TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 3, pp. 981-989, June 2013
DOI: 10.11650/tjm.17.2013.2424
This paper is available online at http://journal.taiwanmathsoc.org.tw

CLASS NUMBER ONE CRITERION FOR SOME NON-NORMAL TOTALLY REAL CUBIC FIELDS

Jun Ho Lee

Abstract. Let $\{K_m\}_{m\geq 4}$ be the family of non-normal totally real cubic number fields defined by the irreducible cubic polynomial $f_m(x) = x^3 - mx^2 - (m+1)x - 1$, where m is an integer with $m \geq 4$. In this paper, we will give a class number one criterion for K_m.

1. INTRODUCTION

It has been known for a long time that there exists close connection between prime producing polynomials and class number one problem for some number fields. Rabinowitsch [9] proved that for a prime number q, the class number of $\mathbb{Q}(\sqrt{1-4q})$ is equal to one if and only if $k^2 + k + q$ is prime for every $k = 0, 1, \ldots, q-2$. For real quadratic fields, many authors [2, 3, 8, 11] considered the connection between prime producing polynomials and class number. For the simplest cubic fields, Kim and Hwang [6] gave a class number one criterion which is related to some prime producing polynomials. The aim of this paper is to give a class number one criterion for some non-normal totally real cubic fields. Its criterion provides some polynomials having almost prime values in a given interval. The method done in this paper is basically same as one in [2, 3, 6].

Let $\zeta_K(s)$ be the Dedekind zeta function of an algebraic number field K and $\zeta_K(s, P)$ be the partial zeta function for the principal ideal class P of K. Then if K a cubic number filed, we have

$$\zeta_K(-1) \leq \zeta_K(-1, P).$$

Halbritter and Pohst [5] developed a method of expressing special values of the partial zeta functions of totally real cubic fields as a finite sum involving norm, trace, and 3-fold Dedekind sums. Their result has been exploited by Byeon [1] to give an explicit

Received September 10, 2012, accepted December 4, 2012.
Communicated by Wen-Ching Li.
2010 *Mathematics Subject Classification*: Primary 11R42; Secondary 11R16, 11R29.
Key words and phrases: Cubic fields, Class number, Zeta function.

formula for the values of the partial zeta functions of the simplest cubic fields. Kim and Hwang [6] gave a class number one criterion for the simplest cubic fields by estimating the value $\zeta_K(-1)$ and combining Byeon's result. In this paper, we will do this kind of work in some non-normal totally real cubic fields. First, we apply Halbritter and Pohst's formula to our cubic fields, and then evaluate the upper bound of $\zeta_K(-1)$ by using Siegel's formula. Finally, combining this computation, we give a class number one criterion for some non-normal totally real cubic fields. Halbritter and Pohst [5] proved:

Theorem 1.1. *Let K be a totally real cubic field with discriminant Δ. For $\alpha \in K$, the conjugates are denoted by α' and α'', respectively. Furthermore, for $\alpha \in K$, let $\mathrm{Tr}(\alpha) := \alpha + \alpha' + \alpha''$ and $\mathrm{N}(\alpha) := \alpha \cdot \alpha' \cdot \alpha''$. Let $\widehat{K} := K(\sqrt{\Delta}), k \in \mathbb{N}, k \geq 2$, and $\{\epsilon_1, \epsilon_2\}$ be a system of fundamental units of K. Define L by $L := \ln|\epsilon_1/\epsilon_1''| \ln|\epsilon_2'/\epsilon_2''| - \ln|\epsilon_1'/\epsilon_1''| \ln|\epsilon_2/\epsilon_2''|$. Let W be an integral ideal of K with basis $\{w_1, w_2, w_3\}$. Let $\rho = \widetilde{w_3}$ for a dual basis $\widetilde{w_1}, \widetilde{w_2}, \widetilde{w_3}$ of W subject to*

$$\mathrm{Tr}(w_i \widetilde{w}_j) = \delta_{ij} (1 \leq i, j \leq 3).$$

For $j = 1, 2$, set

$$E_j = \begin{pmatrix} 1 & 1 & 1 \\ \epsilon_j & \epsilon_j' & \epsilon_j'' \\ \epsilon_1 \epsilon_2 & \epsilon_1' \epsilon_2' & \epsilon_1'' \epsilon_2'' \end{pmatrix}$$

and

$$B_\rho = \begin{pmatrix} \rho w_1 & \rho w_2 & \rho w_3 \\ \rho' w_1' & \rho' w_2' & \rho' w_3' \\ \rho'' w_1'' & \rho'' w_2'' & \rho'' w_3'' \end{pmatrix}.$$

For $\tau_1, \tau_2 \in K$, $\nu = 1, 2$, set

$$M(k, \nu, \tau_1, \tau_2) := 0$$

if $\det E_\nu = 0$, otherwise

$$\begin{aligned} & M(k, \nu, \tau_1, \tau_2) \\ & \quad := \mathrm{sign}(L)(-1)^\nu [\widehat{K} : \mathbb{Q}]^{-1} \frac{(2\pi i)^{3k}}{(3k)!} \mathrm{N}(\rho)^k \\ & \quad \cdot \sum_{m_1=0}^{3k} \sum_{m_2=0}^{3k} \binom{3k}{m_1, m_2} \\ & \quad \cdot \Bigg\{ \frac{\det E_\nu}{|\det(E_\nu B_\rho)|^3} \mathbf{B}(3, m_1, m_2, 3k - (m_1 + m_2), (E_\nu B_\rho)^*, \mathbf{0}) \\ & \quad \cdot \sum_{\kappa_1=0}^{k-1} \sum_{\kappa_2=0}^{k-1} \sum_{\mu_1=0}^{k-1} \sum_{\mu_2=0}^{k-1} \binom{m_1 - 1}{k - 1 - (\kappa_1 + \kappa_2), k - 1 - (\mu_1 + \mu_2)} \end{aligned}$$

$$\cdot\binom{m_2-1}{\kappa_1,\mu_1}\binom{3k-1-(m_1+m_2)}{\kappa_2,\mu_2}$$
$$\cdot\mathrm{Tr}_{\widehat{K}/\mathbb{Q}}(\tau_1^{\kappa_1+\kappa_2}\tau_1'^{\,\mu_1+\mu_2}\tau_1''^{\,3k-2-(m_1+\kappa_1+\kappa_2+\mu_1+\mu_2)}$$
$$\left.\cdot\tau_2^{\kappa_2}\tau_2'^{\,\mu_2}\tau_2''^{\,3k-1-(m_1+m_2+\kappa_2+\mu_2)})\right\},$$

where $(E_\nu B_\rho)^*$ denotes the transposed matrix of $(E_\nu B_\rho)$, and

$$\begin{aligned}
&C(k,\nu,\tau_1,\tau_2)\\
&\quad:=\mathrm{sign}(L)(-1)^{\nu+1}\frac{(2\pi i)^{3k}}{12\cdot(3k-2)(k-1)!^3}\mathrm{N}(\rho)^k\\
&\quad\cdot\widetilde{B}_{3k-2}(0)|\mathrm{det}B_\rho|^{-1}\mathrm{sign}(\mathrm{det}E_\nu)\\
&\quad\cdot\{\mathrm{sign}((\tau_1\tau_2-\tau_1'\tau_2')(\tau_1-\tau_1'))+\mathrm{sign}((\tau_1'\tau_2'-\tau_1''\tau_2'')(\tau_1'-\tau_1''))\\
&\quad+\mathrm{sign}((\tau_1''\tau_2''-\tau_1\tau_2)(\tau_1''-\tau_1))+\mathrm{sign}(\tau_1''(\tau_1-\tau_1')(\tau_2'-\tau_2))\\
&\quad+\mathrm{sign}(\tau_1(\tau_1'-\tau_1'')(\tau_2''-\tau_2'))+\mathrm{sign}(\tau_1'(\tau_1''-\tau_1)(\tau_2-\tau_2''))\\
&\quad+\mathrm{N}(\tau_2)[\mathrm{sign}(\tau_1''(\tau_2-\tau_2')(\tau_1\tau_2-\tau_1'\tau_2'))\\
&\quad+\mathrm{sign}(\tau_1(\tau_2'-\tau_2'')(\tau_1'\tau_2'-\tau_1''\tau_2''))+\mathrm{sign}(\tau_1'(\tau_2''-\tau_2)(\tau_1''\tau_2''-\tau_1\tau_2))]\}.
\end{aligned}$$

Define

$$\begin{aligned}\zeta(k,W,\epsilon_1,\epsilon_2):=&M(k,1,\epsilon_1,\epsilon_2)+M(k,2,\epsilon_2,\epsilon_1)\\&+C(k,1,\epsilon_1,\epsilon_2)+C(k,2,\epsilon_2,\epsilon_1).\end{aligned}$$

Let $\zeta_K(s,K_0)$ be the partial zeta function of an absolute ideal class K_0 of K and $W\in K_0^{-1}$. Then we have

$$\zeta_K(2k,K_0)=\frac{1}{2}\mathrm{Norm}(W)^{2k}\zeta(2k,W,\epsilon_1,\epsilon_2). \tag{1}$$

Remark 1. For $k,l,m\in\mathbb{Z}$,

$$\binom{k}{l,m}:=\begin{cases}\dfrac{k!}{l!m!(k-(l+m))!} & \text{if } k,l,m,k-(l+m)\in\mathbb{N}\cup\{0\}\\ (-1)^{l+m}\binom{l+m}{l} & \text{if } k=-1 \text{ and } l,m\in\mathbb{N}\cup\{0\}\\ 0 & \text{otherwise.}\end{cases}$$

Remark 2. Let $A=(a_{ij})_{n,n}$ be a regular (n,n)-matrix with integral coefficients, $(A_{ij})_{n,n}:=(\mathrm{det}A)A^{-1}$. Let

$$\widetilde{B}_r(x):=\begin{cases}B_r(x-[x]) & \text{if } r=0 \text{ or } r\geq 2 \text{ or } r=1\wedge x\notin\mathbb{Z}\\ 0 & \text{if } r=1\wedge x\in\mathbb{Z},\end{cases}$$

where $B_r(y)$ is defined as usual by $ze^{yz}(e^z-1)^{-1} = \sum_{r=0}^{\infty} B_r(y)z^r/r!$. Then, for $\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{N} \cup \{0\})^n$,

$$\mathbf{B}(n, \mathbf{r}, A, \mathbf{0}) = \sum_{\kappa_1=0}^{|\mathrm{det}A|-1} \cdots \sum_{\kappa_n=0}^{|\mathrm{det}A|-1} \prod_{i=1}^{n} \widetilde{B}_{r_i}(\frac{1}{\mathrm{det}A}\sum_{j=1}^{n} A_{ij}\kappa_j).$$

Next, we introduce Siegel's formula for the values of the Dedekind zeta function of a totally real algebraic number field at negative odd integers.

For an ideal I of the ring of integers $\mathcal{O}_K$, we define the sum of ideal divisors function $\sigma_r(I)$ by

$$\sigma_r(I) = \sum_{J|I} N_{K/\mathbb{Q}}(J)^r, \tag{2}$$

where J runs over all ideals of $\mathcal{O}_K$ which divide I. Note that, if $K = \mathbb{Q}$ and $I = (n)$, our definition coincides with the usual sum of divisors function

$$\sigma_r(n) = \sum_{\substack{d|n \\ d>0}} d^r. \tag{3}$$

Now let K be a totally real algebraic number field. For $l, b = 1, 2, \ldots,$ we define

$$S_l^K(2b) = \sum_{\substack{\nu \in \mathcal{D}_K^{-1} \\ \nu \gg 0 \\ \mathrm{Tr}_{K/\mathbb{Q}}(\nu)=l}} \sigma_{2b-1}((\nu)\mathcal{D}_K), \tag{4}$$

where $\mathcal{D}_K$ is the different of K. At this moment, we remark that this is a finite sum. Siegel [10] proved:

Theorem 1.2. *Let b be a natural number, K a totally real algebraic number field of degree n, and $h = 2bn$. Then*

$$\zeta_K(1-2b) = 2^n \sum_{l=1}^{r} b_l(h) S_l^K(2b). \tag{5}$$

The numbers $r \geq 1$ and $b_1(h), \ldots, b_r(h) \in \mathbb{Q}$ depend only on h. In particular,

$$r = \mathrm{dim}_{\mathbb{C}} \mathcal{M}_h, \tag{6}$$

where $\mathcal{M}_h$ denotes the space of modular forms of weight h. Thus by a well-known formula,

$$r = \begin{cases} [\frac{h}{12}] & \text{if} \quad h \equiv 2 \pmod{12} \\ [\frac{h}{12}] + 1 & \text{if} \quad h \not\equiv 2 \pmod{12}. \end{cases}$$

Now, we will introduce our target fields. Let $m(\geq 4)$ be a rational integer and K_m(or simply K)= $\mathbb{Q}(\alpha)$ be the non-normal totally real cubic number field (whose arithmetic was studied in [7]) associated with the irreducible cubic polynomial

$$f_m(x) = x^3 - mx^2 - (m+1)x - 1 \in \mathbb{Z}[x] \tag{7}$$

of positive discriminant

$$D_m = (m^2 + m - 3)^2 - 32 > 0$$

and with three distinct real roots $\alpha_3 < \alpha_2 < \alpha_1 = \alpha$. We borrow known results for arithmetic of K_m.

Theorem 1.3. (1) *The set* $\{1, \alpha, \alpha^2\}$ *forms an integral basis of the ring* $\mathcal{O}_K$ *of algebraic integers of* K *if and only if one of the following conditions holds true:*

(i) $m \not\equiv 3 \pmod 7$ *and* D_m *is square-free,*

(ii) $m \equiv 3 \pmod 7$, $m \not\equiv 24 \pmod{7^2}$ *and* $\frac{D_m}{7^2}$ *is square-free.*

(2) *The full group of algebraic units of* K_m *is* $<-1, \alpha, \alpha+1>$.

Proof. See [7]. ■

2. Class Number One Criterion for K_m

In this section, to have the value of $\zeta_K(-1, P)$, we apply Theorem 1.1 to K_m. On the other hand, we evaluate the upper bound of $\zeta_K(-1)$ by using Theorem 1.2. Finally, combining these results, we give a class number one criterion for K_m.

We take $W = \mathcal{O}_K = (\alpha)$. Since the ideal class containing $\mathcal{O}_K$ is the principal ideal class P, by (1), we have

$$\zeta_K(2, P) = \frac{1}{2}\,\zeta(2, \mathcal{O}_K, \alpha, \alpha+1).$$

By definition,

$$\zeta(2, \mathcal{O}_K, \alpha, \alpha+1) = M(2, 1, \alpha, \alpha+1) + M(2, 2, \alpha+1, \alpha)$$
$$+C(2, 1, \alpha, \alpha+1) + C(2, 2, \alpha+1, \alpha).$$

Let $\{\widetilde{w_1}, \widetilde{w_2}, \widetilde{w_3}\}$ be a dual basis of $\mathcal{O}_K$. Then, by a simple computation, we get

$$\rho = \widetilde{w_3} = \tfrac{-1}{D_m}\{(m^3 + 5m^2 + 5m + 4) + (2m^3 + 7m^2 + 7m + 9)\alpha$$
$$-2(m^2 + 3m + 3)\alpha^2\}$$

This makes it possible to determine matrices E_1, E_2 and B_ρ. Now, we note that 3-fold Dedekind sum $\mathbf{B}(3, m_1, m_2, 6 - (m_1 + m_2), (E_\nu B_\rho)^*, \mathbf{0})$ vanishes when m_1 or m_2

is odd. Next, we need the computation for trace. This computation is very long but elementary. Combining these data, we have

$$M(2,1,\alpha,\alpha+1) = -(4m^9+54m^8+304m^7+979m^6 \\ +2119m^5+3234m^4+3327m^3+2067m^2+72m-714)\pi^6/2835D_m^{3/2}$$

$$M(2,2,\alpha+1,\alpha) = (4m^9+54m^8+304m^7+985m^6 \\ +2137m^5+3204m^4+3237m^3+2091m^2+144m-714)\pi^6/2835D_m^{3/2}$$

On the other hand, the calculation of $C(2,1,\alpha,\alpha+1)$(resp. $C(2,2,\alpha+1,\alpha)$) is simpler than one of $M(2,1,\alpha,\alpha+1)$(resp. $M(2,2,\alpha+1,\alpha)$). In fact,

$$C(2,1,\alpha,\alpha+1) = \frac{2\pi^6}{45D_m^{3/2}}, \ C(2,2,\alpha+1,\alpha) = -\frac{2\pi^6}{45D_m^{3/2}}.$$

Then, by collecting these results, we have the following theorem.

Theorem 2.1. *Let $m(\geq 4)$ be an integer which satisfies the conditions of Theorem 1.3 and K_m the non-normal totally real cubic field defined by (7). Let P be the principal ideal class of K_m. Then we have*

$$\zeta_K(2,P) = \frac{m(m^5+3m^4-5m^3-15m^2+4m+12)\pi^6}{945(D_m)^{3/2}}.$$

Moreover, by a functional equation,

$$\zeta_K(-1,P) = -\frac{m(m^5+3m^4-5m^3-15m^2+4m+12)}{7560}.$$

Next, by Theorem 1.2, noting that $b_1(8) = -1/504$(cf. [12]), we have

$$\zeta_K(-1) = -\frac{8}{504}S_1^K(2) = -\frac{8}{504}\sum_{\nu\in S_1}\sigma_1((\nu)\mathcal{D}_K),$$

where

$$S_1 := \{\nu\in K \,|\, \nu\in\mathcal{D}_K^{-1}, \nu\gg 0, \mathrm{Tr}_{K/\mathbb{Q}}(\nu)=1\}.$$

Let T be the set of integral points in (s,t)-plane corresponding to S_1 by one-to-one correspondence in [4, Proposition 2.1]. This set has been completely determined in [4, Theorem 2.3] as follows:

$$\begin{aligned} T = \{ \ & (1,1), \quad (1,2), \quad \ldots \quad, \quad (1,m-1), \\ & (2,2), \quad (2,3), \quad \ldots \quad, \quad (2,m), \\ & (3,3), \quad (3,4), \quad \ldots \quad, \quad (3,m), \\ & \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots, \\ & (m-2,m-2), (m-2,m-1), (m-2,m), \\ & (m-1,m) \,\}. \end{aligned}$$

Furthermore, by (26) of [4]

$$N((\nu)\mathcal{D}_K) = |f_m(s,t)|,$$

where

$$f_m(s,t) = (-s^2 + (t+1)s)m^2 + ((t-2)s^2 - (t^2-t)s - (t^2+t))m \\ +(s^3 - 2s^2 - (t^2-3t-1)s + t^3 - t - 1).$$

One can easily check that $f_m(s,t) > 1$ for all $(s,t) \in T$. Therefore, we have the following inequalities

$$\begin{aligned}
\zeta_K(-1) &\leq -\frac{8}{504}\sum_{\nu\in S_1}(1 + N((\nu)\mathcal{D}_K)) \\
&= -\frac{8}{504}\{\sharp S_1 + \sum_{\nu\in S_1} N((\nu)\mathcal{D}_K)\} \\
&= -\frac{8}{504}\{\frac{1}{2}(m^2+m-6) + \sum_{(s,t)\in T} f_m(s,t)\} \\
&= -\frac{8}{504}\{\frac{1}{2}(m^2+m-6) + \sum_{t=1}^{m-1} f_m(1,t) \\
&\quad + \sum_{s=2}^{m-2}\sum_{t=s}^{m} f_m(s,t) + f_m(m-1,m)\} \\
&= -\frac{m(m^5+3m^4-5m^3-15m^2+4m+12)}{7560} \\
&= \zeta_K(-1,P),
\end{aligned} \tag{8}$$

and equality holds in (8) if and only if $(\nu)\mathcal{D}_K$ is a prime ideal for all $\nu \in S_1$. Combining this computation, we give a class number one criterion for K_m.

Theorem 2.2. *Let $m(\geq 4)$ be an integer which satisfies the conditions of Theorem* 1.3 *and K_m the non-normal totally real cubic field defined by* (7). *Then we have*

$$h_K = 1 \text{ if and only if } (\nu)\mathcal{D}_K \text{ is a prime ideal for all } \nu \in S_1.$$

On the other hand, Louboutin [7] showed:

$$m = 4,5,6,8 \text{ gives all the values of } m \text{ such that } h_K = 1.$$

Therefore, we can conclude that

$$m = 4,5,6,8 \text{ if and only if } (\nu)\mathcal{D}_K \text{ is a prime ideal for all } \nu \in S_1.$$

Remark 3. Unlike in the simplest cubic fields which is a Galois extension of $\mathbb{Q}$, $N((\nu)\mathcal{D}_K) = |f_m(s,t)|$ is necessarily not prime where $(\nu)\mathcal{D}_K$ is a prime ideal for each $\nu \in S_1$. For example, $f_m(2,3) = (2m-5)^2$ for the point $(2,3)$ in T, but if $m = 4,5,6,8$, then each integral ideal $(\nu)\mathcal{D}_K$ corresponding to the point $(2,3)$ is prime. Furthermore, $f_m(s,t)$ is a prime for all points only except $(2,3)$ in T when $m = 4,5,6,8$.

Acknowledgments

The author would like to thank the referee for careful reading of the manuscript and helpful comments.

References

1. D. Byeon, Special values of zeta functions of the simplest cubic fields and their applications, *Proc. Japan Acad., Ser. A*, **74** (1998), 13-15.
2. D. Byeon and H. K. Kim, Class number 1 criteria for real quadratic fields of Richaud-Degert type, *J. Number Theory*, **57** (1996), 328-339.
3. D. Byeon and H. K. Kim, Class number 2 criteria for real quadratic fields of Richaud-Degert type, *J. Number Theory*, **62** (1997), 257-272.
4. S. J. Cheon, H. K. Kim and J. H. Lee, Evaluation of the Dedekind zeta functions of some non-normal totally real cubic fields at negative odd integers, *Manuscripta Math.*, **124** (2007), 551-560.
5. U. Halbritter and M. Pohst, On the computation of the values of zeta functions of totally real cubic fields, *J. Number Thoery*, **36** (1990), 266-288.
6. H. K. Kim and H. J. Hwang, Values of zeta functions and class number 1 criterion for the simplest cubic fields, *Nagoya Math. J.*, **160** (2000), 161-180.
7. S. Louboutin, Class number and class group problems for some non-normal totally real cubic number fields, *Manuscripta Math.*, **106** (2001), 411-427.
8. R. A. Mollin and H. C. Williams, On prime valued polynomials and class numbers of real quadratic fields, *Nagoya Math. J.*, **112** (1988), 143-151.
9. G. Rabinowitsch, Eindeutigkeit der zerlegung in primzahlfaktoren in quadratischen zahlkörpern, *J. Reine Angew. Math.*, **142** (1913), 153-164.
10. C. L. Siegel, *Berechnung von Zetafuncktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, 1969, pp. 87-102.
11. H. Yokoi, *Class-number one problem for certaim kind of real quadratic fields*, Proc. Int. Conf. on class numbers and fundamental units, Katata, Japan, 1986, pp. 125-137.
12. D. B. Zagier, On the values at negative integers of the zeta function of a real quadratic field, *Enseignement Math.*, **22** (1976), 55-95.

Jun Ho Lee
School of Mathematics
Korea Institute for Advanced Study
Hoegiro 85, Dongdaemun-gu
Seoul 130-722
Republic of Korea
E-mail: jhleee@kias.re.kr