle point of the set-valued Lagrangian map $L(x, T)$ iff there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that*

(i) $\bar{y} + \bar{T}(\bar{z}) \in O^{intE}(L(S, \bar{T}))$;

(ii) $G(\bar{x}) \subset -P$;

(iii) $-\bar{T}(\bar{z}) \in K\backslash intE$;

(iv) $(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z})) \bigcap intE = \varnothing$.

Proof. Suppose that $(\bar{x}, \bar{T}) \in S \times L^+(Z, Y)$ is a weak E-saddle point of $L(x, T)$. Then there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

(9) $$\bar{y} + \bar{T}(\bar{z}) \in O^{\text{int}E}(L(S, \bar{T})),$$

(10) $$\bar{y} + \bar{T}(\bar{z}) \in \overline{O^{\text{int}E}}(L(\bar{x}, L^+)).$$

Hence, by (9), it follows that (i) holds, and from (10), we have

(11) $$y + T(z) - \bar{y} - \bar{T}(\bar{z}) \notin \text{int}E, \forall y \in F(\bar{x}), \forall T \in L^+(Z, Y), \forall z \in G(\bar{x}).$$

Taking $y = \bar{y}$ and $z = \bar{z}$ in (11), we get

(12) $$T(\bar{z}) - \bar{T}(\bar{z}) \notin \text{int}E, \forall T \in L^+(Z, Y).$$

Now, suppose that $\bar{z} \notin -P$. Noticing that Z is a locally convex topological vector space, we have $(P^+)^+ = \text{cl}P$. By the closeness of P, we have $-\bar{z} \notin (P^+)^+$. It is not difficult to show that there exists $\lambda \in P^+$ such that $\langle \bar{z}, \lambda \rangle > 0$. Taking any fixed $\hat{e} \in \text{int}E$ and define a map $\hat{T} : Z \to Y$ as

$$\hat{T}(z) := \frac{\langle z, \lambda \rangle}{\langle \hat{z}, \lambda \rangle} \hat{e} + \bar{T}(z),\ z \in Z.$$

Obviously, $\hat{T} \in L^+(Z, Y)$, which satisfies $\hat{T}(\bar{z}) - \bar{T}(\bar{z}) = \hat{e} \in \text{int}E$. This contradicts (12), and so $\bar{z} \in -P$. Therefore, $-\bar{T}(\bar{z}) \in K$. On the other hand, taking $T = 0 \in L^+(Z, Y)$ in (12), we have also that $-\bar{T}(\bar{z}) \notin \text{int}E$. So, (iii) holds.

Now, we show that $G(\bar{x}) \subset -P$. If it is not true, then this implies that there exists $z_0 \in G(\bar{x})$ such that $-z_0 \notin P$. We can show that there exists $\lambda_0 \in P^+$ such that $\langle z_0, \lambda_0 \rangle > 0$. Taking a fixed $e_0 \in \text{int}E$, we define a map $T_0 : Z \to Y$ as

$$T_0(z) := \frac{\langle z, \lambda_0 \rangle}{\langle z_0, \lambda \rangle_0} e_0,\ z \in Z.$$

Obviously, $T_0 \in L^+(Z, Y)$. Noticing that $-\bar{T}(\bar{z}) \in K$ and $\text{int}E \in \mathfrak{T}_Y$, we also get

(13) $$T_0(z_0) - \bar{T}(\bar{z}) = e_0 - \bar{T}(\bar{z}) \in \text{int}E + K = \text{int}E.$$

Taking $T = T_0$, $y = \bar{y}$ and $z = z_0$ in (11), we obtain that $T_0(z_0) - \bar{T}(\bar{z}) \notin \text{int}E$, which contradicts (13), and so (ii) holds. Taking $T = 0 \in L^+(Z, Y)$ in (11), we get

$$(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z})) \bigcap \text{int}E = \varnothing,$$

which shows that (iv) holds.

Conversely, from $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$, we obtain that

$$\bar{y} + \bar{T}(\bar{z}) \in F(\bar{x}) + \bar{T}(G(\bar{x})) = L(\bar{x}, \bar{T}). \tag{14}$$

From (ii), then $-T(G(\bar{x})) \subset T(P) \subset K,\ \forall T \in L^+(Z,Y)$. Noticing that $\text{int}E \in \mathfrak{T}_Y$, we have

$$\text{int}E - T(G(\bar{x})) \subset \text{int}E + K = \text{int}E,\ \forall T \in L^+(Z,Y). \tag{15}$$

From (15) and (iv), it follows that

$$(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z})) \bigcap (\text{int}E - T(G(\bar{x}))) = \varnothing,\ \forall T \in L^+(Z,Y).$$

i.e.,

$$(\bar{y} + \bar{T}(\bar{z}) + \text{int}E) \bigcap L(\bar{x}, L^+) = \varnothing.$$

Then

$$\bar{y} + \bar{T}(\bar{z}) \in \overline{O^{\text{int}E}}(L(\bar{x}, L^+)). \tag{16}$$

From (i), (14), and (16), we obtain that

$$\bar{y} + \bar{T}(\bar{z}) \in L(\bar{x}, \bar{T}) \bigcap O^{\text{int}E}(L(S, \bar{T})) \bigcap \overline{O^{\text{int}E}}(L(\bar{x}, L^+)).$$

Therefore, $(\bar{x}, \bar{T})$ is a weak E-saddle point of $L(x,T)$. ∎

Remark 6.1. Theorem 6.1 is a generalization of proposition 4.1 in [14].

Theorem 6.2. *If an ordered pair $(\bar{x}, \bar{T}) \in S \times L^+(Z,Y)$ is a weak E-saddle point of the set-valued Lagrangian map $L(x,T)$ and $0 \in G(\bar{x})$, then there exists $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that $\bar{x}$ is a weak E'-optimal solution of (VP), where $E' := E - \bar{T}(\bar{z})$.*

Proof. Since $(\bar{x}, \bar{T}) \in S \times L^+(Z,Y)$ is a weak E-saddle point of $L(x,T)$ and by Theorem 6.1, then there exists $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that $\bar{y} + \bar{T}(\bar{z}) \in O^{\text{int}E}(L(S, \bar{T}))$. Hence,

$$(\bar{y} + \bar{T}(\bar{z}) - \text{int}E) \bigcap (F(S) + \bar{T}(G(S))) = \varnothing,$$

i.e.

$$(\bar{y} + \bar{T}(\bar{z}) - \bar{T}(G(S)) - \text{int}E) \bigcap F(S) = \varnothing.$$

Taking $E' = E - \bar{T}(\bar{z})$. Obviously, $E' \in \mathfrak{T}_Y$. Noticing that $0 \in G(\bar{x}) \subset G(S)$, we have

$$-\text{int}E' = \bar{T}(\bar{z}) - \text{int}E \subset \bar{T}(\bar{z}) - \bar{T}(G(S)) - \text{int}E.$$

Therefore, $(\bar{y} - \text{int}E') \bigcap F(S) = \varnothing$. From $C \subset S$, we have $(\bar{y} - \text{int}E') \bigcap F(C) = \varnothing$. That is, $\bar{x}$ is a weak E'-optimal solution of (VP). ■

Remark 6.2. Theorem 6.2 is a generalization of Theorem 4.1 in [14].

The following example illustrates the above Theorem 6.2.

Example 6.1. Let $X = Y = Z = \mathbb{R}^2$, $K = P = \mathbb{R}^2_+$, $E = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \geq 3\}$, $S = \{0\} \times [-1, 1]$. The set-valued maps $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are, respectively, defined as

$$F(x_1, x_2) = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) \in \{0\} \times [-x_2, x_2]\}, \forall (x_1, x_2) \in S,$$

$$G(x_1, x_2) = \{(z_1, z_2) \in \mathbb{R}^2 | (z_1, z_2) \in -\lambda(|x_2|, |x_2|), \forall \lambda \in [0, 1]\}, \forall (x_1, x_2) \in S.$$

Let $\bar{x} = (0, 1), \bar{T}(z_1, z_2) = (z_1, z_2) \in L^+(Z, Y)$. Clearly, $(\bar{x}, \bar{T})$ is a weak E-saddle point of the set-valued Langrangian map $L(x, T)$. Then from Theorem 6.2, there exist $\bar{y} = (0, -1) \in F(\bar{x}), \bar{z} = (0, 0) \in G(\bar{x})$ such that $\bar{x}$ is a weak E'-optimal solution of (VP) with $E' = E$.

7. Weak E-Duality

A set-valued map $\Phi : L^+(Z, Y) \rightrightarrows 2^Y$ is defined as

$$\Phi(T) := O^{\text{int}E}(L(S, T)),\ T \in L^+(Z, Y),$$

which is said to be a weak E-dual map of (VP). The vector maximization problem with set-valued map Φ,

$$\text{(VD)} \max \bigcup_{T \in L^+} \Phi(T),$$

is said to be a dual problem of (VP).

Definition 7.1. A point $y \in Y$ is called a feasible point of (VD) if $y \in \bigcup_{T \in L^+} \Phi(T)$. A feasible point $\bar{y}$ is called a weak E-optimal solution of (VD) if

$$y - \bar{y} \notin \text{int}E,\ y \in \bigcup_{T \in L^+} \Phi(T).$$

Theorem 7.1. (Weak E-Duality). *Let $\bar{x}$ be any feasible solution of (VP), $\bar{y}$ be any feasible point of (VD). Then, $\bar{y} \notin F(\bar{x}) + intE$.*

Proof. Since $\bar{y}$ is a feasible point of (VD), then $\bar{y} \in \bigcup_{T \in L^+} \Phi(T)$. This implies that there exists $\bar{T} \in L^+(Z, Y)$ such that $\bar{y} \in \Phi(\bar{T}) = O^{\text{int}E}(L(S, \bar{T})$, i.e.,

$$(\bar{y} - \text{int}E) \bigcap (F(S) + \bar{T}(G(S))) = \varnothing.$$

Then,

$$\bar{y} - y - \bar{T}(z) \notin \text{int}E, \forall y \in F(\bar{x}), \forall z \in G(\bar{x}). \tag{17}$$

On the other hand, since $\bar{x}$ is a feasible solution of (VP), we have $G(\bar{x}) \bigcap (-P) \neq \varnothing$. Then there exists $\bar{z} \in G(\bar{x})$ such that $-\bar{z} \in P$ and so $-\bar{T}(\bar{z}) \in \bar{T}(P) \subset K$. Taking $z = \bar{z}$ in (17), we obtain $\bar{y} - y - \bar{T}(\bar{z}) \notin \text{int}E, \forall y \in F(\bar{x})$. From $-\bar{T}(\bar{z}) \in K$ and $\text{int}E \in \mathfrak{T}_Y$, it follows that $\bar{y} - y \notin \text{int}E, \forall y \in F(\bar{x})$. This implies that $\bar{y} \notin F(\bar{x}) + \text{int}E$. ■

Theorem 7.2. *Let $\bar{x}$ be any feasible solution of (VP), $\bar{y}$ be any feasible point of (VD), where $\bar{y} \in F(\bar{x})$. Then, $\bar{x}$ is a weak E-optimal solution of (VP) and $\bar{y}$ is a weak E-optimal solution of (VD).*

Proof. Suppose that $\bar{y}$ is a feasible point of (VD). Then, we have $\bar{y} \in \bigcup_{T \in L^+} \Phi(T)$. Hence, there exists $\bar{T} \in L^+(Z, Y)$ such that $\bar{y} \in \Phi(\bar{T}) = O^{\text{int}E}(L(S, \bar{T})$, i.e.,

$$\bar{y} - y - \bar{T}(z) \notin \text{int}E, \forall x \in S, \forall y \in F(x), \forall z \in G(x). \tag{18}$$

When $x \in C$, we know that there exists $\tilde{z} \in G(x)$ such that $\tilde{z} \in -P$. Hence, $-\bar{T}(\tilde{z}) \in K$. Furthermore, taking $z = \tilde{z}$ in (18), we can prove that

$$\bar{y} - y \notin \text{int}E, \ \forall x \in C, \forall y \in F(x).$$

From $\bar{y} \in F(\bar{x})$, it follows that $\bar{x}$ is a weak E-optimal solution of (VP). On the other hand, suppose that $\bar{x}$ is a feasible point of (VP). By the assumption $\bar{y} \in F(\bar{x})$ and Theorem 7.1, we obtain

$$y - \bar{y} \notin \text{int}E, \ \forall y \in \bigcup_{T \in L^+} \Phi(T).$$

Thus, $\bar{y}$ is a weak E-optimal solution of (VD). ■

Theorem 7.3. (Strong E-Duality). *Let $(F - \bar{y}, G)$ be nearly $(E \times P)$-subconvexlike on S and (VP) satisfies the generalized Slater constraint qualification. If $(\bar{x}, \bar{y})$ is a weak E-optimal point of (VP) and $0 \in G(\bar{x})$, then $\bar{y}$ is a weak E-optimal solution of (VD).*

Proof. By Theorem 5.1, there exists $T \in L^+$ such that $(\bar{x}, \bar{y})$ is a weak E-optimal point of (UVP). Hence, we obtain that

$$\bar{y} \in F(\bar{x}) \subset F(\bar{x}) + T(G(\bar{x})) \in L(S, T), \forall T \in L^+(Z, Y).$$

and

$$(\bar{y} - \text{int}E) \bigcap L(S, T) = \varnothing, \forall T \in L^+(Z, Y).$$

Then,

$$\bar{y} \in O^{\mathrm{int}E}(L(S,T) = \Phi(T), \forall T \in L^+(Z,Y).$$

Therefore,

$$\bar{y} \in F(\bar{x}) \bigcap \left(\bigcup_{T \in L^+} \Phi(T) \right).$$

Then, $\bar{y}$ is a weak E-optimal solution of (VD) by using Theorem 7.2. ■

8. Concluding Remarks

In this paper, we introduce the concepts of nearly E-subconvexlikeness of set-valued maps via improvement set in a real locally convex Hausdorff topological vector space and establish an alternative theorem. Furthermore we apply the alternative theorem to establish scalarization theorem and Lagrange multiplier theorem of weak E-optimal solution for (VP). We also establish saddle point criteria and duality of weak E-optimal solution for (VP). It is meaningful to discuss some new characterizations of weak E-optimal solution of (VP) and it will be our future research scope.

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