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STOKES' THEOREM ON MANIFOLDS: A KURZWEIL-HENSTOCK APPROACH

Varayu Boonpogkrong

Abstract. In this paper, Stokes' theorem is proved by the Kurzweil-Henstock approach. Sufficient conditions for the existence of the exterior derivative of a k-form in $\mathbb{R}^n$ are given. Concepts of strong differentiability are used in sufficient conditions.

1. INTRODUCTION

In mathematics papers and books, the usual definition of the divergence $\mathrm{div}F$ of a vector field $F = (F_1, F_2, \ldots, F_n)$ in $\mathbb{R}^n$ is given by $\sum_{i=1}^n \partial F_i/\partial x_i$, whereas in physics papers and books, it is given by an exterior derivative

$$(\mathrm{div}F)(p) = \lim_{\mathrm{diam}(I)\to 0} \frac{1}{|I|} \int_{\partial I} F \cdot \hat{n}\, ds,$$

where I is an interval containing the point p with surface ∂I and $\hat{n}$ is the exterior normal to ∂I. Recently, this physical definition of the divergence has been used by Acker, Macdonald, Hubbard and Boonpogkrong; see [1, 3, 5, 9].

In this paper, we shall use the physical definition of an exterior derivative and k-forms to prove Stokes' theorem by the Kurzweil-Henstock approach.

2. PRELIMINARIES

For any fixed positive integer n, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Let $S \subset \mathbb{R}^n$; the boundary and outer Lebesgue measure of S are denoted by ∂S and $|S|$ respectively. Let $x \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$; the norm $\|x\|$ is defined by

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$\|x\| = \sum_{i=1}^{n} |x_i|$. Let $\eta > 0$; $B(x,\eta)$ or $B_\eta(x)$ denote $\{y \mid \|x-y\| < \eta\}$. Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ be k linearly independent vectors in $\mathbb{R}^n$, and $P_a(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ be a k-parallelogram spanned by $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$, where the point a is one of the corners. We say $P_a(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ is anchored at the point a. We may use $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ instead of $P_a(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$. Let E be a k-parallelogram $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ in $\mathbb{R}^n$. A partition P of E is a finite family of non-overlapping k-subparallelogram $\{I_i\}_{i=1}^m$ whose union is E. We should stress that if $I_i = P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$, then $\vec{u}_j$ and $\vec{w}_j$ are parallel for all j. In this paper, a parallelogram is called an interval. A division D of E is a finite family of point-interval pairs $\{(x_i, I_i)\}_{i=1}^m$ such that $\{I_i\}_{i=1}^m$ is a partition of E. Let $\delta(x)$ be a positive function defined on E. A point-interval pair (x, I) is said to be Henstock δ-fine if $x \in I \subset B(x, \delta(x))$. We remark that we may assume that the point x is one of the corners of k-subparallelogram. Suppose x may not belong to I. Then (x, I) is said to be McShane δ-fine. A division D of E is said to be Henstock δ-fine if each point-interval pair in D is Henstock δ-fine. Similarly we can define McShane δ-fine divisions.

In this section, we only consider n-parallelograms in $\mathbb{R}^n$. Let E be an n-parallelogram in $\mathbb{R}^n$ and $f : E \to \mathbb{R}$. Let $D = \{(x_i, I_i)\}_{i=1}^m$ be a δ-fine division (Henstock or McShane) of E. We denote the Riemann sum $\sum_{i=1}^m f(x_i)\, |I_i|$ by $S(f, D, \delta)$, where $|I_i|$ is the volume of I_i. In this paper, a division $D = \{(x_i, I_i)\}_{i=1}^m$ will often be written as $D = \{(x, I)\}$, in which (x, I) represents a typical point-interval pair in D. The corresponding Riemann sum will be written as $(D) \sum f(x)\, |I|$.

Definition 2.1. Let $f : E \to \mathbb{R}$. Then f is said to be Kurzweil-Henstock integrable to $A \in \mathbb{R}$ on E if for each $\epsilon > 0$, there exists a positive function δ on E such that whenever $D = \{(x, I)\}$ is a Henstock δ-fine division of E, we have

$$|S(f, D, \delta) - A| \le \epsilon.$$

We denote A as $\int_E f$.

Definition 2.2. In the above Definition 2.1, if "a Henstock δ-fine division" is replaced by "a McShane δ-fine division". Then f is said to be MsShane integrable on E. We denote A as $(L) \int_E f$.

It is known that (i) f is McShane integrable on E if and only if f is Lebesgue integrable on E; (ii) if f is McShane integrable on E, then f is Kurzweil-Henstock integrable on E; see [7].

In this paper, we only consider Kurzweil-Henstock integrals.

3. Integration of k-Forms in $\mathbb{R}^n$

Now we shall consider k-parallelograms in $\mathbb{R}^n$. Let β be a function that maps $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ to the $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ component of the signed k-dimensional volume of $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$, which is given by the determinant of the $k \times k$ matrix formed by selecting rows $i_1, i_2, \ldots, i_k$ of the matrix whose columns are the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$. The function β is denoted by $dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, which is called an elementary k-form on $\mathbb{R}^n$. It is known, see [5], that

$$\sum_{j=1}^{k}(-1)^{j-1} dx_{i_j}(\vec{v}_j)(dx_{i_1} \wedge \ldots \wedge \widehat{dx}_{i_j} \wedge \ldots \wedge dx_{i_k})(P(\vec{v}_1, \ldots, \hat{\vec{v}}_j, \ldots, \vec{v}_k))$$
$$= (dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})(P(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)).$$

We use the notation $(\vec{v}_1, \ldots, \hat{\vec{v}}_j, \ldots, \vec{v}_k)$ for $(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_k)$.

Let us have $F : P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k) \to \mathbb{R}$ and $P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ a k-subparallelogram of $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$. We assume that $\vec{u}_j$ and $\vec{w}_j$ are parallel for all j. Hence the signed volumes of $(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and $(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ are of the same sign. Let

$$h(x, P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)) = F(x)\left[(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)\right].$$

Then h is a point-parallelogram function. Using the Kurzweil-Henstock approach, we can define an integral of h over $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$,

$$\int_{P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)} F(x)(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}),$$

denoted by $\int_{P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)} h$, called the Kurzweil-Henstock integral of h. More precisely, for every $\epsilon > 0$, there exists $\delta(x) > 0$ such that whenever $\{(x^j, P(\vec{u}_1^j, \ldots, \vec{u}_k^j))\}_{j=1}^q$ is a Henstock δ-fine division of $P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$, we have

$$\left|\sum_{j=1}^{q} F(x^j)(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P(\vec{u}_1^j, \ldots, \vec{u}_k^j) - \int_{P(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)} h\right| \le \epsilon.$$

We may assume that x^j is one of the corners of $P(\vec{u}_1^j, \vec{u}_2^j, \ldots, \vec{u}_k^j)$.

In the above, $F(x)(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})$ or briefly $F\,(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})$ is also called an elementary k-form on $\mathbb{R}^n$, denoted by φ in the following.

In the following, let $I = \{1, 2, \ldots, k+1\}$, $I_j = I \setminus \{j\}$, then $V(I)$ denotes $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$ and $V(I_j)$ denotes $(\vec{v}_1, \ldots, \hat{\vec{v}}_j, \ldots, \vec{v}_{k+1})$. Let $I^* = \{i_1, i_2, \ldots, i_k\}$, then $dX(I^*)$ denotes $dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$ and $dX(I_j^*)$ denotes $dx_{i_1} \wedge \ldots \wedge \widehat{dx}_{i_j} \wedge \ldots \wedge dx_{i_k}$.

The oriented boundary $\partial P_a(V(I))$ of an oriented $(k+1)$-parallelogram $P_a(V(I))$

is composed of its $2(k+1)$ faces, each of the form $P_{a+\vec{v}_i}(V(I_i))$ or $P_a(V(I_i))$. Then

$$
\begin{aligned}
&\int_{\partial P_a(V(I))} \varphi \\
(1)\qquad &= \sum_{i=1}^{k+1} (-1)^{i-1} \int_{P_{a+\vec{v}_i}(V(I_i)) - P_a(V(I_i))} \varphi \\
&= \sum_{i=1}^{k+1} (-1)^{i-1} \int_{P_a(V(I_i))} (F(x+\vec{v}_i) - F(x))(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}).
\end{aligned}
$$

In this paper, for convenience, $x + \vec{v}^{\,T}$ is always written as $x + \vec{v}$.

4. Exterior Derivative of a k-Form in $\mathbb{R}^n$

To understand the exterior derivative, first we consider the directional derivative of a function $F : \mathbb{R}^n \to \mathbb{R}$, where F is called a 0-form. Let $x \in \mathbb{R}^n$ and $\vec{v}$ a vector in $\mathbb{R}^n$ be given. We define dF as follows:

$$(dF)(x, \vec{v}) = \lim_{h \to 0} \frac{F(x+h\vec{v}) - F(x)}{h}.$$

It is well-known that if $\vec{v} = (v_1, v_2, \ldots, v_n)^T$, then

$$\lim_{h \to 0} \frac{F(x+h\vec{v}) - F(x)}{h} = [DF(x)] \cdot \vec{v} = \sum_{j=1}^{n} (\partial_j F(x)) v_j.$$

We write

$$dF = \sum_{j=1}^{n} (\partial_j F)(dx_j)$$

and $(dF)(x, \vec{v}) = \sum_{j=1}^{n} (\partial_j F(x))(dx_j(\vec{v})) = \sum_{j=1}^{n} (\partial_j F(x)) v_j$.

$dF = \sum_{j=1}^{n} (\partial_j F)(dx_j)$ is called the exterior derivative of F and dF is called a 1-form.

Now we shall define the exterior derivative $d\varphi$ of an elementary k-form φ, which is given by

$$\varphi = F\left(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}\right).$$

Note that φ is a point-parallelogram function $\varphi(x, P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)) =$ $F(x)\left[(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}) P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)\right]$.

Let $x \in \mathbb{R}^n$ and a $(k+1)$-parallelogram $P(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$ be given. The exterior derivative $d\varphi$ is defined as follows

$$d\varphi = \lim_{\substack{P_x(U(I)) \subset B(x, \delta(x)) \\ \delta(x) \to 0}} \frac{\int_{\partial P_x(U(I))} \varphi}{SV(P_x(U(I)))},$$

where $\vec{u}_j = h_j\vec{v}_j$, $U(I) = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{k+1})$, $I = \{1, 2, \ldots, k+1\}$, $SV(P_x(U(I)))$ is the signed $(k+1)$-dimensional volume of $P_x(U(I))$ and $P_x(V(I)) = P_x(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$.

The exterior derivative $d\varphi$ is a point-parallelogram function and the parallelograms here are $(k+1)$-parallelograms. More precisely, for each $\epsilon > 0$, there exists $\delta(x) > 0$ such that for any $(k+1)$-parallelogram $P_x(U(I)) \subset B(x, \delta(x))$, we have

$$\left| \int_{\partial P_x(U(I))} \varphi - (d\varphi)(x, P_x(U(I))) \right| < \epsilon |SV(P_x(U(I)))|.$$

Theorem 4.2 after Definition 4.1 shows that $d\varphi = dF \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, which is a $(k+1)$-form. The $(k+1)$-form $d\varphi$ takes point-$(k+1)$-parallelogram $(x, P_x(U(I)))$ and returns a number.

The concept of strong differentiability used in [3] shall be used again in this section.

Definition 4.1. Let $F : \mathbb{R}^n \to \mathbb{R}$. Then F is said to be strongly Henstock differentiable at x with respect to a $(k+1)$-parallelogram $P(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$ with derivative $A(x)$ if (i) F is classical (Fréchet) differentiable at x; (ii) for each $\epsilon > 0$, there exists $\delta(x) > 0$ such that for every $(k+1)$-parallelogram $P_x(U(I)) \subset B(x, \delta(x))$ and, for $i = 1, 2, \ldots, k+1$, $z \in P_x(U(I_i))$, we have

$$|F(z + \vec{u}_i) - F(z) - A(x) \cdot (\vec{u}_i)| \le \epsilon \, \|\vec{u}_i\|,$$

where $P_x(U(I))$ is given in the definition of $d\varphi$.

It is clear that

$$A = (\partial_1 F, \partial_2 F, \ldots, \partial_n F).$$

Suppose $P_x(V(I))$ is replaced by $P(V(I))$, where $P(V(I))$ may not be anchored at the point x. Then F is said to be strongly McShane differentiable at x.

An example given in [3, section 8, remark (viii)] shows that there exists a function F which is strongly Henstock differentiable, but is not C^1.

Theorem 4.2. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous and $\varphi = F(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})$. Suppose that F is strongly Henstock differentiable at x with respect to a $(k+1)$-parallelogram $P(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$. Then the exterior derivative $d\varphi$ exists and $d\varphi = dF \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, which is a $(k+1)$-form.*

Proof. Let $I = \{1, 2, \ldots, k+1\}$ and $V(I) = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$. Let $\vec{u}_j = h_j\vec{v}_j$, where $0 < h_j \le 1$, $j = 1, 2, \ldots, k+1$ and $U(I) = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{k+1})$. By definition,

$$d\varphi = \lim_{\substack{P_x(U(I)) \subset B(x, \delta(x)) \\ \delta(x) \to 0}} \frac{\int_{\partial P_x(U(I))} \varphi}{SV(P_x(U(I)))}.$$

We may assume that $SV(P_x(U(I)))$ is always positive in this proof.

Now, we shall show that

$$\lim_{\substack{P_x(U(I))\subset B(x,\delta(x))\\ \delta(x)\to 0}} \frac{\int_{\partial P_x(U(I))}\varphi}{SV(P_x(U(I)))} = dF\wedge dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k}.$$

First, consider

$$\int_{P_x(U(I_j))}(F(z+\vec{u}_j)-F(z))(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k}),$$

where $U(I_j)=(\vec{u}_1,\ldots,\hat{\vec{u}}_j,\ldots,\vec{u}_{k+1})$. By given, F is strongly Henstock differentiable at x. Hence we have, for any $z\in P_x(U(I_j))$

$$|F(z+\vec{u}_j)-F(z)-A(x)\cdot\vec{u}_j|\le\epsilon\|\vec{u}_j\|,$$

where $A=(\partial_1F,\partial_2F,\ldots,\partial_nF)$. Thus

$$\begin{aligned}&|(F(z+\vec{u}_j)-F(z))(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})\\&-(A(x)\cdot\vec{u}_j)(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})|\\&\le\epsilon\|\vec{u}_j\||(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})|.\end{aligned}$$

Hence

$$\begin{aligned}&\Bigg|\int_{P_x(U(I_j))}(F(z+\vec{u}_j)-F(z))(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})\\&\qquad-(A(x)\cdot\vec{u}_j)\int_{P_x(U(I_j))}(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})\Bigg|\\&\le\ \epsilon\ \|\vec{u}_j\|\int_{P_x(U(I_j))}|(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})|.\end{aligned}$$

Note that if $\vec{u}_j=(u_{j_1},\ldots,u_{j_n})^T$, then

$$A(x)\cdot\vec{u}_j=\sum_{l=1}^n\partial_lF(x)u_{j\,l}=\sum_{l=1}^n\partial_lF(x)(dx_l)(\vec{u}_j).$$

Thus

$$\begin{aligned}&\sum_{j=1}^{k+1}(-1)^{j-1}(A(x)\cdot\vec{u}_j)(dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})P_x(U(I_j))\\&=\sum_{j=1}^{k+1}(-1)^{j-1}\left[\sum_{l=1}^n\partial_lF(x)(dx_l)(\vec{u}_j)\right](dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})P_x(U(I_j))\\&=(dF\wedge dx_{i_1}\wedge dx_{i_2}\wedge\ldots\wedge dx_{i_k})P_x(U(I)).\end{aligned}$$

Recall that $dF \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$ is the wedge product of dF and $dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$.

Similarly, we have

$$\begin{aligned}
&\sum_{j=1}^{k+1}(-1)^{j-1}\|\vec{u}_j\||(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P_x(U(I_j))| \\
=&\sum_{j=1}^{k+1}(-1)^{j-1}\|h_j\vec{v}_j\||(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P_x(U(I_j))| \\
=&\sum_{j=1}^{k+1}(-1)^{j-1}h_j\|\vec{v}_j\||(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})(h_1h_2\cdots \widehat{h_j} \cdots h_{k+1})P_x(V(I_j))| \\
=&(h_1h_2\cdots h_{k+1})\sum_{j=1}^{k+1}(-1)^{j-1}\|\vec{v}_j\||(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P_x(V(I_j))| \\
=&\ (h_1h_2\cdots h_{k+1})Q(V(I)) \\
=&\ (h_1h_2\cdots h_{k+1})|SV(P_x(V(I)))|\frac{Q(V(I))}{|SV(P_x(V(I)))|} \\
=&\ |SV(P_x(U(I)))|R(V(I)).
\end{aligned}$$

In the above, $Q(V(I))$ and $R(V(I))$ are defined accordingly and $R(V(I))$ is a fixed value since $P_x(V(I))$ is fixed. In the above, we use the fact that $\vec{u}_j = h_j\vec{v}_j$ and $h_1h_2\cdots h_{k+1}SV(P_x(V(I))) = SV(P_x(U(I)))$. Applying equation (1) with $P_a(V(I))$ replaced by $P_x(U(I))$ and $I = \{1, 2, \ldots, k+1\}$, we have

$$\left|\int_{\partial P_x(U(I))}\varphi - (dF \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})P_x(U(I))\right| \\ \le \epsilon\ R(V(I))|SV(P_x(U(I)))|$$

whenever $P_x(U(I)) \subset B(x, \delta(x))$.

Thus

$$d\varphi = dF \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}. \qquad \blacksquare$$

5. Stokes' Theorem in $\mathbb{R}^n$

A similar proof of the following theorem is given in [3, 9]. The proof is intuitive and natural.

Theorem 5.1. *Let $E = P(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$ be a $(k+1)$-parallelogram. Let φ be a k-form in $\mathbb{R}^n$. Suppose the exterior derivative $d\varphi$ exists on a $(k+1)$-parallelogram*

E with respect to $(k+1)$-parallelogram $P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{k+1})$ where $\vec{v}_i$ and $\vec{w}_i$ are parallel for all i. Then $d\varphi$ is Kurzweil-Henstock integrable on E and

$$\int_E d\varphi = \int_{\partial E} \varphi.$$

Proof. Suppose the exterior derivative $d\varphi$ exists on E. Hence for each $x \in E$ and each $\epsilon > 0$, there exists $\delta(x) > 0$ such that whenever an $(k+1)$-parallelogram I with $x \in I \subset B(x, \delta(x))$, we have

$$\left| d\varphi(I) - \int_{\partial I} \varphi \right| \leq \epsilon\, |I|.$$

In the above, I is of the form $P(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{k+1})$ and more precisely, $d\varphi(I)$ should be written as $d\varphi(x, I)$.

Let $D = \{(x, I)\}$ be a Henstock δ-fine division of E. Then we have

$$\left| (D) \sum \left\{ d\varphi(I) - \int_{\partial I} \varphi \right\} \right| \leq \epsilon\, (D) \sum |I|.$$

Therefore

$$\left| (D) \sum d\varphi(I) - \int_{\partial E} \varphi \right| \leq \epsilon\, |E|.$$

Consequently $d\varphi$ is Kurzweil-Henstock integrable on E and

$$\int_E d\varphi = \int_{\partial E} \varphi. \qquad \blacksquare$$

6. Integral on Manifolds

The Kurzweil-Henstock integration on Manifolds has been studied in [2, 3]. For easy reference, we give a brief introduction here.

In this section, $\mathbb{H}^n$ denotes the upper half-space in $\mathbb{R}^n$, which consists of those $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which $x_n \geq 0$. A non-empty subset M of $\mathbb{R}^n$ is said to be a k-manifold if for each $x \in M$, there exist an open subset V of M containing x, an open subset U of $\mathbb{R}^k$ (or $\mathbb{H}^k$) and a homeomorphism mapping $\alpha : U \to V$, i.e., α is a bijection and both α and α^{-1} are continuous, and $D\alpha(y)$ has rank k for each $y \in U$,

where $D\alpha = \begin{pmatrix} \dfrac{\partial \alpha_1}{\partial y_1} & \dfrac{\partial \alpha_1}{\partial y_2} & \cdots & \dfrac{\partial \alpha_1}{\partial y_k} \\ \dfrac{\partial \alpha_2}{\partial y_1} & \dfrac{\partial \alpha_2}{\partial y_2} & \cdots & \dfrac{\partial \alpha_2}{\partial y_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \alpha_n}{\partial y_1} & \dfrac{\partial \alpha_n}{\partial y_2} & \cdots & \dfrac{\partial \alpha_n}{\partial y_k} \end{pmatrix}$ and $\alpha(y) = (\alpha_1(y), \alpha_2(y), \ldots, \alpha_n(y))$,

$y = (y_1, y_2, ..., y_k)$. Such an α is called a chart. If the mapping $\alpha : U \to V$ is a C^1-diffeomorphism, i.e., α is a bijection and both α and α^{-1} are of C^1-class, then M is said to be a differentiable k-manifold. Let M be a manifold. A finite collection $\Theta = \{\alpha_j\}_{j=1}^m$ of charts, where $\alpha_j : U_j \to V_j$, is said to be an atlas if the union of all V_j is M. Let $\alpha : U \to V$ be a chart and $I \subseteq U$ be a k-parallelogram in $\mathbb{R}^k$. Let $I^\alpha = \alpha(I)$, which is called a tile. Here I^α can be viewed as a distorted k-parallelogram.

A partial partition $P = \{I_i^{\alpha_{s_i}}\}_{i=1}^m$ of M is a finite collection of non-overlapping distorted k-parallelogram. If the union of $\{I_i^{\alpha_{s_i}}\}_{i=1}^m$ is M, then P is said to be a partition of M. A partial division D of M is a finite collection of point-distorted k-parallelogram pairs $\{(x_i, I_i^{\alpha_{s_i}})\}_{i=1}^m$ such that $\{I_i^{\alpha_{s_i}}\}_{i=1}^m$ is a partial partition of M. If $\{I_i^{\alpha_{s_i}}\}_{i=1}^m$ is a partition of M, then D is said to be a division of M.

Let δ be a positive function on M and $x \in M$. A point-distorted k-parallelogram pair (x, I^α) is said to be Henstock δ-fine if $x \in I^\alpha \subset B(x, \delta(x))$. A partial division D of M is said to be a Henstock δ-fine partial division of M if each point-distorted k-parallelogram pair in D is Henstock δ-fine. If, in addition, D is a division of M, then D is said to be a Henstock δ-fine division of M. Similarly, we can define McShane δ-fine and McShane δ-fine division, see Section 2.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a chart and $x \in M$ with $\alpha(y) = x$. Let $\vec{v}_i = (\partial_i\alpha_1(y), \partial_i\alpha_2(y), \ldots, \partial_i\alpha_n(y))^T$ and $\vec{u}_i = h_i\vec{v}_i$, where $0 < h_i \leq 1$. Let $J = \{1, 2, \ldots, k\}$, $\boldsymbol{V}(J) = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ and $\boldsymbol{U}(J) = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$. Then the volume of I^α can be approximated by the volume of the k-parallelogram $P_x(\boldsymbol{U}(J))$ induced by $\boldsymbol{U}(J)$. The volume of $P_x(\boldsymbol{U}(J))$ is given by

$$\left[\det\Big([D\alpha(y)]^T \cdot D\alpha(y)\Big)\right]^{\frac{1}{2}} |I|.$$

A k-form $\varphi = F(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$ defined on M is said to be α *parameterisable* if the closure of $\text{supp}F$ can be parameterised by one chart α, i.e., $\alpha : U \to V \supset \overline{\text{supp}F}$. In the following, $\text{supp}F$ is denoted by $\text{supp}\varphi$.

Definition 6.1. Let M be a compact differentiable k-manifold with atlas Θ. An α parameterisable k-form φ defined on M is said to be *KH-integrable* to real number A on M associated with chart α if for every $\epsilon > 0$, there exists a positive function δ defined on M such that for every Henstock δ-fine partial division $D = \{(x_i, I_i^\alpha)\}_{i=1}^m$ of M covering $\overline{\text{supp}\varphi}$ with $x_i \in \overline{\text{supp}\varphi}$, for each i, we have

$$|S(\varphi, \delta, D) - A| \leq \epsilon,$$

where

$$S(\varphi, \delta, D) = \sum_{i=1}^m F(x_i)(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})P_{x_i}(\boldsymbol{U}^i(J))$$

and $P_{x_i}(\boldsymbol{U}^i(J))$ is the k-parallelogram corresponding to I_i^α as mentioned before Definition 6.1. We denote A by $(KH)\int_M \varphi$.

The value of the integral does not depend on α, more precisely, if a k-form φ is KH-integrable on M with respect to a chart α and another chart β, then the values of these two integrals are equal, see [3]. Hence the integral value is uniquely determined. Independence of the integral with respect to a chart is not required. Furthermore, if F is continuous on M, then φ is KH-integrable on M. We remark that in [3] Corollary 1, the claim that f is HK-integrable with respect to chart α if and only if f is HK-integrable with respect to any other chart α' is not correct.

Let M be a compact differentiable k-manifold with an atlas Θ. Let $\alpha : U \to V$ be a chart in the atlas Θ and $\varphi = F(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$ be a k-form defined on M. Then φ is said to be KH-integrable on V if $F\chi_V(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$, denoted by $\varphi\chi_V$, is KH-integrable on M. Suppose φ is KH-integrable on V. Then, $(F\omega)(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$, denoted by $\varphi\omega$, is KH-integrable on V if ω is of class C^∞ and $\text{supp}\omega \subseteq V$. Here we use the fact that if g is Kurzweil-Henstock integrable on a compact interval $E^* \subseteq \mathbb{R}^k$ and $\omega : E^* \to \mathbb{R}$ is of class C^∞, then $g\omega$ is Kurzweil-Henstock integrable on E^*; see [6, 8].

In general, a δ-fine division may not exist on a compact manifold with more than one chart. For example, let M be a unit sphere in $\mathbb{R}^3$ and U be an open unit disk in $\mathbb{R}^2$. Let α_1 be a function mapping the open unit disk U to the upper half of the unit sphere M defined by $\alpha_1(t_1, t_2) = (t_1, t_2, \sqrt{1 - t_1^2 - t_2^2})$; and $\alpha_2, \ldots, \alpha_6$ be functions mapping the open unit disk U to the lower, right, left, front and back half of the unit sphere M defined in a similar way. Clearly M is a compact 2-manifold with atlas $\Theta = \{\alpha_j\}_{j=1}^6$. Suppose a δ-fine division exists on M. Then there exist two nonoverlapping distorted intervals from different charts such that their common points form a non-degenerated curve in $\mathbb{R}^3$. Suppose that the two distorted intervals are $\alpha_1(I)$ and $\alpha_5(I)$. Then $(t_1, t_2, \sqrt{1 - t_1^2 - t_2^2}) = (\sqrt{1 - s_1^2 - s_2^2}, s_1, s_2)$ on the common curve. We may assume that s_1 and t_1 are constants; s_2 and t_2 are variables. Then $t_2 = s_1$ and $s_2 = \sqrt{1 - t_1^2 - t_2^2} = \sqrt{1 - t_1^2 - s_1^2}$, i.e., s_2 and t_2 are also constants. Thus the two distorted intervals have only one common point. It leads to a contradiction. Therefore a δ-fine division does not exist on M. So we shall use a partition of unity in the following Definition 6.2.

A partition of unity $\{\omega_j\}_{j=1}^m$, where each ω_j is of class C^∞ (see [4, p. 298]) and $\text{supp}\omega_j = \overline{\text{supp}\omega_j}$, on a manifold with an atlas $\Theta = \{\alpha_j\}_{j=1}^m$ is said to be dominated by Θ if for each j, $\text{supp}\omega_j \subset V_j$, where $\alpha_j : U_j \to V_j$.

Definition 6.2. Let M be a compact differentiable k-manifold and $\Theta = \{\alpha_j\}_{j=1}^m$ an atlas of M with $\alpha_j : U_j \to V_j$. Let $\{\omega_j\}_{j=1}^m$ be a partition of unity dominated by atlas Θ on M. Suppose that a k-form φ is KH-integrable on each V_j. Then the KH-integral of φ on M is defined by

$$(KH)\int_M \varphi = \sum_{j=1}^{m}(KH)\int_M \varphi\omega_j.$$

Suppose $d\varphi$ is KH-integrable on M and φ is KH-integrable on ∂M. Using the Henstock Lemma; see [8, p. 81] or following the proof of Lemma 3 in [3], we can prove that for each $\epsilon > 0$, there exists $\delta(x) > 0$ such that whenever $D = \{(x, I^\alpha)\}$ is a Henstock δ-fine division of M, we have

(2) $$(D)\sum |d\varphi(x, I^\alpha) - d\varphi(x, P_x(\boldsymbol{U}(J)))| < \epsilon.$$

(3) $$(D)\sum |\int_{\partial P_x(\boldsymbol{U}(J))}\varphi - \int_{\partial I^\alpha}\varphi| < \epsilon.$$

7. Stokes' Theorem on Manifolds

In this section, we consider compact oriented differentiable $(k+1)$-manifolds M with atlas Θ and boundary ∂M in $\mathbb{R}^n$. It is known that the boundary ∂M is a k-dimensional manifold without boundary. We assume that the atlas Θ is orientation-preserving.

Let $\varphi = F(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$ be a k-form in $\mathbb{R}^n$, where F is continuous and suppose that the exterior derivative $d\varphi$ exists in the following sense:

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a chart, $x \in M$ with $\alpha(y) = x$ and $\vec{v}_i = (\partial_i\alpha_1(y), \partial_i\alpha_2(y), \ldots, \partial_i\alpha_n(y))^T$. Let $J = \{1, 2, \ldots, k+1\}$, $\boldsymbol{V}(J) = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k+1})$. Let $\vec{u}_i = h_i\vec{v}_i$, where $0 < h_i \le 1$, $i = 1, 2, \ldots, k+1$ and $\boldsymbol{U}(J) = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{k+1})$.

$$d\varphi = \lim_{\substack{P_x(\boldsymbol{U}(J))\subset B(x,\delta(x))\\ \delta(x)\to 0}} \frac{\int_{\partial P_x(\boldsymbol{U}(J))}\varphi}{SV(P_x(\boldsymbol{U}(J)))}.$$

We stress that when taking limit, the chart is fixed. Recall that $SV(P_x(\boldsymbol{U}(J)))$ is the signed $(k+1)$-dimensional volume of $P_x(\boldsymbol{U}(J))$.

More precisely, for each $\epsilon > 0$, there exists $\delta(x) > 0$ such that when $P_x(\boldsymbol{U}(J)) \subset B(x, \delta(x))$, we have

$$\left| d\varphi(x, P_x(\boldsymbol{U}(J))) - \int_{\partial P_x(\boldsymbol{U}(J))}\varphi \right| \le \epsilon\ |SV(P_x(\boldsymbol{U}(J)))|.$$

Lemma 7.1. *Let M be a compact oriented differentiable $(k+1)$-manifold and $\varphi = F(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$ a k-form in $\mathbb{R}^n$, where F is continuous. Suppose that the exterior derivative $d\varphi$ exists with respect to the chart α and $d\varphi$ is α-parametrisable. Then $d\varphi$ is Kurzweil-Henstock integrable on M and*

$$\int_M d\varphi = \int_{\partial M}\varphi.$$

Proof. Assume the chart $\alpha : U \to V$. Let $\epsilon > 0$. Then, by the definition of $d\varphi$, there exists $\delta(x) > 0$ on V such that when $P_x(\boldsymbol{U}(J)) \subset B(x, \delta(x))$, we have

$$\left| d\varphi(x, P_x(\boldsymbol{U}(J))) - \int_{\partial P_x(\boldsymbol{U}(J))} \varphi \right| \leq \epsilon \left| SV(P_x(\boldsymbol{U}(J))) \right|.$$

We may assume that $B(x, \delta(x)) \subset V$ and inequality (2) and (3) hold. Let $D = \{(x, I^\alpha)\}$ be a Henstock δ-fine partial division covering $\overline{\text{supp} d\varphi}$ with $x \in \overline{\text{supp} d\varphi}$. We may assume that D is a division of M, since if $I^\alpha \cap \overline{\text{supp} d\varphi} = \emptyset$, then $\int_{I^\alpha} d\varphi = 0$ and $\int_{\partial I^\alpha} \varphi = 0$.

Therefore

$$\begin{aligned}
&\left| (D) \sum d\varphi\, (x, I^\alpha) - \int_{\partial M} \varphi \right| \\
&\quad = \left| (D) \sum d\varphi(x, I^\alpha) - (D) \sum \int_{\partial I^\alpha} \varphi \right| \\
&\quad \leq \left| (D) \sum (d\varphi(x, I^\alpha) - d\varphi(x, P_x(\boldsymbol{U}(J)))) \right| \\
&\qquad + \left| (D) \sum \left(\int_{\partial P_x(\boldsymbol{U}(J))} \varphi - \int_{\partial I^\alpha} \varphi \right) \right| \\
&\qquad + \left| (D) \sum \left(d\varphi(x, P_x(\boldsymbol{U}(J))) - \int_{\partial P_x(\boldsymbol{U}(J))} \varphi \right) \right| \\
&\quad \leq \left| (D) \sum (d\varphi(x, I^\alpha) - d\varphi(x, P_x(\boldsymbol{U}(J)))) \right| \\
&\qquad + \left| (D) \sum \left(\int_{\partial P_x(\boldsymbol{U}(J))} \varphi - \int_{\partial I^\alpha} \varphi \right) \right| + \epsilon\, (D) \sum \left| SV(P_x(\boldsymbol{U}(J))) \right| \\
&\quad \leq 2\epsilon + \epsilon\, \beta,
\end{aligned}$$

where β is a constant.

Hence

$$\int_M d\varphi = \int_{\partial M} \varphi. \qquad \blacksquare$$

Theorem 7.2. *Let M be a compact oriented differentiable $(k+1)$-manifold in $\mathbb{R}^n$ with atlas Θ and $\varphi = F(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$ a k-form in $\mathbb{R}^n$, where F is continuous. Suppose that the exterior derivative $d\varphi$ and $d(\varphi\gamma)$ exist on M for any $\gamma \in C^\infty$ with respect to the atlas Θ. Then*

$$\int_M d\varphi = \int_{\partial M} \varphi.$$

Proof. Let $\{\gamma_i\}_{i=1}^m$ be a partition of unity dominated by atlas $\Theta = \{\alpha_i\}_{i=1}^m$ with $\alpha_i : U_i \to V_i$.

Then, for each i, $\overline{\text{supp}\varphi\gamma_i} \subseteq \overline{\text{supp}\gamma_i}$. Applying Lemma 7.1 to $\varphi\gamma_i$, which is α_i-parametrisable with $\alpha_i : U_i \to V_i$, we have

$$\int_M d(\varphi\gamma_i) = \int_{\partial M} \varphi\gamma_i.$$

Note that $d\varphi = \sum_{i=1}^m d(\varphi\gamma_i)$. Thus,

$$\int_M d\varphi = \sum_{i=1}^m \int_M d(\varphi\gamma_i) = \sum_{i=1}^m \int_{\partial M} \varphi\gamma_i = \int_{\partial M} \varphi. \qquad \blacksquare$$

The strong Henstock differentiability of F can be defined similarly on Manifold as in Definition 4.1 for an n-dimensional space.

Theorem 7.3. *Let M be a compact oriented differentiable $(k+1)$-manifold in $\mathbb{R}^n$ with atlas Θ and $\varphi = F(dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k})$ a k-form in $\mathbb{R}^n$, where F is continuous. Suppose that F is strongly Henstock differentiable on M with respect to $(k+1)$-parallelograms induced by the atlas Θ. Then the exterior derivative $d\varphi$ exists with respect to the atlas Θ and $d\varphi = dF \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$ on M.*

The proof of Theorem 7.3 is similar to that of Theorem 4.2.

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Varayu Boonpogkrong
Department of Mathematics and Statistics
Prince of Songkla University
Hat Yai 90110
Thailand
and
Centre of Excellence in Mathematics
CHE, Si Ayutthaya Rd.
Bangkok 10400
Thailand
E-mail: varayu.b@psu.ac.th