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WEIGHTED REPRESENTATION FUNCTIONS ON $\mathbb{Z}_m$

Quan-Hui Yang and Yong-Gao Chen*

Abstract. Let m, k_1, and k_2 be three integers with $m \geq 2$. For $A \subseteq \mathbb{Z}_m$ and $n \in \mathbb{Z}_m$, let $\hat{r}_{k_1,k_2}(A,n)$ denote the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. In this paper, we characterize all m, k_1, k_2, and A for which $\hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n) = \hat{r}_{k_1,k_2}(A,n)$ for all $n \in \mathbb{Z}_m$. As a corollary, we prove that there exists $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n) = \hat{r}_{k_1,k_2}(A,n)$ for all $n \in \mathbb{Z}_m$ if and only if $2d \mid m$, where $d = (k_1,m)(k_2,m)/(k_1,k_2,m)^2$. We also pose several problems for further research.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R_1(A,n)$, $R_2(A,n)$, $R_3(A,n)$ denote the number of solutions of $a + a' = n, a, a' \in A$, $a + a' = n, a, a' \in A, a < a'$, and $a + a' = n, a, a' \in A, a \leqslant a'$ respectively. We usually call them representation functions. Representation functions first appeared in the famous Erdős-Turán conjecture (see [13]) and are named so by Nathanson (see [18]) about forty years later. After that, they are studied by Erdős, Sárközy and Sós in a series of papers [8-12]. Representation functions have recently been extensively studied by many authors (see [1, 7, 14-16, 19-21, 23-25]) and are still a fruitful area of research in additive number theory.

For $i \in \{1,2,3\}$, Sárközy asked whether there are sets A and B with infinite symmetric difference such that $R_i(A,n) = R_i(B,n)$ for all sufficiently large integers n. It is known that the answer is negative for $i = 1$ (see Dombi [6]). Dombi [6] for $i = 2$ and the second author and Wang [4] for $i = 3$ proved that there exists a set $A \subseteq \mathbb{N}$ such that $R_i(A,n) = R_i(\mathbb{N} \setminus A, n)$ for all $n \geq n_0$. Later, Lev [17], Sándor [22] and Tang [24] provided several simple and nice proofs. The second author and Tang [3]

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*Corresponding author.

determined those A for which $R_i(A, n) = R_i(\mathbb{N} \setminus A, n) \geq 1$ for all $n \geq n_0$. Recently, the second author [2] determined those A for which $R_i(A, n) = R_i(\mathbb{N} \setminus A, n) \geq cn$ for all $n \geq n_0$.

Given any two positive integers $k_1 \leq k_2$ and any set A of nonnegative integers, let $r_{k_1,k_2}(A, n)$ denote the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. Cilleruelo and Rué [5] proved that if $k_2 \geq k_1 \geq 2$, then $r_{k_1,k_2}(A, n)$ cannot be constant from some point on. Recently, the authors [26] proved that there exists a set $A \subseteq \mathbb{N}$ such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(\mathbb{N} \setminus A, n)$ for all sufficiently large integers n if and only if $k_1 \mid k_2$ and $k_2 > k_1$. In this paper, we study the modular version of this property.

First we give some notation here. For a positive integer m, let $\mathbb{Z}_m$ be the set of residue classes modulo m. Given any t integers $k_1, \ldots, k_t$, any set $A \subseteq \mathbb{Z}_m$, and any $n \in \mathbb{Z}_m$, let $\hat{r}_{k_1,\cdots,k_t}(A, n)$ denote the number of solutions of the equation $n = k_1a_1 + \cdots + k_ta_t$ with $a_1, \ldots, a_t \in A$. For $d \mid m$, the set A is said *uniformly distributed modulo* d if $|\{x : x \in A, x \equiv i \pmod d\}| = |A|/d$ for all $i = 0, 1, \ldots, d-1$. In this paper, we characterize all m, k_1, k_2, and A for which $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$. The following results are proved.

Theorem 1. *Let m, k_1, and k_2 be three integers with $m \geq 2$, $A \subseteq \mathbb{Z}_m$, and $d = (k_1, m)(k_2, m)/(k_1, k_2, m)^2$. Then $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if $|A| = m/2$ and A is uniformly distributed modulo d.*

Corollary 1. *Let the notation be as in Theorem 1. Then there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if $2d \mid m$.*

For a nonzero integer k, let $v_2(k) = t$ if $2^t \mid k$ and $2^{t+1} \nmid k$.

Corollary 2. *Let the notation be as in Theorem 1. Then there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if m is even and one of the following statements is true: (i) $2 \mid k_1 + k_2$; (ii) $2 \nmid k_1 + k_2$ and $v_2(k_1k_2) < v_2(m)$.*

Motivated by Lev [17] and the authors [26], we now pose the following problems for further research.

Problem 1. Given any integers m, k_1 and k_2 with $m \geq 2$, determine all pairs of subsets $A, B \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(B, n)$ for all $n \in \mathbb{Z}_m$.

Problem 2. For $t \geq 3$, find all $t+1$-tuples $(m, k_1, \ldots, k_t)$ of integers for which there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,\ldots,k_t}(A, n) = \hat{r}_{k_1,\ldots,k_t}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$.

2. Proofs

For $T \subseteq \mathbb{Z}_m$ and $x \in \mathbb{Z}_m$, let

$$S_T(x) = \sum_{t \in T} e^{2\pi i t x/m}.$$

Let $A \subseteq \mathbb{Z}_m$ and $B = \mathbb{Z}_m \setminus A$. Then

$$\hat{r}_{k_1,k_2}(A,n) = \frac{1}{m}\sum_{x=0}^{m-1} S_A(k_1x)S_A(k_2x)e^{-2\pi i n x/m}$$

for all $n \in \mathbb{Z}_m$. Let $g_A(x) = S_A(k_1x)S_A(k_2x) - S_B(k_1x)S_B(k_2x)$. Thus

$$\hat{r}_{k_1,k_2}(A,n) - \hat{r}_{k_1,k_2}(B,n) = \frac{1}{m}\sum_{x=0}^{m-1} g_A(x)e^{-2\pi i n x/m} \tag{1}$$

for all $n \in \mathbb{Z}_m$.

In order to prove Theorem 1, we need the following Lemmas.

Lemma 1. *Let $m, k_1,$ and k_2 be three integers with $m \geq 2$. If $\hat{r}_{k_1,k_2}(A,n) = \hat{r}_{k_1,k_2}(B,n)$ for all $n \in \mathbb{Z}_m$, then m is even and $|A| = m/2$.*

Proof. If $\hat{r}_{k_1,k_2}(A,n) = \hat{r}_{k_1,k_2}(B,n)$ for all $n \in \mathbb{Z}_m$, then

$$|A|^2 = \sum_{n\in\mathbb{Z}_m} \hat{r}_{k_1,k_2}(A,n) = \sum_{n\in\mathbb{Z}_m} \hat{r}_{k_1,k_2}(B,n) = |B|^2.$$

Hence $|A| = |B|$. Therefore, $m = |A| + |B|$ is even and $|A| = m/2$. ∎

Lemma 2. *If $m \nmid k_i x$ $(i = 1, 2)$, then $g_A(x) = 0$.*

Proof. Since $m \nmid k_i x$ $(i = 1, 2)$, it follows that

$$S_A(k_1x) + S_B(k_1x) = \sum_{j=0}^{m-1} e^{2\pi i k_1 x j/m} = 0,$$
$$S_A(k_2x) + S_B(k_2x) = \sum_{j=0}^{m-1} e^{2\pi i k_2 x j/m} = 0.$$

Hence $g_A(x) = S_A(k_1x)S_A(k_2x) - S_B(k_1x)S_B(k_2x) = 0$. ∎

Lemma 3. *If $|A| = m/2$ and $m \mid k_i x$ $(i = 1, 2)$, then $g_A(x) = 0$.*

Proof. Since $m \mid k_i x\ (i = 1, 2)$, it follows that

$$S_A(k_1x) = |A| = S_A(k_2x) \quad \text{and} \quad S_B(k_1x) = |B| = S_B(k_2x).$$

Thus $g_A(x) = |A|^2 - |B|^2$. By $|A| = m/2$ we have $|B| = m/2$. Therefore, $g_A(x) = 0$. ■

Lemma 4. *If k and ℓ are two integers, then*

$$\sum_{\substack{x=0\\ m|kx}}^{m-1} S_T(\ell x)e^{-2\pi inx/m} = (k,m) \sum_{\substack{t\in T\\ (k,m)|\ell t-n}} 1.$$

Proof. Let $d = (k, m)$. Then

$$\sum_{\substack{x=0\\ m|kx}}^{m-1} S_T(\ell x)e^{-2\pi inx/m} = \sum_{\substack{x=0\\ m|kx}}^{m-1} \sum_{t\in T} e^{2\pi i(\ell t-n)x/m}$$

$$= \sum_{s=0}^{d-1} \sum_{t\in T} e^{2\pi i(\ell t-n)s/d} = \sum_{t\in T} \sum_{s=0}^{d-1} e^{2\pi i(\ell t-n)s/d} = d \sum_{\substack{t\in T\\ d|\ell t-n}} 1.$$ ■

Proof of Theorem 1. Let $d_1 = (k_1, m), d_2 = (k_2, m)$, and $d_3 = (d_1, d_2)$. Then $d = d_1d_2/d_3^2$. By Lemma 1 we may assume that m is even and $|A| = |B| = m/2$. From equality (1), by Lemmas 2-4, we have

$$\begin{aligned}
&\hat{r}_{k_1,k_2}(A, n) - \hat{r}_{k_1,k_2}(B, n)\\
&= \frac{1}{m} \sum_{\substack{x=0\\ m\nmid k_1x, m\nmid k_2x}}^{m-1} g_A(x)e^{-2\pi inx/m} + \frac{1}{m} \sum_{\substack{x=0\\ m|k_1x}}^{m-1} g_A(x)e^{-2\pi inx/m}\\
&\quad + \frac{1}{m} \sum_{\substack{x=0\\ m|k_2x}}^{m-1} g_A(x)e^{-2\pi inx/m} - \frac{1}{m} \sum_{\substack{x=0\\ m|k_1x, m|k_2x}}^{m-1} g_A(x)e^{-2\pi inx/m}\\
&= \frac{1}{m} \sum_{\substack{x=0\\ m|k_1x}}^{m-1} g_A(x)e^{-2\pi inx/m} + \frac{1}{m} \sum_{\substack{x=0\\ m|k_2x}}^{m-1} g_A(x)e^{-2\pi inx/m}\\
&= \frac{1}{2} \sum_{\substack{x=0\\ m|k_1x}}^{m-1} \left(S_A(k_2x) - S_B(k_2x)\right) e^{-2\pi inx/m}
\end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{x=0\\ m|k_2x}}^{m-1}\left(S_A(k_1x)-S_B(k_1x)\right)e^{-2\pi inx/m}$$

$$=\frac{1}{2}d_1\left(\sum_{\substack{a\in A\\ d_1|k_2a-n}}1-\sum_{\substack{b\in B\\ d_1|k_2b-n}}1\right)+\frac{1}{2}d_2\left(\sum_{\substack{a\in A\\ d_2|k_1a-n}}1-\sum_{\substack{b\in B\\ d_2|k_1b-n}}1\right).$$

It follows that

$$\hat{r}_{k_1,k_2}(A,n)=\hat{r}_{k_1,k_2}(B,n) \tag{2}$$

is equivalent to

$$d_1\sum_{\substack{a\in A\\ d_1|k_2a-n}}1+d_2\sum_{\substack{a\in A\\ d_2|k_1a-n}}1=d_1\sum_{\substack{b\in B\\ d_1|k_2b-n}}1+d_2\sum_{\substack{b\in B\\ d_2|k_1b-n}}1. \tag{3}$$

First, we prove the necessity in Theorem 1. Suppose that (2) holds for all integers n. Then (3) holds for all integers n. Thus, replacing n by d_1n in (3), we have

$$d_1\sum_{\substack{a\in A\\ d_1|k_2a-d_1n}}1+d_2\sum_{\substack{a\in A\\ d_2|k_1a-d_1n}}1=d_1\sum_{\substack{b\in B\\ d_1|k_2b-d_1n}}1+d_2\sum_{\substack{b\in B\\ d_2|k_1b-d_1n}}1 \tag{4}$$

for all integers n. Let $d_i=d_3d_i'$ and $k_i=d_3k_i'\,(i=1,2)$. Since $d_3=(d_1,d_2)$ and $d_i=(k_i,m)$, we see that d_i' and k_i' are integers $(i=1,2)$. By (4), we have

$$d_1\sum_{\substack{a\in A\\ d_1'|k_2'a}}1+d_2\sum_{\substack{a\in A\\ d_2'|k_1'a-d_1'n}}1=d_1\sum_{\substack{b\in B\\ d_1'|k_2'b}}1+d_2\sum_{\substack{b\in B\\ d_2'|k_1'b-d_1'n}}1. \tag{5}$$

Since $(d_1,k_2)=(k_1,m,k_2)=d_3$, it follows that $(d_1',k_2')=1$. Similarly, we have that $(d_2',k_1')=1$. Thus the sum of the two sides of (5) is

$$d_1\sum_{t\in\mathbb{Z}_m,d_1'|k_2't}1+d_2\sum_{t\in\mathbb{Z}_m,d_2'|k_1't-d_1'n}1=d_1\sum_{t\in\mathbb{Z}_m,d_1'|t}1+d_2\sum_{t\in\mathbb{Z}_m,d_2'|t}1=2d_3m. \tag{6}$$

By (5) and (6), we have

$$d_1\sum_{\substack{a\in A\\ d_1'|k_2'a}}1+d_2\sum_{\substack{a\in A\\ d_2'|k_1'a-d_1'n}}1=d_3m \quad \text{for all integers } n. \tag{7}$$

Since $(d_1, d_2) = d_3$, we see that $(d'_1, d'_2) = 1$. Hence there exists an integer t_1 such that $d'_1 t_1 \equiv 1 \pmod{d'_2}$. Thus $k'_1 a - d'_1 t_1 k'_1 n \equiv k'_1(a-n) \pmod{d'_2}$. By (7), replacing n by $t_1 k'_1 n$, and $(d'_2, k'_1) = 1$, we have

$$d_2 \sum_{a\in A, d'_2 | a-n} 1 = d_3 m - d_1 \sum_{a\in A, d'_1 | k'_2 a} 1,$$

so that

$$\sum_{\substack{a\in A\\ d'_2|a-n_1}} 1 = \sum_{\substack{a\in A\\ d'_2|a-n_2}} 1 \quad \text{for all integers } n_1, n_2. \tag{8}$$

Hence A is uniformly distributed modulo d'_2. Similarly, A is uniformly distributed modulo d'_1. Since $(d'_1, d'_2) = 1$, the set A is uniformly distributed modulo $d'_1 d'_2 = d_1 d_2 / d_3^2 = d$.

Now we prove the sufficiency in Theorem 1. Suppose that A is uniformly distributed modulo $d'_1 d'_2 = d$. Then A is uniformly distributed modulo d'_1. So $|\{a \in A : d'_1 | a - n\}| = |A|/d'_1$ for all integers n. Since $(k'_2, d'_1) = 1$, it follows that $|\{a \in A : d'_1 | k'_2 a - n\}| = |A|/d'_1$ for all integers n. That is, $|\{a \in A : d_1 | k_2 a - d_3 n\}| = d_3 |A|/d_1$ for all integers n. Similarly, $|\{a \in A : d_2 | k_1 a - d_3 n\}| = d_3|A|/d_2$ for all integers n. Hence

$$d_1 \sum_{\substack{a\in A\\ d_1|k_2 a-d_3 n}} 1 + d_2 \sum_{\substack{a\in A\\ d_2|k_1 a-d_3 n}} 1 = 2d_3|A| \quad \text{for all integers } n. \tag{9}$$

Since A is uniformly distributed modulo d, the set $B = \mathbb{Z}_m \setminus A$ is also uniformly distributed modulo d. Similarly, we have that

$$d_1 \sum_{\substack{b\in B\\ d_1|k_2 b-d_3 n}} 1 + d_2 \sum_{\substack{b\in B\\ d_2|k_1 b-d_3 n}} 1 = 2d_3|B| \quad \text{for all integers } n. \tag{10}$$

Noting that $|A| = |B|$, by (9) and (10), the equality (3) holds for all integers n with $d_3 | n$. If $d_3 \nmid n$, by $d_3 \mid d_1$, $d_3 \mid d_2$, and $d_2 \mid k_2$, we have $d_1 \nmid k_2 a - n$. Similarly, if $d_3 \nmid n$, then $d_2 \nmid k_1 a - n$, $d_2 \nmid k_1 b - n$, and $d_1 \nmid k_2 b - n$. So (3) holds trivially for all integers n with $d_3 \nmid n$. Thus (3) holds for all integers n. Therefore, (2) holds for all $n \in \mathbb{Z}_m$. ∎

Proof of Corollary 1. Suppose that there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$. By Theorem 1, $|A| = m/2$ and A is uniformly distributed modulo d. So m is even and $d \mid m/2$. Thus $2d \mid m$. Conversely, suppose that $2d \mid m$. Let

$$A = \bigcup_{i=0}^{d-1} \left\{ i + d\ell : \ell = 1, \ldots, \frac{m}{2d} \right\}.$$

Then $|A| = m/2$ and A is uniformly distributed modulo d. By Theorem 1, $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$. ■

Proof of Corollary 2. By Corollary 1, there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if $2d \mid m$.

Since $d_1 \mid m$ and $d_2 \mid m$, by $(d_1/d_3, d_2/d_3) = 1$ and $d = d_1d_2/d_3^2$ we have $d \mid m$. So $2d \mid m$ is equivalent to $v_2(2d) \le v_2(m)$. Without loss of generality, we assume that $v_2(k_1) \le v_2(k_2)$. Noting that

$$\begin{aligned} v_2(2d) &= 1 + v_2(d_1) + v_2(d_2) - 2v_2(d_3) \\ &= 1 + \min\{v_2(k_2), v_2(m)\} - \min\{v_2(k_1), v_2(m)\}, \end{aligned}$$

the inequality $v_2(2d) \le v_2(m)$ is equivalent to that m is even and one of (i) and (ii) is true. Therefore, there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1,k_2}(A, n) = \hat{r}_{k_1,k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if m is even and one of (i) and (ii) is true. ■

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Quan-Hui Yang and Yong-Gao Chen
School of Mathematical Sciences and Institute of Mathematics
Nanjing Normal University
Nanjing 210023
P. R. China
E-mail: yangquanhui01@163.com
ygchen@njnu.edu.cn