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GENERALIZED INTEGRATION OPERATORS BETWEEN BLOCH-TYPE SPACES AND $F(p, q, s)$ SPACES

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Abstract. Let $H(\mathbb{D})$ denote the space of all holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Let φ be a holomorphic self-map of $\mathbb{D}$, n be a positive integer and $g \in H(\mathbb{D})$. In this paper, we investigate the boundedness and compactness of a generalized integration operator

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta, \quad z \in \mathbb{D},$$

between Bloch-type spaces and $F(p, q, s)$ spaces.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let μ be a weight, that is, μ is a positive continuous function on $\mathbb{D}$. The Bloch-type $\mathcal{B}_\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$b_\mu(f) = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

With the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$, it becomes a Banach space. The little Bloch-type space $\mathcal{B}_{\mu,0}$ is a subspace of $\mathcal{B}_\mu$ consisting of those $f \in \mathcal{B}_\mu$ such that

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0.$$

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When $\mu(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, the Bloch-type space becomes the α-Bloch space $\mathcal{B}^\alpha$ (see [28][22][6][18]) and the quantity $b_\mu(f)$ is denoted by $b_\alpha(f)$, while the little Bloch-type space $\mathcal{B}_{\mu,0}$ becomes the little α-Bloch space $\mathcal{B}_0^\alpha$.

A positive continuous function ν on the interval $[0,1)$ is called normal (see [17]) if there are $\delta \in [0,1)$ and a, b, $0 < a < b$ such that

$\frac{\nu(r)}{(1-r)^a}$ is decreasing on $[\delta, 1)$ and $\lim\limits_{r\to 1} \frac{\nu(r)}{(1-r)^a} = 0$;

$\frac{\nu(r)}{(1-r)^b}$ is increasing on $[\delta, 1)$ and $\lim\limits_{r\to 1} \frac{\nu(r)}{(1-r)^b} = \infty$.

If we say that a function $\nu : \mathbb{D} \to [0,1)$ is normal we also assume that it is radial, i. e. $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$.

Let $0 < p, s < \infty$, $-2 < q < \infty$. A function $f \in H(\mathbb{D})$ is said to belong to general function space $F(p,q,s) = F(p,q,s)(\mathbb{D})$ (see [26]) if

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q h^s(z,a)\,dA(z) < \infty,$$

where $h(z,a) = \ln|\varphi_a(z)|^{-1}$ is the Green's function for $\mathbb{D}$ with logarithmic singularity at a. And $f \in H(\mathbb{D})$ is said to belong to $F_0(p,q,s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q h^s(z,a)\,dA(z) = 0.$$

The space $F(p,q,s)$ is called a general function space because we can get many function spaces from which, such as $BMOA$ space, Q_p space, Bergman space, Hardy space, Bloch space, if we take special parameters of p, q, s. if $q + s \leq -1$, then $F(p,q,s)$ is the space of constant functions.

Let φ be an analytic self-map of $\mathbb{D}$, then the composition operator on $H(\mathbb{D})$ is given by

$$C_\varphi f = f \circ \varphi.$$

Composition operators acting on various spaces of analytic functions have been the object for recent years, especially the problems of relating operator-theoretic properties of C_φ to function theoretic properties of φ. See the book of Cowen and MacCluer [4] and Shapiro [15] for discussions of composition operators on classical spaces of analytic functions.

Assume that $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic map of the unit disk $\mathbb{D}$, for $f \in H(\mathbb{D})$, define

$$I_g f(z) = \int_0^z f(\zeta) g'(\zeta)\,d\zeta, \quad z \in \mathbb{D}.$$

This operator is called Riemann-Stieltjes operator (or Extended-Cesàro operator). Ch. Pommerenke [13] initiated the study of Riemman-Stieltjes operator I_g on H^2, where he showed that I_g is bounded on H^2 if and only if g is in $BMOA$. This was extended to other Hardy spaces $H^p (1 \leq p < \infty)$ in [1] and [2] where compactness of I_g on

H^p and Schatten class membership of I_g on H^2 was also completely characterized in terms of the symbol g.

In this paper, we consider an integration operator $I_{g,\varphi}^{(n)}$ which is defined as

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

This operator is called the generalized integral operator, which was introduced in [16] and studied in [20, 16]. Also, the operator $I_{g,\varphi}^{(n)}$ is a generalization of the Rimann-Stieltjes operator I_g induced by g. In fact, the operator $I_{g,\varphi}^{(n)}$ can induce many known operators. For example, when $n = 1$, $I_{g,\varphi}^{(n)}$ reduces to an integration operator recently studied by S. Stević, S. Li, X. Zhu and W. Yang in [7, 8, 9, 10, 19, 24, 31]. When $n = 1$ and $g(z) = \varphi'(z)$, we obtain the composition operator C_φ defined as $C_\varphi f = f \circ \varphi - f(\varphi(0))$, $f \in H(\mathbb{D})$. Let D be the differentiation operator, $n = m + 1$ and $g(z) = \varphi'(z)$, then we get the operator $C_\varphi D^m f(z) = f^{(m)}(\varphi(z)) - f^{(m)}(\varphi(0))$ which was studied in [5, 11, 30].

In [16], S. D. Sharma and A. Sharmat have characterized the boundedness and compactness of generalized integration operators $I_{g,\varphi}^{(n)}$ from Bloch type spaces to weighted $BMOA$ spaces by using logarithmic Carleson measure characterization of the weighted $BMOA$ spaces. This paper is devoted to investigating the boundedness and compactness of generalized integration operators between Bloch-type spaces and $F(p, q, s)$ spaces.

Throughout this paper, we will use the letter C to denote a generic positive constant that can change its value at each occurrence. The notation $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

Here we quote some auxiliary results which will be used in the proofs of the main results in this paper.

Lemma 2.1. ([28]). *For $\alpha > 0$, if $f \in \mathcal{B}^\alpha$, then*

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1; \\ \|f\|_{\mathcal{B}^\alpha} \ln \dfrac{2}{1-|z|^2}, & \alpha = 1; \\ \dfrac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1. \end{cases}$$

and

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}},$$

for some C independent of f.

Lemma 2.2. ([23]). *For $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, if $f \in F(p, q, s)$, then*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{F(p,q,s)}}{(1-|z|^2)^{\frac{2+q-p}{p}+n}},$$

for some C independent of f.

Lemma 2.3. ([32]). *Let $0 < p < \infty$ and suppose that n_k is an increasing sequence of positive integers with Hadamard gaps, that is,*

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1,$$

for all k. Then there exists constants C_1 and C_2 depending on p and λ, such that

$$C_1 \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |\sum_{k=1}^{\infty} a_k e^{in_k\theta}|^p d\theta \right)^{1/p} \leq C_2 \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2},$$

for any scalars $a_1, a_2, \cdots$ with $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

Lemma 2.4. ([21]). *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

1. *The point evaluation functions on X are continuous.*
2. *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
3. *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Let $\alpha > 0$, then by [14] there are two holomorphic functions f_1, $f_2 \in \mathcal{B}^\alpha$, such that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{(1-|z|^2)^\alpha}, \quad z \in \mathbb{D}.$$

If we choose $g_1(z) = f_1(z) - zf_1'(0)$ and $g_2(z) = f_2(z) - zf_2'(0)$, then by the well known result (see [28])

$$(1-|z|^2)^{\alpha+1}|f''(z)| + |f'(0)| \asymp (1-|z|^2)^\alpha |f'(z)|,$$

we obtain that $g_1, g_2 \in \mathcal{B}^\alpha$ and

$$|g_1''(z)| + |g_2''(z)| \geq \frac{C}{(1-|z|^2)^{\alpha+1}}, \quad z \in \mathbb{D}.$$

Proceeding this way, then we have the following result

Lemma 2.5. ([29]). *Let $\alpha > 0$, then there are two holomorphic functions h_1, $h_2 \in \mathcal{B}^\alpha$, such that*

$$|h_1^{(n)}(z)| + |h_2^{(n)}(z)| \geq \frac{C}{(1-|z|^2)^{\alpha+n-1}}, \quad z \in \mathbb{D}.$$

3. Boundedness and Compactness of $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha(or\ \mathcal{B}_0^\alpha) \to F(p,q,s)$

In this section, we study the boundedness and compactness of the operators $I_{g,\varphi}^{(n)}$: $\mathcal{B}^\alpha(or\mathcal{B}_0^\alpha) \to F(p,q,s)$.

Theorem 3.1. *Let $g \in H(\mathbb{D})$, n be a positive integer and φ be a holomorphic self-map of $\mathbb{D}$, $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\alpha > 0$. Then the following statements are equivalent*

(i) $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \to F(p,q,s)$ *is bounded;*

(ii) $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \to F(p,q,s)$ *is bounded;*

(iii)

$$M_1 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} h^s(z,a)dA(z) < \infty.$$

Moreover, if the operator $I_{g,\varphi}^{(n)}$: $\mathcal{B}^\alpha(or\mathcal{B}_0^\alpha) \to F(p,q,s)$ is bounded, then the following relationship holds

$$\|I_{g,\varphi}^{(n)}\|_{\mathcal{B}^\alpha\to F(p,q,s)} \asymp \|I_{g,\varphi}^{(n)}\|_{\mathcal{B}_0^\alpha\to F(p,q,s)} \asymp M_1^{1/p}. \tag{1}$$

Proof. $(i) \Rightarrow (ii)$ This implication is clear.

$(ii) \Rightarrow (iii)$ Assume that (ii) holds. We adopt the methods of Theorem 16 in [3], as well as its extension in [12] and [25].

Let $r_j \in (1/2, 1)$ such that $r_j \to 1$ as $j \to \infty$, and let

$$f_{j,\theta}(z) = \sum_{k=1}^{\infty} a_k z^{2^k} = \sum_{k=1}^{\infty} 2^{k(\alpha-1)}(r_j e^{i\theta} z)^{2^k}.$$

Since $2^{k(1-\alpha)}|a_k| \to 0$ as $k \to \infty$ and $r_j \in (0,1)$, by Theorem 1 in [22], we have that $f_{j,\theta} \in \mathcal{B}_0^\alpha$ and $\|f_{j,\theta}\|_{\mathcal{B}^\alpha} \le C < \infty$, where $C > 0$ is a constant independent of n and θ. Since $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \to F(p,q,s)$ is bounded, it follows that

$$\int_{\mathbb{D}} |f_{j,\theta}^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q h^s(z,a)dA(z) \le \|I_{g,\varphi}^{(n)} f_{j,\theta}\|_{F(p,q,s)}^p \le C\|I_{g,\varphi}^{(n)}\|^p.$$

Integrating above inequality with respect to θ, applying Fubini's Theorem, Lemma 2.3 and the inequality

$$\Pi_{m=0}^{n-1}(2^k - m) \ge \frac{2^{nk}}{n!},\ n \in \mathbb{N},\ k \in \mathbb{N},\ 2^k \ge n,$$

we obtain

$$\begin{aligned}
&C\|I_{g,\varphi}^{(n)}\|^p_{\mathcal{B}_0^\alpha \to F(p,q,s)} \\
&\ge \int_{\mathbb{D}} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{k \ge [\log_2 n]}^{\infty} \Pi_{m=0}^{n-1}(2^k - m) 2^{k(\alpha-1)} (\varphi(z))^{2^k - n} (r_j e^{i\theta})^{2^k} \Big|^p \\
&\times d\theta |g(z)|^p (1-|z|^2)^q h^s(z,a) dA(z) \\
&\ge \int_{\mathbb{D}} \left(\sum_{k \ge [\log_2 n]}^{\infty} \left(\Pi_{m=0}^{n-1}(2^k - m)\right)^2 2^{2k(\alpha-1)} (r_j|\varphi(z)|)^{2(2^k-n)} \right)^{p/2} \\
&\times |g(z)|^p (1-|z|^2)^q h^s(z,a) dA(z) \\
&\ge \frac{C}{(n!)^p} \int_{\mathbb{D}} \left(\sum_{k \ge [\log_2 n]}^{\infty} 2^{2k(\alpha+n-1)} (r_j|\varphi(z)|)^{2(2^k-n)} \right)^{p/2} \\
&\times |g(z)|^p (1-|z|^2)^q h^s(z,a) dA(z),
\end{aligned}$$

where $[x]$ is the greatest integer less than or equal to x. By lemma 3.1 of [12] there is a constant C depending only on α and n such that

$$\sum_{k=1}^{\infty} 2^{2\alpha k} r^{2^{k+1}} \ge \frac{C}{(1-r^2)^{2\alpha}},$$

for $r \in (e^{-\frac{\alpha}{2}}, 1)$. Now since $\alpha + n - 1 > 0$, there is an $r_0' \in (e^{-\frac{\alpha+n-1}{2}}, 1)$ such that

$$\sum_{1 \le k < [\log_2 n]} 2^{2k(\alpha+n-1)} (r_j|\varphi(z)|)^{2^{k+1}} \le \frac{C}{2(1-|r_j\varphi(z)|^2)^{2(\alpha+n-1)}}$$

for all $r_0' \le r_j|\varphi(z)| < 1$. Thus we have

$$\sum_{k \ge [\log_2 n]} 2^{2k(\alpha+n-1)} (r_j|\varphi(z)|)^{2^{k+1}} \ge \frac{C}{2(1-|r_j\varphi(z)|^2)^{2(\alpha+n-1)}}.$$

Using Fatou's lemma,

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}\setminus\triangle(0,r_0')}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}h^s(z,a)dA(z)<\infty, \tag{2}$$

and also

$$\sup_{a\in\mathbb{D}}\int_{\triangle(0,r_0')}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}h^s(z,a)dA(z)<\infty, \tag{3}$$

since $\frac{z^n}{n!}\in\mathcal{B}_0^\alpha$. Hence, (iii) follows from (2) and (3).

$(iii)\Rightarrow(i)$ Suppose that (iii) is true, then by Lemma 2.1 we have that

$$\begin{aligned}\|I_{g,\varphi}^{(n)}f\|_{F(p,q,s)}^p&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)\\&\le C\|f\|_{\mathcal{B}^\alpha}^p\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}h^s(z,a)dA(z).\end{aligned}$$

Thus $I_{g,\varphi}^{(n)}:\mathcal{B}^\alpha\to F(p,q,s)$ is bounded.

From the proofs of $(ii)\Rightarrow(iii)$ and $(iii)\Rightarrow(i)$ the relationship (1) follows.

Lemma 3.2. *Let $g\in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$, n be a positive integer, $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$ and $\alpha>0$. Then $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ is weakly compact if and only if it is compact.*

Proof. It is clear that $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ is weakly compact if and only if $(I_{g,\varphi}^{(n)})^*:(F(p,q,s))^*\to(\mathcal{B}_0^\alpha)^*$ is weakly compact. Since $(\mathcal{B}_0^\alpha)^*\cong A^1$ (the Bergman space, see [27]), and A^1 satifies the Schur property, it follows that it is equivalent to $(I_{g,\varphi}^{(n)})^*:(F(p,q,s))^*\to(\mathcal{B}_0^\alpha)^*$ is compact, which is equivalent to $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ is compact as desired.

Theorem 3.3. *Let $g\in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$, n be a positive integer, $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$ and $\alpha>0$. Then the following statements are equivalent*

(i) $I_{g,\varphi}^{(n)}:\mathcal{B}^\alpha\to F(p,q,s)$ *is compact;*

(ii) $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ *is compact;*

(iii) $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ *is weakly compact;*

(iv)

$$M:=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)<\infty \tag{4}$$

and

$$\lim_{\rho\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\rho}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}}h^s(z,a)dA(z)=0. \tag{5}$$

Proof. From Lemma 3.2 the equivalent $(ii)\Leftrightarrow(iii)$ holds.

$(i)\Rightarrow(ii)$ This implication is clear.

$(ii)\Rightarrow(iv)$ If $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ is compact, then obviously it is bounded and hence condition (4) holds since $\frac{z^n}{n!}\in\mathcal{B}_0^\alpha$. Let

$$f_k(z)=z^{k+n},\ k\in\mathbb{N}.$$

It is easy to see that $\{f_k\}\subset\mathcal{B}_0^\alpha$ is a norm bounded sequence, and for each $k\in\mathbb{N}$, $f_k\to 0$ as $k\to\infty$ on any compact subset of $\mathbb{D}$.

Hence by Lemma 2.4 we have

$$\lim_{k\to\infty}\|I_{g,\varphi}^{(n)}f_k\|_{F(p,q,s)}=0,$$

that is,

$$\lim_{k\to\infty}\sup_{a\in\mathbb{D}}\left(\frac{(k+n)!}{k!}\right)^p\int_{\mathbb{D}}|\varphi(z)|^{kp}|g(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)=0.$$

From this we have that for every $\varepsilon>0$ there is a $k_0\in\mathbb{N}$ such that for each $\rho\in(0,1)$

$$\rho^{pk_0}k_0^{pn}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\rho}|g(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)<\varepsilon.$$

And hence we have that for every $\varepsilon>0$, there is a $k_0\in\mathbb{N}$ such that for $\rho^{pk_0}k_0^{pn}>c_0>0$

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\rho}|g(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)<\frac{\varepsilon}{c_0}. \tag{6}$$

Let $f\in B_{\mathcal{B}_0^\alpha}$, then since $I_{g,\varphi}^{(n)}:\mathcal{B}_0^\alpha\to F(p,q,s)$ is compact, we have that for every $\varepsilon>0$, there is an $r_0\in(0,1)$ such that for every $r\in(r_0,1)$

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(I_{g,\varphi}^{(n)}(f-f_r))'(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)<\varepsilon. \tag{7}$$

Hence for a fixed $r\in(r_0,1)$, by (6) and (7) we have that for $\rho>c_0^{\frac{1}{pk_0}}$

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\rho}|(I_{g,\varphi}^{(n)}f)'(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)\\&\le C\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\rho}|(I_{g,\varphi}^{(n)}(f-f_r))'(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)\\&\quad+C\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\rho}|(I_{g,\varphi}^{(n)}f_r)'(z)|^p(1-|z|^2)^qh^s(z,a)dA(z)\end{aligned}$$

$$\leq C\varepsilon + C\|f_r\|_\infty^p \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>\rho} |g(z)|^p(1-|z|^2)^q h^s(z,a)dA(z)$$
$$\leq C\varepsilon(1+\|f_r\|_\infty^p/c_0).$$

From which we have that for every $f \in B_{\mathcal{B}_0^\alpha}$ there is a $\rho_0 \in (0,1)$, $\rho_0 = \rho_0(f,\varepsilon)$, such that for $\rho \in (\rho_0, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>\rho} |(I_{g,\varphi}^{(n)} f)'(z)|^p(1-|z|^2)^q h^s(z,a)dA(z) < \varepsilon. \tag{8}$$

The compactness of $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \to F(p,q,s)$ implies that the ball $B_{\mathcal{B}_0^\alpha}$ is mapped by $I_{g,\varphi}^{(n)}$ into a relatively compact subset of $F(p,q,s)$. Thus for every $\varepsilon > 0$ there exists a finite collection of functions $\hat{f}_1, \cdots, \hat{f}_{N_1}$ in the unit ball $B_{\mathcal{B}_0^\alpha}$ such that for each $f \in B_{\mathcal{B}_0^\alpha}$, there is a $j \in \{1,2,\cdots,N_1\}$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)}(f-\hat{f}_j))'(z)|^p(1-|z|^2)^q h^s(z,a)dA(z) < \varepsilon. \tag{9}$$

Also, by (8) we have that for $\rho_1 = \max_{1\leq j\leq N_1} \rho_0(\hat{f}_j, \varepsilon)$ and $\rho \in (\rho_1, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>\rho} |(I_{g,\varphi}^{(n)} \hat{f}_j)'(z)|^p(1-|z|^2)^q h^s(z,a)dA(z) < \varepsilon \tag{10}$$

for every $j \in \{1,\cdots,N_1\}$. Hence by (9) and (10) we have

$$\int_{|\varphi(z)|>\rho} |f^{(n)}(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q h^s(z,a)dA(z) < 2^{p+1}\varepsilon. \tag{11}$$

for every $f \in B_{\mathcal{B}_0^\alpha}$, $a \in \mathbb{D}$ and $\rho \in (\rho_1, 1)$. Applying (11) to delay functions $(f_j)_r = f_j(rz), j = 1,2$ of the functions in Lemma 2.5, and hence using Fatou's lemma we can easily obtain that (5) holds.

$(iv) \Rightarrow (i)$ Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^\alpha$ converging to 0 on compact subsets of $\mathbb{D}$ as $k \to \infty$. By (5) we have that for every $\varepsilon > 0$ there is a $\rho_0 \in (0,1)$ such that for $\rho \in (\rho_0, 1)$ we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>\rho} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p(\alpha+n-1)}} h^s(z,a)dA(z) < \varepsilon. \tag{12}$$

The uniform convergence of $\{f_k\}$ on compact subsets of $\mathbb{D}$ along with Cauchy's estimate implies that $\{f_k^{(n)}\}$ also converges to zero on compact subsets of $\mathbb{D}$ as $k \to \infty$. Hence

$$\lim_{k\to\infty} \sup_{|w|\leq\rho} |f_k^{(n)}(w)| = 0. \tag{13}$$

By Lemma 2.1, (4), (12) and (13) we have

$$\begin{aligned}\|I_{g,\varphi}^{(n)} f_k\|_{F(p,q,s)}^p &\le C \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)} f_k)'(z)|^p (1-|z|^2)^q h^s(z,a)\,dA(z)\\ &\le C \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le\rho} |f_k^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q h^s(z,a)\,dA(z)\\ &+ C \sup_{a\in\mathbb{D}} \int_{\rho<|\varphi(z)|<1} |f_k^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q h^s(z,a)\,dA(z)\\ &\le CM \sup_{|z|\le\rho} |f_k^{(n)}(z)|^p + C\varepsilon \sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}_\alpha}^p.\end{aligned}$$

Hence, from which we can easily obtain that $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \to F(p,q,s)$ is compact.

4. Boundedness and Compactness of $I_{g,\varphi}^{(n)} : F(p,q,s)(or\ F_0(p,q,s)) \to \mathcal{B}_\mu$

In this section, we investigate the boundedness and compactness of the operators $I_{g,\varphi}^{(n)} : F(p,q,s)(or\ F_0(p,q,s)) \to \mathcal{B}_\mu$.

Theorem 4.1. *Let $g \in H(\mathbb{D})$, n be a positive integer and φ be a holomorphic self-map of $\mathbb{D}$, $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, μ be a normal on $[0,1)$. Then the following statements are equivalent*

- (i) $I_{g,\varphi}^{(n)} : F(p,q,s) \to \mathcal{B}_\mu$ *is bounded;*
- (ii) $I_{g,\varphi}^{(n)} : F_0(p,q,s) \to \mathcal{B}_\mu$ *is bounded;*
- (iii)

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \infty. \tag{14}$$

Moreover, if the operator $I_{g,\varphi}^{(n)} : F(p,q,s)(or\ F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded, then the following relationship holds

$$\|I_{g,\varphi}^{(n)}\|_{F(p,q,s)\to\mathcal{B}_\mu} \asymp \|I_{g,\varphi}^{(n)}\|_{F_0(p,q,s)\to\mathcal{B}_\mu} \asymp M_2. \tag{15}$$

Proof. $(i) \Rightarrow (ii)$ This implication is clear.

$(ii) \Rightarrow (iii)$ Assume that $I_{g,\varphi}^{(n)} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded, then for $f(z) = \frac{z^n}{n!} \in F_0(p,q,s)$, we have that

$$\sup_{z\in\mathbb{D}} \mu(|z|)|g(z)| < \infty. \tag{16}$$

For $w \in \mathbb{D}$, let

$$f_w(z) = \begin{cases} \dfrac{1-|\varphi(w)|^2}{(1-z\overline{\varphi(w)})^{\frac{2+q}{p}}} & p \neq q+2; \\ \ln \dfrac{e}{1-z\overline{\varphi(w)}} & p = q+2. \end{cases}$$

Then $f_w(z) \in F_0(p,q,s)$ and $\sup_{w\in\mathbb{D}} \|f_w\|_{F(p,q,s)} \le C$ for some constant $C > 0$, moreover,

$$|f_w^{(n)}(\varphi(w))| \asymp \frac{|\varphi(w)|^n}{(1-|\varphi(w)|^2)^{\frac{2+q-p}{p}+n}} \tag{17}$$

Therefore for every $w \in \mathbb{D}$, we have

$$\begin{aligned} \frac{\mu(|z|)|g(z)||\varphi(z)|^n}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} &\asymp \mu(|w|)|f_w^{(n)}(\varphi(w))g(w)| \\ &\le \sup_{z\in\mathbb{D}} \mu(|z|)|(I_{g,\varphi}^{(n)}f_w)'(z)| \\ &\le \|I_{g,\varphi}^{(n)}f_w\|_{\mathcal{B}_\mu} \preceq \|I_{g,\varphi}^{(n)}\|_{F_0(p,q,s)\to\mathcal{B}_\mu} < \infty. \end{aligned} \tag{18}$$

By (16), we have

$$\frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \le \left(\frac{4}{3}\right)^n \left(\frac{4}{3}\right)^{\frac{2+q-p}{p}} \mu(|z|)|g(z)| < \infty, \tag{19}$$

for $z \in \mathbb{D}$ such that $|\varphi(z)| \le 1/2$. And from (18), we obtain that

$$\frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \le 2^n \frac{\mu(|z|)|g(z)||\varphi(z)|^n}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \infty, \tag{20}$$

for $z \in \mathbb{D}$ such that $|\varphi(z)| > 1/2$. Thus combing (19) with (20) we get the condition (14).

$(iii) \Rightarrow (i)$ Assume that (iii) holds. For $f \in F(p,q,s)$, by Lemma 2.2 we can see that

$$\begin{aligned} \|I_{g,\varphi}^{(n)}f\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(|z|)|(I_{g,\varphi}^{(n)}f)'(z)| = \sup_{z\in\mathbb{D}} \mu(|z|)|f^{(n)}(\varphi(z))||g(z)| \\ &\le C\|f\|_{F(p,q,s)} \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}, \end{aligned}$$

from which it follows that $I_{g,\varphi}^{(n)} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded.

Also, from the proofs of $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$, we can see that (15) holds.

Theorem 4.2. *Let $g \in H(\mathbb{D})$, n be a positive integer and φ be a holomorphic self-map of $\mathbb{D}$, $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, μ be a normal on $[0,1)$. Then the following statements are equivalent* *(i)* $I_{g,\varphi}^{(n)} : F(p,q,s) \to \mathcal{B}_\mu$ *is compact;*

(ii) $I_{g,\varphi}^{(n)} : F_0(p,q,s) \to \mathcal{B}_\mu$ *is compact;*

(iii) $g \in \mathcal{B}_\mu$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = 0.$$

Proof. $(i) \Rightarrow (ii)$ This implication is obvious.

$(ii) \Rightarrow (iii)$ Suppose that $I_{g,\varphi}^{(n)} : F_0(p,q,s) \to \mathcal{B}_\mu$ is compact, then $I_{g,\varphi}^{(n)} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded, and hence $g \in \mathcal{B}_\mu$ from the proof of Theorem 4.1. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$, and

$$f_k(z) = \begin{cases} \dfrac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})^{\frac{2+q}{p}}} & p \neq q+2; \\ \left(\ln \dfrac{e}{1-|\varphi(z_k)|^2}\right)^{-1} \left(\ln \dfrac{e}{1-z\overline{\varphi(z_k)}}\right)^2 & p = q+2. \end{cases}$$

Then $f_k \in F_0(p,q,s)$ and f_k uniformly converges to zero on any compact subset of $\mathbb{D}$. From Lemma 2.4, we have

$$\lim_{k\to\infty} \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}_\mu} = 0.$$

Moreover, by (17) we have

$$\begin{aligned} \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(|z|)|(I_{g,\varphi}^{(n)} f_k)'(z)| = \sup_{z\in\mathbb{D}} \mu(|z|)|f_k^{(n)}(\varphi(z))||g(z)| \\ &\geq \mu(|z_k|)|f_k^{(n)}(\varphi(z_k))||g(z_k)| \\ &\geq C\frac{\mu(|z_k|)|g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}}. \end{aligned}$$

Therefore

$$\lim_{k\to\infty} \frac{\mu(|z_k|)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} = \lim_{k\to\infty} \frac{\mu(|z_k|)|g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} = 0,$$

which implies that

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = 0.$$

$(iii) \Rightarrow (i)$ Suppose that (iii) holds, then it is easy to see that (14) holds and hence $I_{g,\varphi}^{(n)} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $F(p,q,s)$

and $f_k \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $k \to \infty$. By (iii), for every $\varepsilon > 0$, there is a positive number $\delta \in (0,1)$ such that when $\delta < |\varphi(z)| < 1$,

$$\frac{\mu(|z|)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \varepsilon. \tag{21}$$

From (21), $g \in \mathcal{B}_\mu$ and Lemma 2.2, we have

$$\begin{aligned}
\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(|z|)|(I_{g,\varphi}^{(n)} f_k)'(z)| = \sup_{z\in\mathbb{D}} \mu(|z|)|f_k^{(n)}(\varphi(z))||g(z)| \\
&\le \sup_{|\varphi(z)|\le\delta} \mu(|z|)|f_k^{(n)}(\varphi(z))||g(z)| + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|f_k^{(n)}(\varphi(z))||g(z)| \\
&\le \|g\|_{\mathcal{B}_\mu} \sup_{|w|\le\delta} |f_k^{(n)}(w)| + C\varepsilon \sup_{k\in\mathbb{N}} \|f_k\|_{F(p,q,s)}.
\end{aligned}$$

Note that $\{w \in \mathbb{D} : |w| \le \delta\}$ is a compact subset of $\mathbb{D}$, then using the Cauchy's estimate we have that

$$\lim_{k\to\infty} \sup_{|w|\le\delta} |f_k^{(n)}(w)| = 0.$$

Hence $\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}_\mu} \le C\varepsilon$, and thus by the arbitrariness of the positive number ε it follows that

$$\lim_{k\to\infty} \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{B}_\mu} = 0.$$

From which and Lemma 2.4, we can see that $I_{g,\varphi}^{(n)} : F(p,q,s) \to \mathcal{B}_\mu$ is compact.

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