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MULTIPLICITY OF SOLUTIONS FOR PERIODIC AND NEUMANN PROBLEMS INVOLVING THE DISCRETE $p(\cdot)$-LAPLACIAN

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Abstract. Using critical point theory, we study the multiplicity of solutions for some periodic and Neumann boundary value problems involving the discrete $p(\cdot)$-Laplacian.

1. INTRODUCTION

Let T be a positive integer, $[a, b]$ be the discrete interval $\{a, a+1, \ldots, b\}$ for $a, b \in \mathbb{N}$ $(a < b)$ and λ be a positive parameter. For given $s \in (1, \infty)$, h_s will stand for the homeomorphism defined by $h_s(x) = |x|^{s-2}x$, for all $x \in \mathbb{R}$. Also, let $p : [0, T] \to (1, \infty)$, $q : [1, T] \to (1, \infty)$, $r : [1, T] \to (0, \infty)$ and $b : [1, T] \to \mathbb{R}$ be given functions. We denote

$$\mathcal{A}_k(x) := -\Delta_{p(k-1)} x(k-1) + r(k) h_{p(k)}(x(k)), \quad (k \in [1, T],\ x \in \mathbb{R}),$$

where $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator and $\Delta_{p(\cdot)}$ is the discrete $p(\cdot)$-Laplacian operator, i.e.,

$$\Delta_{p(k-1)} x(k-1) := \Delta(h_{p(k-1)}(\Delta x(k-1))) = h_{p(k)}(\Delta x(k)) - h_{p(k-1)}(\Delta x(k-1)).$$

In this paper we use the critical point theory [22], [19], to obtain the existence of multiple nontrivial solutions for the one-parameter periodic, respectively, Neumann problems:

$$(\mathcal{P}_P) \qquad \begin{cases} \mathcal{A}_k(x) = f(k, x(k)) + \lambda b(k) h_{q(k)}(x(k)), & (\forall)\ k \in [1, T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases}$$

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and

$$(\mathcal{P}_N) \qquad \begin{cases} \mathcal{A}_k(x) = f(k, x(k)) + \lambda b(k) h_{q(k)}(x(k)), & (\forall)\ k \in [1, T], \\ \Delta x(0) = 0 = \Delta x(T), \end{cases}$$

with $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ a continuous function.

Also, we deal with the existence of at least two positive solutions for the problems:

$$(\mathcal{P}'_P) \qquad \begin{cases} \mathcal{A}_k(x) = \lambda f(k, x(k)), & (\forall)\ k \in [1, T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases}$$

and

$$(\mathcal{P}'_N) \qquad \begin{cases} \mathcal{A}_k(x) = \lambda f(k, x(k)), & (\forall)\ k \in [1, T], \\ \Delta x(0) = 0 = \Delta x(T). \end{cases}$$

Throughout the paper, we assume that the variable exponent p satisfies

$$p(0) = p(T) \tag{1.1}$$

whenever we refer to the periodic problems $(\mathcal{P}_P)$ and $(\mathcal{P}'_P)$. From now on, we also employ the notations:

$$p^- = \min_{k \in [0,T]} p(k), \quad p^+ = \max_{k \in [0,T]} p(k) \ \text{ and } \ \underline{p} = \min_{k \in [1,T]} p(k), \quad \overline{p} = \max_{k \in [1,T]} p(k)$$

and we shall assume that

$$\overline{q} < p^-. \tag{1.2}$$

In recent years, equations involving the discrete p-Laplacian operator, subjected to classical or less classical boundary conditions, have been widely studied by many authors using various techniques. The variational method appears as being a very fruitful one. In this direction we mention the papers [1, 2, 6, 10, 14, 15, 23, 25, 26]. In [18], using a variational approach, the author has obtained the existence of periodic solutions for systems involving a general discrete ϕ-Laplacian operator.

Also, boundary value problems with discrete $p(\cdot)$-Laplacian were studied in recent time; we refer the reader to [11, 13, 16, 17, 20, 21]. Existence results for discrete $p(\cdot)$-Laplacian equations subjected to a general potential type boundary condition were obtained in [5], using Szulkin's critical point theory [24]. By mountain pass type arguments and the Karush-Kuhn-Tucker theorem, in [12], the existence of at least two positive solutions is established, in the case of Dirichlet boundary conditions. In the recent paper [4], the authors have obtained the existence of ground state and saddle point solutions for problems $(\mathcal{P}'_P)$ and $(\mathcal{P}'_N)$ with $\lambda = 1$; also, they give an alternative variational proof of the upper and lower solutions theorem for both of the problems.

Here, we obtain the existence of at least two nontrivial solutions for problems $(\mathcal{P}_P)$, $(\mathcal{P}_N)$ (see Theorems 3.5 and 3.6) in the presence of an Ambrosetti-Rabinowitz type condition. In this view, we employ some ideas from [3], combined with specific technicalities due to the discrete and anisotropic character of the problems. Also, for sufficiently large values of the parameter λ, the existence of at least two positive solutions for problems $(\mathcal{P}'_P)$, $(\mathcal{P}'_N)$ is established in Theorems 4.1 and 4.4.

The rest of the paper is organized as follows. The functional framework and the variational setting are presented in Section 2. In Section 3 we establish the existence of at least two nontrivial solutions for problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$, if λ is small enough. Sections 4 is devoted to problems $(\mathcal{P}'_P)$ and $(\mathcal{P}'_N)$; we obtain the existence of at least two positive solutions, for λ sufficiently large.

2. The Functional Framework

To establish the main results we shall use a variational approach. With this aim, to treat the periodic problems $(\mathcal{P}_P)$ and $(\mathcal{P}'_P)$, we introduce the space

$$X_P := \{x : [0, T+1] \to \mathbb{R} \mid x(0) = x(T+1)\},$$

while in the case of Neumann problems $(\mathcal{P}_N)$ and $(\mathcal{P}'_N)$, we shall use

$$X_N := \{x : [0, T+1] \to \mathbb{R}\}.$$

Further, for the convenience in notations we generically denote by X one of the spaces X_P or X_N. The space X will be endowed with the Luxemburg norm

$$\|x\|_{\eta,p(\cdot)} = \inf\left\{\nu > 0 : \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \left|\frac{\Delta x(k-1)}{\nu}\right|^{p(k-1)} + \eta \sum_{k=1}^{T} \frac{1}{p(k)} \left|\frac{x(k)}{\nu}\right|^{p(k)} \le 1\right\},$$

for some $\eta > 0$. Also, we shall make use of the usual sup-norm

$$\|x\|_\infty = \max_{k\in[0,T+1]} |x(k)| \quad (x \in X).$$

It is easy to check that for all $x \in X$ and any $\eta > 0$, one has

$$\|x\|^{p^+}_{\eta,p(\cdot)} \le \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \eta \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} \tag{2.1}$$
$$\le \|x\|^{p^-}_{\eta,p(\cdot)}, \text{ if } \|x\|_{\eta,p(\cdot)} < 1$$

and

$$\|x\|_{\eta,p(\cdot)}^{p^-} \leq \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \eta \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} \leq \|x\|_{\eta,p(\cdot)}^{p^+}, \text{ if } \|x\|_{\eta,p(\cdot)} > 1. \tag{2.2}$$

Now, let $\varphi_{X,\lambda} : X \to \mathbb{R}$ be defined by

$$\varphi_{X,\lambda}(x) = \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)}|x(k)|^{p(k)} - \lambda \sum_{k=1}^{T} \frac{b(k)}{q(k)}|x(k)|^{q(k)},$$

for all $x \in X$. Standard arguments show that $\varphi_{X,\lambda} \in C^1(X, \mathbb{R})$ and

$$\begin{aligned} &\langle \varphi'_{X,\lambda}(x), y\rangle \\ &= \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) \\ &+ \sum_{k=1}^{T} r(k) h_{p(k)}(x(k))y(k) - \lambda \sum_{k=1}^{T} b(k) h_{q(k)}(x(k))y(k), \quad (\forall)\ x \in X. \end{aligned} \tag{2.3}$$

Next, denoting by $F : [1, T] \times \mathbb{R} \to \mathbb{R}$ the primitive of f, i.e.,

$$F(k, t) = \int_0^t f(k, \tau)d\tau, \quad (\forall)\ k \in [1, T],\ (\forall)\ t \in \mathbb{R},$$

we define

$$\mathcal{F}_X(x) = \sum_{k=1}^{T} F(k, x(k)), \quad (\forall)\ x \in X.$$

It is a simple matter to see that $\mathcal{F}_X \in C^1(X, \mathbb{R})$ and

$$\langle \mathcal{F}'_X(x), y\rangle = \sum_{k=1}^{T} f(k, x(k))y(k), \quad (\forall)\ x, y \in X. \tag{2.4}$$

The energy functional corresponding to problem $(\mathcal{P}_P)$ (resp. $(\mathcal{P}_N)$) is

$$\Phi_X(x) = \varphi_{X,\lambda}(x) - \mathcal{F}_X(x), \quad (\forall)\ x \in X,$$

with $X = X_P$ (resp. $X = X_N$). Therefore, $\Phi_X \in C^1(X, \mathbb{R})$ and by (2.3) and (2.4), one has

$$\langle \Phi'_X(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) + \sum_{k=1}^{T} r(k) h_{p(k)}(x(k))y(k)$$

$$(2.5) \qquad -\lambda \sum_{k=1}^{T} b(k) h_{q(k)}(x(k)) y(k) - \sum_{k=1}^{T} f(k, x(k)) y(k), \quad (\forall)\ x, y \in X.$$

In the case of the problem $(\mathcal{P}'_P)$ (resp. $(\mathcal{P}'_N)$), instead of $\varphi_{X,\lambda}$, one works with $\varphi_X : X \to \mathbb{R}$ given by

$$(2.6) \qquad \varphi_X(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)} |x(k)|^{p(k)}$$

and the corresponding Euler-Lagrange functional will be now

$$\Psi_X(x) = \varphi_X(x) - \lambda \mathcal{F}_X(x), \quad (\forall)\ x \in X,$$

with $X = X_P$ (resp. $X = X_N$). Also, $\Psi_X \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle \Psi'_X(x), y \rangle = & \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1)) \Delta y(k-1) \\ & + \sum_{k=1}^{T} r(k) h_{p(k)}(x(k)) y(k) - \lambda \sum_{k=1}^{T} f(k, x(k)) y(k), \quad (\forall)\ x, y \in X.\end{aligned}$$

The search of solutions of problem $(\mathcal{P}_P)$ (resp. $(\mathcal{P}'_P)$) reduces to finding critical points of the energy functional Φ_{X_P} (resp. Ψ_{X_P}) by the following

Proposition 2.1. (see [4, Proposition 2.1]). *Assume that hypothesis* (*1.1*) *holds true. A function $x \in X_P$ is solution of problem $(\mathcal{P}_P)$ (resp. $(\mathcal{P}'_P)$) if and only if it is a critical point of Φ_{X_P} (resp. Ψ_{X_P}).*

Also, we have

Proposition 2.2. (see [4, Proposition 2.3]). *A function $x \in X_N$ is solution of problem $(\mathcal{P}_N)$ (resp. $(\mathcal{P}'_N)$) if and only if it is a critical point of Φ_{X_N} (resp. Ψ_{X_N}).*

3. Nontrivial Solutions for Problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$

Assuming an Ambrosetti-Rabinowitz type condition and controlling the asymptotic behavior of the primitive $F(k, \cdot)$ near 0, we obtain that problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$ have at least two nontrivial solutions, for small enough values of the parameter λ. To do this we need an abstract result, which is proved in [3].

Let $(Y, \|\cdot\|)$ be a real Banach space and $I \in C^1(Y, \mathbb{R})$. For $\sigma > 0$, we shall denote $B_\sigma = \{y \in Y : \ \|y\| < \sigma\}$ and by $\overline{B}_\sigma$ its closure.

Proposition 3.1. (see [3, Proposition 2]). *Suppose that I satisfies the Palais-Smale* (*in short,* (*PS*)) *condition, together with*

(i_1) $I(0) = 0$ *and there exists* $\rho > 0$ *such that*

$$-\infty < \inf_{\overline{B}_\rho} I < 0 < \inf_{\partial B_\rho} I;$$

(i_2) $I(e) \leq 0$ *for some* $e \in Y \setminus \overline{B}_\rho$.

Then I *has at least two nontrivial critical points.*

Toward the application of Proposition 3.1, we first have to know that the energy functional Φ_X satisfies the (PS) condition.

Lemma 3.2. *Assume that (1.2) holds true. If there are constants* $\theta > p^+$ *and* $\rho > 0$ *such that*

$$0 < \theta F(k,t) \leq t f(k,t), \quad (\forall)\ k \in [1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho, \tag{3.1}$$

then Φ_X *satisfies the (PS) condition and*

$$\Phi_X(c) \to -\infty \quad \text{as } |c| \to \infty,\ c \in \mathbb{R},$$

for any $\lambda > 0$.

Proof. Let $\{x_n\} \subset X$ be a sequence for which $\{\Phi_X(x_n)\}$ is bounded and

$$\Phi'_X(x_n) \to 0, \qquad \text{as } n \to \infty. \tag{3.2}$$

Since X is finite dimensional, it suffices to prove that $\{x_n\}$ is bounded. Without loss of generality, we may assume that $\|x_n\|_{\underline{r},p(\cdot)} > 1$, for all $n \in \mathbb{N}$. Using (3.1), we deduce that, for all $n \in \mathbb{N}$, it holds

$$\begin{aligned}\theta \mathcal{F}_X(x_n) - \langle \mathcal{F}'_X(x_n), x_n\rangle &= \sum_{k=1}^{T} [\theta F(k, x_n(k)) - x_n(k) f(k, x_n(k))] \\ &\leq \sum_{|x_n(k)| \leq \rho} [\theta F(k, x_n(k)) - x_n(k) f(k, x_n(k))] \\ &\leq \sum_{k=1}^{T} \max_{|x| \leq \rho} |\theta F(k,x) - x f(k,x)| =: C_1.\end{aligned} \tag{3.3}$$

From (1.2), (2.2), (2.3), (2.5) and (3.3), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&\left(\frac{\theta}{p^+} - 1\right) p^- \|x_n\|_{\underline{r},p(\cdot)}^{p^-} \\ &\leq \left(\frac{\theta}{p^+} - 1\right) \left[\sum_{k=1}^{T+1} |\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T} r(k) |x_n(k)|^{p(k)}\right] \\ &\leq \theta \left(\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)} |x_n(k)|^{p(k)}\right)\end{aligned}$$

$$
\begin{aligned}
&-\left(\sum_{k=1}^{T+1}|\Delta x_n(k-1)|^{p(k-1)}+\sum_{k=1}^{T}r(k)|x_n(k)|^{p(k)}\right)\\
&=\theta\Phi_X(x_n)-\langle\Phi'_X(x_n),x_n\rangle+\sum_{k=1}^{T}[\theta F(k,x_n(k))-x_n(k)f(k,x_n(k))]\\
&\quad+\lambda\sum_{k=1}^{T}\left[\theta\frac{b(k)}{q(k)}|x_n(k)|^{q(k)}-b(k)|x_n(k)|^{q(k)}\right]\\
&\le\theta\Phi_X(x_n)-\langle\Phi'_X(x_n),x_n\rangle+\lambda\left(\frac{\theta}{\underline{q}}-1\right)\sum_{k=1}^{T}|b(k)||x_n(k)|^{q(k)}+C_1\\
&\le\theta\Phi_X(x_n)-\langle\Phi'_X(x_n),x_n\rangle\\
&\quad+\lambda\|b\|_\infty\left(\frac{\theta}{\underline{q}}-1\right)\sum_{k=0}^{T+1}\left(|x_n(k)|^{\overline{q}}+|x_n(k)|^{\underline{q}}\right)+C_1.
\end{aligned}
$$

Then, the equivalence of the norms on X yields

$$
\begin{aligned}
&\left(\frac{\theta}{p^+}-1\right)p^-\|x_n\|_{\underline{r},p(\cdot)}^{p^-}\\
&\le\theta\Phi_X(x_n)-\langle\Phi'_X(x_n),x_n\rangle\\
&\quad+\lambda C_2\|b\|_\infty\left(\frac{\theta}{\underline{q}}-1\right)\left(\|x_n\|_{\underline{r},p(\cdot)}^{\overline{q}}+\|x_n\|_{\underline{r},p(\cdot)}^{\underline{q}}\right)+C_1\\
&\le\theta\Phi_X(x_n)+\|\Phi'_X(x_n)\|\|x_n\|_{\underline{r},p(\cdot)}+C_3\left(\|x_n\|_{\underline{r},p(\cdot)}^{\overline{q}}+\|x_n\|_{\underline{r},p(\cdot)}^{\underline{q}}\right)+C_1,
\end{aligned}
\tag{3.4}
$$

with $C_2>0$ and $C_3:=\lambda C_2\|b\|_\infty\left(\frac{\theta}{\underline{q}}-1\right)>0$.

As $\{\Phi_X(x_n)\}$ is bounded and on account of (3.2), from (3.4), we get that $\{x_n\}$ is bounded and hence, Φ_X satisfies the (PS) condition.

It is easy to check that, by virtue of (3.1), there exist $a_1,a_2>0$ such that

$$
F(k,t)\ge a_1|t|^\theta-a_2,\quad (\forall)\ k\in[1,T],\ (\forall)\ t\in\mathbb{R}. \tag{3.5}
$$

From (3.5), we infer

$$
\begin{aligned}
\Phi_X(c)&=\sum_{k=1}^{T}\frac{r(k)}{p(k)}|c|^{p(k)}-\sum_{k=1}^{T}F(k,c)-\lambda\sum_{k=1}^{T}\frac{b(k)}{q(k)}|c|^{q(k)}\\
&\le\frac{\overline{r}T}{p^-}|c|^{p^+}-a_1T|c|^\theta+a_2T-\frac{\lambda|c|^{\underline{q}}}{\overline{q}}\sum_{\{k\in[1,T];\ b(k)\ge0\}}b(k)\\
&\quad-\frac{\lambda|c|^{\overline{q}}}{\underline{q}}\sum_{\{k\in[1,T];\ b(k)<0\}}b(k),
\end{aligned}
$$

for all $c \in \mathbb{R}$, with $|c| > 1$. Then, since $\theta > p^+ > \overline{q} \geq \underline{q}$, the result follows. ■

Lemma 3.3. *Assume hypothesis (1.2) and that* $\underline{b} > 0$. *If either*

$$\liminf_{t \to 0^-} \frac{F(k,t)}{|t|^{p(k)}} \geq 0, \quad (\forall)\ k \in [1,T] \tag{3.6}$$

or

$$\liminf_{t \to 0^+} \frac{F(k,t)}{t^{p(k)}} \geq 0, \quad (\forall)\ k \in [1,T], \tag{3.7}$$

then

$$\inf_{B_\zeta} \Phi_X < 0, \tag{3.8}$$

for all $\zeta, \lambda > 0$.

Proof. First, note that $\underline{b} > 0$ means $b > 0$ on $[1,T]$. Let us suppose that (3.6) holds true. A similar argument works under assumption (3.7). Condition (3.6) means that

$$\lim_{\varepsilon \to 0^+} \inf_{t \in (-\varepsilon, 0)} \frac{F(k,t)}{|t|^{p(k)}} \geq 0, \quad (\forall)\ k \in [1,T].$$

This yields the existence of some $\varepsilon_1 > 0$ so that

$$F(k,t) \geq -|t|^{p(k)}, \quad (\forall)\ k \in [1,T],\ (\forall)\ t \in (-\varepsilon_1, 0], \tag{3.9}$$

and we may assume that $\zeta < \varepsilon_1$. For $c \in (-\zeta, 0) \subset (-\varepsilon_1, 0]$, using (3.9), (1.2) and $\underline{b} > 0$, we estimate Φ_X as follows

$$\begin{aligned}
\Phi_X(c) &= \sum_{k=1}^{T} \frac{r(k)}{p(k)} |c|^{p(k)} - \sum_{k=1}^{T} F(k,c) - \lambda \sum_{k=1}^{T} \frac{b(k)}{q(k)} |c|^{q(k)} \\
&\leq \frac{\overline{r}T}{p^-} |c|^{p^-} + T|c|^{p^-} - \frac{\lambda \underline{b} T}{\overline{q}} |c|^{\overline{q}} = |c|^{\overline{q}} T \left[\left(\frac{\overline{r}}{p^-} + 1 \right) |c|^{p^- - \overline{q}} - \frac{\lambda \underline{b}}{\overline{q}} \right] < 0,
\end{aligned}$$

provided that $|c| \in (0,1)$ is small enough, which imply (3.8) and the proof is complete. ■

Lemma 3.4. *If*

$$\limsup_{|t| \to 0} \frac{p(k)F(k,t)}{|t|^{p(k)}} < \underline{r}, \quad (\forall)\ k \in [1,T], \tag{3.10}$$

then there exist $\rho, \lambda_0 > 0$ *such that*

$$\inf_{\partial B_\rho} \Phi_X > 0, \tag{3.11}$$

for all $\lambda \in (0, \lambda_0)$.

Proof. By the equivalence of the norms on X, for each $\eta > 0$, there exists same $C_\eta > 0$, such that

$$\|x\|_\infty \le C_\eta \|x\|_{\eta,p(\cdot)}, \quad (\forall)\ x \in X. \tag{3.12}$$

From (3.10), we can find constants $\sigma \in (0, \underline{r})$, $\rho \in (0,1)$ and $C_\sigma > 0$ such that

$$F(k,t) \le \frac{\underline{r}-\sigma}{p(k)}|t|^{p(k)}, \quad (\forall)\ k \in [1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \le \rho C_\sigma. \tag{3.13}$$

Let $x \in X$, with $\|x\|_{\sigma,p(\cdot)} = \rho$, be arbitrarily chosen. By (3.12) and (3.13) one has

$$F(k,x(k)) \le \frac{\underline{r}-\sigma}{p(k)}|x(k)|^{p(k)}, \quad (\forall)\ k \in [1,T],$$

which implyies that

$$\mathcal{F}_X(x) \le (\underline{r}-\sigma)\sum_{k=1}^{T}\frac{1}{p(k)}|x(k)|^{p(k)}$$

and hence, from (2.1) and the equivalence of the norms on X, we get

$$\begin{aligned}
\Phi_X(x) &\ge \sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta x(k-1)|^{p(k-1)} + \underline{r}\sum_{k=1}^{T}\frac{1}{p(k)}|x(k)|^{p(k)} \\
&\quad +(\sigma-\underline{r})\sum_{k=1}^{T}\frac{1}{p(k)}|x(k)|^{p(k)} - \lambda\sum_{k=1}^{T}\frac{b(k)}{q(k)}|x(k)|^{q(k)} \\
&\ge \|x\|_{\sigma,p(\cdot)}^{p^+} - \frac{\lambda\|b\|_\infty}{\underline{q}}\sum_{k=0}^{T+1}\left(|x(k)|^{\overline{q}} + |x(k)|^{\underline{q}}\right) \ge \\
&\quad \|x\|_{\sigma,p(\cdot)}^{p^+} - \frac{\lambda C\|b\|_\infty}{\underline{q}}\left(\|x\|_{\sigma,p(\cdot)}^{\overline{q}} + \|x\|_{\sigma,p(\cdot)}^{\underline{q}}\right) = \rho^{p^+} - \frac{\lambda C\|b\|_\infty}{\underline{q}}\left(\rho^{\underline{q}} + \rho^{\overline{q}}\right),
\end{aligned}$$

with $C > 0$. Setting

$$\lambda_0 := \frac{\underline{q}\rho^{p^+}}{\|b\|_\infty\left(\rho^{\underline{q}} + \rho^{\overline{q}}\right)C} > 0,$$

one has

$$\Phi_X(x) \ge \frac{\|b\|_\infty\left(\rho^{\underline{q}} + \rho^{\overline{q}}\right)C}{\underline{q}}(\lambda_0 - \lambda) =: c_\lambda > 0,$$

for arbitrary $\lambda \in (0, \lambda_0)$ and (3.11) follows. ∎

Theorem 3.5. *Assume (1.1), (1.2), (3.10) and that* $\underline{b} > 0$. *If there are constants* $\theta > p^+$ *and* $\rho > 0$ *such that (3.1) holds and either (3.6) or (3.7) is satisfied, then* *there exists $\lambda_0 > 0$ such that problem $(\mathcal{P}_P)$ has at least two nontrivial solutions for any $\lambda \in (0, \lambda_0)$.*

Proof. The conclusion follows from Proposition 3.1, Lemmas 3.2 - 3.4 with $X = X_P$, and Proposition 2.1. ■

In the same way, but with $X = X_N$ and Proposition 2.2 instead of Proposition 2.1, we obtain the following

Theorem 3.6. *Assume hypotheses (1.2), (3.10) and that $\underline{b} > 0$. If there are constants $\theta > p^+$ and $\rho > 0$ such that (3.1) is satisfied and either (3.6) or (3.7) holds true, then there exists $\lambda_0 > 0$ such that problem $(\mathcal{P}_N)$ has at least two nontrivial solutions for any $\lambda \in (0, \lambda_0)$.*

Remark 3.7. (i) On account of [3, Remark 1(i)] it is easy to see that under the hypotheses of Theorem 3.5 (resp. Theorem 3.6), if, in addition, $f(k, \cdot)$ is odd for all $k \in [1, T]$, then $(\mathcal{P}_P)$ (resp. $(\mathcal{P}_N)$) has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

(ii) It is worth to point out that if q =constant, then Theorems 3.5 and 3.6 remain valid with the weaker hypothesis $\sum_{k=1}^{T} b(k) > 0$ instead of $\underline{b} > 0$.

Example 3.8. If $\theta > p^+$ and $\underline{b} > 0$, then there exists $\lambda_0 > 0$ such that the Neumann problem

$$\begin{cases} \mathcal{A}_k(x) = kh_\theta(x(k)) + \lambda b(k)h_{q(k)}(x(k)), & (\forall)\ k \in [1, T], \\ \Delta x(0) = 0 = \Delta x(T) \end{cases}$$

has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

4. Two Positive Solutions for Problems $(\mathcal{P}'_P)$ and $(\mathcal{P}'_N)$

In this Section we deal with the existence of positive solutions for problems $(\mathcal{P}'_P)$ and $(\mathcal{P}'_N)$, for sufficiently large values of the parameter λ. The main tool in obtaining such a result will be the classical Mountain Pass Theorem [22].

Theorem 4.1. *Assume (1.1) and that there is some $\xi > 0$ such that $f(k, \cdot) > 0$ on $(0, \xi)$ and $f(k, \xi) = 0$, for all $k \in [1, T]$. If*

$$\lim_{t \to 0} \frac{f(k, t)}{|t|^{p(k)-1}} = 0, \quad (\forall)\ k \in [1, T], \tag{4.1}$$

then there exists $\lambda^\star > 0$ such that for all $\lambda > \lambda^\star$, problem $(\mathcal{P}'_P)$ has at least two positive solutions.

Proof. We set $\tilde{f}(k,t) = f(k,t)$ for $0 \le t \le \xi$ and $\tilde{f}(k,t) = 0$ otherwise ($k \in [1,T]$), and let us consider the problem

$$(\tilde{\mathcal{P}}'_P) \qquad \begin{cases} \mathcal{A}_k(x) = \lambda \tilde{f}(k, x(k)), \quad (\forall)\ k \in [1,T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T). \end{cases}$$

Its energy functional is

$$\tilde{\Psi}_{X_P}(x) = \varphi_{X_P}(x) - \lambda \sum_{k=1}^{T} \tilde{F}(k, x(k)), \quad (\forall)\ x \in X_P,$$

where φ_{X_P} is defined in (2.6) and $\tilde{F}$ is the primitive of $\tilde{f}$.

For $x \in X_P$, setting $x_+ := \max\{x, 0\}$ and $x_- := \min\{x, 0\}$, it is straightforward to check that

$$|\Delta x(k)||\Delta x_\pm(k)| = \Delta x(k)\Delta x_\pm(k) \text{ and } |\Delta x(k)| \ge |\Delta x_\pm(k)|,$$

which imply

$$\begin{aligned} &|\Delta x_\pm(k)|^{p(k)} \\ &= |\Delta x_\pm(k)|^{p(k)-1}|\Delta x_\pm(k)| \le |\Delta x(k)|^{p(k)-1}|\Delta x_\pm(k)| \\ &= |\Delta x(k)|^{p(k)-2}|\Delta x(k)||\Delta x_\pm(k)| = |\Delta x(k)|^{p(k)-2}\Delta x(k)\Delta x_\pm(k), \end{aligned} \tag{4.2}$$

for all $k \in [0,T]$.

Now, assume that x is a nontrivial solution of problem $(\tilde{\mathcal{P}}'_P)$. Then, using (1.1), (4.2), the summation by parts formula and the fact that $r > 0$ on $[1,T]$, we obtain

$$\begin{aligned} 0 &= \langle \tilde{\Psi}'_{X_P}(x), (x-\xi)_+ \rangle \\ &= \sum_{k=1}^{T} \mathcal{A}_k(x)(x(k)-\xi)_+ - \lambda \sum_{k=1}^{T} \tilde{f}(k, x(k))(x(k)-\xi)_+ \\ &= \sum_{k=0}^{T} h_{p(k)}(\Delta x(k))\Delta(x(k)-\xi)_+ + \sum_{k=1}^{T} r(k) h_{p(k)}(x(k))(x(k)-\xi)_+ \\ &= \sum_{k=0}^{T} |\Delta(x(k)-\xi)|^{p(k)-2}\Delta(x(k)-\xi)\Delta(x(k)-\xi)_+ \\ &\quad + \sum_{k=1}^{T} r(k) h_{p(k)}(x(k))(x(k)-\xi)_+ \\ &\ge \sum_{k=0}^{T} |\Delta(x(k)-\xi)_+|^{p(k)} + \sum_{k=1}^{T} r(k)|x(k)|^{p(k)-2}[(x(k)-\xi)_+]^2, \end{aligned}$$

which implies that $x \leq \xi$. In a similar way, with x_- instead of $(x-\xi)_+$, we obtain that $x \geq 0$. Moreover, if there exists $k_0 \in [0, T+1]$ such that $x(k_0)=0$, then using that x solves problem $(\tilde{\mathcal{P}}'_P)$, by a simple computation we get that $x(k)=0$, for all $k \in [0, T+1]$, a contradiction because x is assumed to be nontrivial. Hence, $0 < x \leq \xi$. By virtue of these remarks and Proposition 2.1, to find positive solutions of problem $(\mathcal{P}'_P)$, it suffices to produce nontrivial critical points of the functional $\tilde{\Psi}_{X_P}$.

Next, we show that $\tilde{\Psi}_{X_P}$ satisfies the (PS) condition. With this aim, let $\{x_n\} \subset X_P$ be a sequence with $\tilde{\Psi}_{X_P}(x_n) \leq M$, for all $n \in \mathbb{N}$. Since X_P is finite dimensional, it suffices to prove that $\{x_n\}$ is bounded. The boundedness of $\tilde{f}$ implies that there exists $C_1 > 0$ such that

$$|\tilde{F}(k,t)| \leq C_1|t|, \quad (\forall)\ k \in [1,T],\ (\forall)\ t \in \mathbb{R}$$

and using the equivalence of the norms on X_P and (2.2), if $\|x_n\|_{\underline{r},p(\cdot)} > 1$, one has

$$\begin{aligned} M &\geq \tilde{\Psi}_{X_P}(x_n) \geq \sum_{k=1}^{T+1} \frac{|\Delta x_n(k-1)|^{p(k-1)}}{p(k-1)} \\ &\quad + \underline{r} \sum_{k=1}^{T} \frac{|x_n(k)|^{p(k)}}{p(k)} - \lambda C_1 \sum_{k=1}^{T} |x_n(k)| \\ &\geq \|x_n\|_{\underline{r},p(\cdot)}^{p^-} - \lambda C_1 T \|x_n\|_\infty \geq \|x_n\|_{\underline{r},p(\cdot)}^{p^-} - \lambda C_1 C_2 T \|x_n\|_{\underline{r},p(\cdot)}, \end{aligned} \tag{4.3}$$

with C_2 a positive constant. Thus, we get that $\{x_n\}$ is bounded in X_P.

Also, note that (see (4.3))

$$\tilde{\Psi}_{X_P}(x) \geq \|x\|_{\underline{r},p(\cdot)}^{p^-} - \lambda C_1 C_2 T \|x\|_{\underline{r},p(\cdot)}, \quad (\forall)\ x \in X_P,\ \|x\|_{\underline{r},p(\cdot)} > 1,$$

which implies that $\tilde{\Psi}_{X_P}$ is bounded from below. Hence, by [22, Theorem 2.7],

$$c_\lambda = \inf_{x \in X_P} \tilde{\Psi}_{X_P}(x)$$

is a critical value of $\tilde{\Psi}_{X_P}$, for all $\lambda > 0$.

Let $y \in X_P \setminus \{0\}$ such that $y(k) \in (0,\xi)$, for all $k \in [1,T]$. As

$$\sum_{k=1}^{T} \tilde{F}(k, y(k)) > 0,$$

we can find $\lambda^\star > 0$ sufficiently large, such that $\tilde{\Psi}_{X_P}(y) < 0$, for all $\lambda > \lambda^\star$. For such λ, the functional $\tilde{\Psi}_{X_P}$ has a critical value $c_\lambda < 0$ and a corresponding nontrivial critical point x_1, which is a positive solution of problem $(\mathcal{P}'_P)$.

Next, we shall prove that $\tilde{\Psi}_{X_P}$ has a "mountain pass" geometry, namely:

(a) *there exist* $\alpha, \rho > 0$ *such that* $\tilde{\Psi}_{X_P}|_{\partial B_\rho} \geq \alpha$;

(b) $\tilde{\Psi}_{X_P}(e) \le 0$ *for some* $e \in X_P \setminus \overline{B}_\rho$,

where B_ρ denote the open ball in X_P of radius ρ centered at 0 and $\overline{B}_\rho$ its closure.

As in the proof of Lemma 3.4, by the equivalence of the norms on X_P, for any $\eta > 0$, there is some $C_\eta > 0$ such that (3.12) holds true. From (4.1), we can find constants $\sigma \in (0, \underline{r})$, $\rho \in (0, \min\{1, \|x_1\|_{\sigma,p(\cdot)}/2\})$ and $C_\sigma > 0$ such that

$$\tilde{F}(k,t) \le \frac{\underline{r} - \sigma}{\lambda p(k)}|t|^{p(k)}, \quad (\forall)\ k \in [1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \le \rho C_\sigma. \tag{4.4}$$

For $x \in X_P$ with $\|x\|_{\sigma,p(\cdot)} \le \rho$, from (3.12) and (4.4), one has

$$\tilde{F}(k,x(k)) \le \frac{\underline{r} - \sigma}{\lambda p(k)}|x(k)|^{p(k)}, \quad (\forall)\ k \in [1,T],$$

which, together with (2.1), imply

$$\begin{aligned}\tilde{\Psi}_{X_P}(x) &= \varphi_{X_P}(x) - \lambda \sum_{k=1}^{T} \tilde{F}(k,x(k)) \ge \varphi_{X_P}(x) + (\sigma - \underline{r}) \sum_{k=1}^{T} \frac{1}{p(k)}|x(k)|^{p(k)} \\ &\ge \sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta x(k-1)|^{p(k-1)} + \sigma \sum_{k=1}^{T} \frac{1}{p(k)}|x(k)|^{p(k)} \ge \|x\|_{\sigma,p(\cdot)}^{p^+}\end{aligned}$$

and condition *(a)* is fulfilled with $\alpha = \rho^{p^+}$. Since $\|x_1\|_{\sigma,p(\cdot)} > \rho$ and $\tilde{\Psi}_{X_P}(x_1) < 0$, condition *(b)* is also satisfied. Consequently, the Mountain Pass Theorem yields a second nontrivial critical point x_2 of $\tilde{\Psi}_{X_P}$ such that $\tilde{\Psi}_{X_P}(x_2) > 0 > \tilde{\Psi}_{X_P}(x_1)$. Clearly x_2 is distinct from x_1. Hence, x_2 is a second positive solution of problem $(\mathcal{P}'_P)$ and the proof is complete. ■

Remark 4.2. It is worth to point out that Theorem 3.1 proved in [6] for p =constant is an immediate consequence of Theorem 4.1.

Example 4.3. If (1.1) holds, then there exists $\lambda^\star > 0$ such that, for every $\lambda > \lambda^\star$, the problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda k \left(x^\theta(k) - x^{\theta+1}(k)\right), & (\forall)\ k \in [1,T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases}$$

has at least two positive solutions for any $\theta > p^+ - 1$.

For the Neumann problem $(\mathcal{P}'_N)$, as in Theorem 4.1 by no longer than "mutatis mutandis" arguments, we have the following

Theorem 4.4. *If there is some* $\xi > 0$ *such that* $f(k,\cdot) > 0$ *on* $(0,\xi)$ *and* $f(k,\xi) = 0$*, for all* $k \in [1,T]$ *and (4.2) holds true, then there exists* $\lambda^\star > 0$ *such that for all* $\lambda > \lambda^\star$*, problem* $(\mathcal{P}'_N)$ *has at least two positive solutions in* X_N.

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