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A NOTE ON THE GENERALIZED HIGHER-ORDER q-BERNOULLI NUMBERS AND POLYNOMIALS WITH WEIGHT α

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Abstract. In this paper we give some interesting equation of p-adic q-integrals on $\mathbb{Z}_p$. From those p-adic q-integrals, we present a systemic study of some families of extended Carlitz q-Bernoulli numbers and polynomials with weight α in p-adic number field.

1. Introduction

For q-Bernoulli numbers and polynomials, several results have been studied by Carlitz (see [1, 2]), T. Kim (see [4-11]), Y. Simsek (see [12-16]), and H. Ozden (see [12]). Bernoulli numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Recently, many mathematicians have studied in the area of Bernoulli numbers and polynomials. T.Kim (see [5]) introduced the weight q-Bernoulli numbers and polynomials with motivation for weight α, properties and identities. In this paper, we research some properties of a new type of q-Bernoulli numbers and polynomials with weight α and some relations of higher order q-Bernoulli polynomials with weight α to attach χ. Also in this paper, if we take $\alpha = 1$, then [4] is the special case of this paper.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume

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that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1,3,5,13,14]) .}$$

The q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$

and the Gaussian q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}. \tag{1}$$

Note that

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

From (1) we easily see that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n-k}\binom{n}{k-1} + \binom{n}{k}_q.$$

For a fixed positive interger f , $(f,p)=1$, let

$$X = X_f = \lim_{\overleftarrow{N}}(Z/fp^N\mathbb{Z}), \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<fp^N \\ (a,p)=1}} (a + fp^N\mathbb{Z}_p)$$

and

$$a + fp^N\mathbb{Z}_| = \{x \in X | x \equiv a (mod \quad fp^N),$$

where $a \in \mathbb{Z}$ and $0 \leq a < fp^N$ see([1,2,5,6,10,12]). We say that f is an uniformly differential function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y}$$

have a $\lim_{(x,y)\to(a,a)} F_f(x,y) = f'(a)$. For $f \in UD(\mathbb{Z}_p)$ let us begin with the expression

$$\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x = \sum_{0\leq x<p^N} f(x)\mu_q(x+p^N\mathbb{Z}_p)$$

representing a q-analogue of the Riemann sum for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit ($N \to \infty$) of the sums (if exists). The p-adic q-integral (or q-Volkenborn integrals of $f \in UD(\mathbb{Z}_p)$ is defined by

(2) $$I_q(f) = \int_X f(x)d\mu_q(x) = \int_{Z_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{0\le x<p^N} f(x)q^x.$$

Carlitz's q-Bernoulli numbers $\beta_{k,q}$ can be defined recursively by $\beta_{0,q} = 1$ and by the value that

$$q(q\beta_q + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $\beta_q^i = \beta_{i,q}$

It is well known that

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \int_X [x]_q^n d\mu_q(x), n \in Z_+$$

and

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y) = \int_X [y + x]_q^n d\mu_q(y), n \in Z_+$$

where $\beta_{n,q}(x)$ are called the n-th Carlitz's q-Bernoulli polynomials. Let χ be the Difichlet's character with conductor $f \in \mathbb{N}$. Then the generalized Carlitz's q-Bernoulli numbers with weight α attached to χ are defined as follows:

$$\beta_{n,\chi,q} = \int_X \chi(x)[x]_{q^\alpha}^n d\mu_q(x).$$

In this paper, we present a systemic study of some families of multiple Carlitz's q-Bernoulli numbers and polynomials with weight α by using the integral equations of p-adic q-integrals on $\mathbb{Z}_p$. From the p-adic q-integrals on $\mathbb{Z}_p$ we derive some interesting formula for the higher-order Calitz's q-Bernoulli numbers and polynomials with weight α in the p-adic number field.

2. On the Generalized Higher-order q-Bernoulli Numbers and Polynomials with Weight α

Our primary goal of this section is to define q-Bernoulli numbers $\beta_{n,q}^{(\alpha)}$ and polynomials $\beta_{n,q}^{(\alpha)}(x)$ with weight α. We also find generating functions of q-Bernoulli numbers $\beta_{n,q}^{(\alpha)}$ and polynomials $\beta_{n,q}^{(\alpha)}(x)$ with weight α.

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, Bernoulli polynomials with weight α, $\beta_{n,q}^{(\alpha)}(x)$ are defined by

(3) $$\beta_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} q^{-y}[x + y]_{q^\alpha}^n d\mu_q(y).$$

From (3), we see that

$$\beta_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} q^{-y}[x+y]_{q^\alpha}^n d\mu_q(y)$$
$$= \Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{l=0}^{n} \binom{n}{l}(-q^{\alpha x})^l \frac{\alpha l}{[\alpha l]_q} \tag{4}$$
$$= -\frac{n\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{(x+m)\alpha}[x+m]_{q^\alpha}^{n-1}.$$

Note that for $\alpha = 1$ and $n \in \mathbb{N}$,

$$\lim_{q\to 1}\beta_{n,q}^{(1)}(x) = -n\sum_{m=0}^{\infty}(x+m)^{n-1} = B_n(x), \tag{5}$$

where $B_n(x)$ are called the n-th ordinary Bernoulli polynomials. In particular, $x = 0$, $\beta_{n,q}^{(\alpha)}(0) = \beta_{n,q}^{(\alpha)}$ are called the n-th q-Bernoulli numbers.

By (4) and (5), one has the following lemma.

Lemma 1. *For $n \geq 0$, one has*

$$\beta_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} q^{-y}[x+y]_{q^\alpha}^n d\mu_q(y)$$
$$= -\frac{n\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{(x+m)\alpha}[x+m]_{q^\alpha}^{n-1} \tag{6}$$
$$= \Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{l=0}^{n} \binom{n}{l}(-q^{\alpha x})^l \frac{\alpha l}{[\alpha l]_q}.$$

We consider the q-Bernoulli polynomials with weight α of order $r \in \mathbb{N}$ as below.

$$\sum_{n=0}^{\infty}\beta_{n,q}^{(\alpha,r)}(x)\frac{t^n}{n!}$$
$$= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times} q^{-(x_1+x_2+\cdots+x_r)} e^{[x+x_1+x_2+\cdots+x_r]_{q^\alpha} t} d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{7}$$

By (7), one has as below.

$$\beta_{n,q}^{(\alpha,r)}(x)$$
$$= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times} q^{-(x_1+x_2+\cdots+x_r)}[x+x_1+x_2+\cdots+x_r]_{q^\alpha}^n d\mu_q(x_1)\cdots d\mu_q(x_r) \tag{8}$$
$$= \left(\tfrac{1}{1-q^\alpha}\right)^n \textstyle\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\left(\tfrac{\alpha l}{[\alpha l]_q}\right)^r.$$

In the special case $x = 0$, $\beta_{n,q}^{(\alpha,r)}(0) = \beta_{n,q}^{(\alpha,r)}$ is regarded as the q-extension of Bernoulli numbers with weight α of order r. For $f \in \mathbb{N}$, one has

(9)
$$\begin{aligned}
&\beta_{n,q}^{(\alpha,r)}(x)\\
&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times} q^{-(x_1+x_2+\cdots+x_r)}[x+x_1+x_2+\cdots+x_r]_{q^\alpha}^n d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=[f]_q^{-r}\Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{a_1,a_2,\cdots,a_r=0}^{f-1}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(x+a_1+a_2+\cdots+a_r)\alpha l}\Big(\frac{\alpha l}{[\alpha l]_{q^f}}\Big)^r\\
&=\frac{[f]_q^{n-r}}{[\alpha]_q^n}\sum_{a_1,a_2,\cdots,a_r=0}^{f-1}\beta_{n,q^f}^{(\alpha,r)}\Big(\frac{x+a_1+a_2+\cdots+a_r}{f}\Big).
\end{aligned}$$

By (8) and (9), we have the following theorem.

Theorem 2. *For $r\in\mathbb{Z}_+$, $f\in\mathbb{N}$, one has*

$$\begin{aligned}
\beta_{n,q}^{(\alpha,r)}(x)&=[f]_q^{-r}\Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{a_1,a_2,\cdots,a_r=0}^{f-1}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(x+a_1+a_2+\cdots+a_r)\alpha l}\Big(\frac{\alpha l}{[\alpha l]_{q^f}}\Big)^r\\
&=\frac{[f]_q^{n-r}}{[\alpha]_q^n}\sum_{a_1,a_2,\cdots,a_r=0}^{f-1}\beta_{n,q^f}^{(\alpha,r)}\Big(\frac{x+a_1+a_2+\cdots+a_r}{f}\Big).
\end{aligned}$$

Let χ be the primitive Dirichlet's character with conductor $f\in\mathbb{N}$. Then the generalized q-Bernoulli polynomials with weight α attached to χ are defined by

(10)
$$\sum_{n=0}^{\infty}\beta_{n,\chi,q}^{(\alpha)}(x)\frac{t^n}{n!}=\int_X \chi(y)q^{-y}e^{[x+y]_{q^\alpha}t}d\mu_q(y).$$

From (10),

(11)
$$\begin{aligned}
\beta_{n,\chi,q}^{(\alpha)}(x)&=\int_{\mathbb{X}}\chi(y)q^{-y}[x+y]_{q^\alpha}^n d\mu_q(y)\\
&=\sum_{a=0}^{f-1}\chi(a)\lim_{\mathbb{N}\to\infty}\frac{1}{[fp^N]_q}\sum_{y=0}^{p^N-1}[x+a+fy]_{q^\alpha}^n\\
&=\Big(\frac{1}{1-q^\alpha}\Big)^n\sum_{a=0}^{f-1}\chi(a)\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l(x+a)}\frac{1}{[f]_q}\frac{\alpha l}{[\alpha l]_{q^f}}\\
&=-\frac{\alpha n}{[\alpha]_q}\sum_{a=0}^{f-1}\chi(a)\sum_{m=0}^{\infty}q^{\alpha(x+a+fm)}[x+a+fm]_{q^\alpha}^{n-1}\\
&=-\frac{\alpha n}{[\alpha]_q}\sum_{m=0}^{\infty}\chi(a)q^{\alpha(x+m)}[x+m]_{q^\alpha}^{n-1}.
\end{aligned}$$

we can give the generating function for the generalized q-Bernoulli polynomials with weight α attached to χ as below.

$$F_{\chi,q}(x,t) = -t\sum_{m=0}^{\infty}\chi(m)q^{\alpha(x+m)}e^{[x+m]_{q^\alpha}t} = \sum_{n=0}^{\infty}\beta_{n,\chi,q}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{12}$$

From (2), (10) we note that

$$\begin{aligned}
\beta_{n,\chi,q}^{(\alpha)}(x) &= \int_{\mathbb{X}}\chi(y)q^{-y}[x+y]_{q^\alpha}^n d\mu_q(y) \\
&= \frac{1}{[f]_q}\sum_{a=0}^{f-1}\chi(a)\lim_{N\to\infty}\frac{1}{[p^N]_{q^f}}\sum_{y=0}^{p^N-1}[x+a+fy]_{q^\alpha}^n \\
&= \frac{[f]_{q^\alpha}^n}{[\alpha]_q}\sum_{a=0}^{f-1}\chi(a)\beta_{n,q^f}^{(\alpha)}\Big(\frac{x+a}{f}\Big).
\end{aligned} \tag{13}$$

In particular, $x=0$, $\beta_{n,\chi,q}^{(\alpha)}(0) = \beta_{n,\chi,q}^{(\alpha)}$ are called the n-th generalized q-Bernoulli numbers with weight α attached to χ.

Let us consider the higher-order q-Bernoulli polynomials with weight α attached to χ as follows;

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,\chi,q}^{(\alpha,r)}(x)\frac{t^n}{n!} \\
&= \underbrace{\int_X\cdots\int_X}_{r-times}\prod_{i=1}^{r}\chi(x_i)q^{-(x_1+\cdots+x_r)}e^{[x+x_1+\cdots+x_r]_{q^\alpha}t}d\mu_q(x_1)\cdots d\mu_q(x_r),
\end{aligned} \tag{14}$$

where $\beta_{n,\chi,q}^{(\alpha,r)}(x)$ are called the n-th generalized q- Bernoulli polynomials with weight α of order r attached to χ.

From (14) we note that

$$\begin{aligned}
&\beta_{n,\chi,q}^{(\alpha,r)}(x) \\
&= \underbrace{\int_X\cdots\int_X}_{r-times}\prod_{i=1}^{r}\chi(x_i)q^{-(x_1+\cdots+x_r)}[x+x_1+\cdots+x_r]_{q^\alpha}{}^n d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \Big(\frac{1-q^{\alpha f}}{1-q^\alpha}\Big)^n[f]_q^{-r} \\
&\quad \sum_{a_1,\cdots,a_r=0}^{f-1}\prod_{i=1}^{r}\chi(a_i)\Big(\frac{1}{1-q^{\alpha f}}\Big)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l(x+a_1+\cdots+a_r)}\Big(\frac{\alpha l}{[\alpha l]_{q^\alpha}}\Big)^r
\end{aligned} \tag{15}$$

$$= \Big(\frac{[\alpha f]_q}{[\alpha]_q}\Big)^n [f]_q^{-r} \sum_{a_1,\cdots a_r=0}^{f-1} \prod_{i=1}^{r} \chi(a_i) \beta_{n,q^f}^{(\alpha,r)} \Big(\frac{x+a_1+\cdots+a_r}{f}\Big).$$

In particular, $x=0$, $\beta_{n,\chi,q}^{(\alpha,r)}(0)=\beta_{n,\chi,q}^{(\alpha,r)}$ are called the n-th generalized q-Bernoulli numbers with weight α attached to χ.

By (14), (15) one has the following theorem.

Theorem 3. *Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$.* For $n \in \mathbb{Z}_+$, $r \in \mathbb{N}$, one has

$$\beta_{n,\chi,q}^{(\alpha,r)} = \Big(\frac{[\alpha f]_q}{[\alpha]_q}\Big)^n [f]_q^{-r} \sum_{a_1,\cdots a_r=0}^{f-1} \Pi_{i=1}^{r} \chi(a_i) \beta_{n,q^f}^{(\alpha,r)} \Big(\frac{x+a_1+\cdots+a_r}{f}\Big).$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, we introduce the extended higher-order q-Bernoulli polynomials with weight α as follows;

$$\beta_{n,q}^{(h,\alpha,r)}(x) = \underbrace{\int_X \cdots \int_X}_{r-times} q^{\sum_{j=1}^{r}(h-j-1)x_j} [x+x_1+\cdots+x_r]_{q^\alpha}{}^n d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{16}$$

From (12), we note that

$$\beta_{n,q}^{(h,\alpha,r)}(x) = \Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{h+\alpha l-1}{r}}{\binom{h+\alpha l-1}{r}_q} \frac{r!}{[r]_q!},$$

and

$$\beta_{n,q}^{(h,\alpha,r)}(x) = [f]_{q^\alpha}^n [f]_q^{-r} \sum_{a_1,\cdots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j} \beta_{n,q^f}^{(h,\alpha,r)} \Big(\frac{x+a_1+\cdots+a_r}{f}\Big). \tag{17}$$

In the special case, $x=0$, $\beta_{n,q}^{(h,\alpha,r)}(0)=\beta_{n,q}^{(h,\alpha,r)}$ are called the n-th Bernoulli numbers with weight α of order r.

By (17), we obtain the following theorem.

Theorem 4. *For $h \in \mathbb{Z}$, $r \in \mathbb{N}$, one has*

$$\beta_{n,q}^{(h,\alpha,r)}(x) = \Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{h+\alpha l-1}{r}}{\binom{h+\alpha l-1}{r}_q} \frac{r!}{[r]_q!},$$

and

$$\beta_{n,q}^{(h,\alpha,r)}(x)=[f]_{q^\alpha}^n[f]_q^{-r}\sum_{a_1,\cdots,a_r=0}^{f-1}q^{\sum_{j=1}^r(h-j)a_j}\beta_{n,q^f}^{(h,\alpha,r)}\Big(\frac{x+a_1+\cdots+a_r}{f}\Big). \tag{18}$$

Let χ be the primitive Dirichlet's character with conductor $f\in\mathbb{N}$. Then we consider the generalized (h,q)-Bernoulli polynomials with weight α attached to χ of order r as follows;

$$\begin{aligned}&\beta_{n,\chi,q}^{(h,\alpha,r)}(x)\\ &=\underbrace{\int_X\cdots\int_X}_{r-times}\Big(\prod_{j=1}^r\chi(x_j)\Big)q^{\sum_{j=1}^r(h-j-1)x_j}[x+x_1+\cdots+x_r]_{q^\alpha}{}^n d\mu_q(x_1)\cdots d\mu_q(x_r).\end{aligned} \tag{19}$$

From (19) with some calculation, we note that

$$\beta_{n,\chi,q}^{(h,\alpha,r)}(x)=\frac{[f]_{q^\alpha}^n}{[f]_q^r}\sum_{a_1\cdots a_r=0}^{f-1}\Big(\prod_{j=1}^r(\chi(a_i))\Big)q^{\sum_{j=1}^r(h-j)a_i}\beta_{n,q^f}^{(h,\alpha,r)}\Big(\frac{x+a_1+\cdots+a_r}{f}\Big). \tag{20}$$

In the special case $x=0$, $\beta_{n,\chi,q}^{(h,\alpha,r)}(0)=\beta_{n,\chi,q}^{(h,\alpha,r)}$ are called the n-th generalized (h,q)-Bernoulli numbers with weight α attached to χ of order r.

By (19), (20) and $q^{\alpha(x_1+x_2+\cdots+x_r)}=(q^\alpha-1)[x_1+x_2+\cdots+x_r]_{q^\alpha}+1$, we see that

$$\begin{aligned}&\beta_{n,\chi,q}^{(h,\alpha,r)}\\ &=\int_{Z_p}\cdots\int_{Z_p}\prod_{j=1}^r\chi(x_j)[x_1+\cdots+x_r]_{q^\alpha}^n q^{\alpha(x_1\cdots+x_r)}q^{\sum_{j=0}^r(h-j-1-\alpha)x_i}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &=(q^\alpha-1)\beta_{n+1,\chi,q}^{(h-\alpha,\alpha,r)}+\beta_{n,\chi,q}^{(h-\alpha,\alpha,r)}.\end{aligned} \tag{21}$$

By (16), we get similar property as above for $\beta_{n,q}^{(h,\alpha,r)}$.

$$\beta_{n,q}^{(h,\alpha,r)}=(q^\alpha-1)\beta_{n+1,q}^{(h-\alpha,\alpha,r)}+\beta_{n,q}^{(h-\alpha,\alpha,r)}. \tag{22}$$

From (16) and (22), we derive as below.

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{(\alpha n-2)x_1+(\alpha n-3)x_2+\cdots+(\alpha n-r-1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^r(-j-1)x_j}q^{\alpha n(x_1+x_2+\cdots+x_r)}d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^r(-j-1)x_j}\sum_{l=0}^n\binom{n}{l}(q^\alpha-1)^l[x_1+\cdots+x_r]_{q^\alpha}^l d\mu_q(x_1)\cdots d\mu_q(x_r)\\ &=\sum_{l=0}^n\binom{n}{l}(q^\alpha-1)^l\beta_{l,q}^{(0,\alpha,r)},\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{(\alpha n-2)x_1+(\alpha n-3)x_2+\cdots+(\alpha n-r-1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p} q^{(\alpha n-2)x_1}d\mu_q(x_1)\cdot\int_{\mathbb{Z}_p} q^{(\alpha n-3)x_2}d\mu_q(x_2)\cdots\int_{\mathbb{Z}_p} q^{(\alpha n-r-1)x_r}d\mu_q(x_r)\\
&=\frac{\binom{\alpha n-1}{r}}{\binom{\alpha n-1}{r}_q}\frac{r!}{[r]_q!}.
\end{aligned}\tag{24}$$

Also, by simple calculation we obtain as below.

$$\sum_{j=0}^{n}\binom{n}{j}(q^\alpha-1)^j\int_{\mathbb{Z}_p}[x]_{q^\alpha}^j q^{(h-2)x}d\mu_q(x)=\frac{\alpha n+h-1}{[\alpha n+h-1]_q}.\tag{25}$$

By (22), (23), (24) and (25), we get the following theorem.

Theorem 5. *For $h\in\mathbb{Z}$, $r\in\mathbb{N}$ and $n\in\mathbb{Z}+$, one has*

$$\beta_{n,q}^{(h,\alpha,r)}=(q^\alpha-1)\beta_{n+1,q}^{(h-\alpha,\alpha,r)}+\beta_{n,q}^{(h-\alpha,\alpha,r)}$$

and

$$\beta_{n,\chi,q}^{(h,\alpha,r)}=(q^\alpha-1)\beta_{n+1,\chi,q}^{(h-\alpha,\alpha,r)}+\beta_{n,\chi,q}^{(h-\alpha,\alpha,r)}.$$

furthermore, we get

$$\sum_{j=0}^{n}\binom{n}{j}(q^\alpha-1)^j\int_{\mathbb{Z}_p}[x]_{q^\alpha}^j q^{(h-2)x}d\mu_q(x)=\frac{\alpha n+h-1}{[\alpha n+h-1]_q},$$

$$\sum_{l=0}^{n}\binom{n}{l}(q^\alpha-1)^l\beta_{l,q}^{(0,\alpha,r)}=\frac{\binom{\alpha n-1}{r}}{\binom{\alpha n-1}{r}_q}\frac{r!}{[r]_q!}.$$

From now on, we study $\beta_{n,q}^{(0,\alpha,r)}(x)$ in the special case $h=0$.

$$\begin{aligned}
&\beta_{n,q}^{(0,\alpha,r)}(x)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_{q^\alpha}^n q^{(-2x_1-3x_2-\cdots-(r-1)x_r)}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\Big(\frac{1}{1-q^\alpha}\Big)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{\binom{\alpha l-1}{r}r!}{\binom{\alpha l-1}{r}[r]_q!}.
\end{aligned}\tag{26}$$

Hence, from (26) we have the following theorem.

Theorem 6. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$(1-q^{\alpha})^n \beta_{n,q}^{(0,\alpha,r)}(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l-1}{r}}{\binom{\alpha l-1}{r}_q} \frac{r!}{[r]_q!}.$$

By (24) and using multivariate p-adic q-integral on $\mathbb{Z}_p$, we have as below.

$$\begin{aligned}
& q^{\alpha n x} \frac{\binom{\alpha n-1}{r}}{\binom{\alpha n-1}{r}_q} \frac{r!}{[r]_q!} \\
&= q^{\alpha n x} \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r-times} q^{(\alpha n-2)x_1+(\alpha n-3)x_2+\cdots+(\alpha n-r-1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\alpha n(x+x_1+\cdots+x_r)} q^{-2x_1-3x_2-\cdots-(r+1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+(q^{\alpha-1})[x+x_1+\cdots+x_r]_{q^{\alpha}})^n q^{-2x_1-3x_2-\cdots-(r+1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} (q^{\alpha}-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x+x_1+\cdots+x_r]_{q^{\alpha}}^l q^{-2x_1-\cdots-(r+1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} (q^{\alpha}-1)^l \beta_{l,q}^{(0,\alpha,r)}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 7. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$q^{\alpha n x} \frac{\binom{\alpha n-1}{r}}{\binom{\alpha n-1}{r}_q} \frac{r!}{[r]_q!} = \sum_{l=0}^{n} \binom{n}{l} (q^{\alpha}-1)^l \beta_{l,q}^{(0,\alpha,r)}.$$

We consider the other expression for $\beta_{n,q}^{(0,\alpha,r)}(x)$ as below.

$$\begin{aligned}
\beta_{n,q}^{(0,\alpha,r)}(x) &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r-times} [x+x_1+\cdots+x_r]_{q^{\alpha}}^n q^{-2x_1-\cdots-(r+1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \frac{[f]_{q^{\alpha}}^n}{[f]_q^r} \sum_{a_1,\cdots,a_r=0}^{f-1} q^{\sum_{j=1}^{r} j a_j} \beta_{n,q^f}^{(0,\alpha,r)}\Big(\frac{x+a_1+\cdots+a_r}{f}\Big).
\end{aligned} \tag{27}$$

From the multivariate p-adic q-integral on $\mathbb{Z}_p$, one has

(28)
$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times}[x+x_1+\cdots+x_r]_{q^\alpha}^n q^{-2x_1-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}([x]_{q^\alpha}+q^{\alpha x}[x_1+x_2+\cdots+x_r]_{q^\alpha})^n\\
&\qquad q^{-2x_1-3x_2-\cdots-(r+1)x^r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\alpha l x}[x_1+\cdots x_r]_{q^\alpha}^l\\
&\qquad q^{-2x_1-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r),
\end{aligned}$$

and

(29)
$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{Z_p}}_{r-times}[x+y+x_1+\cdots+x_r]_{q^\alpha}^n q^{-2x_1-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{l=0}^{n}\binom{n}{l}[y]_{q^\alpha}^{n-l}q^{\alpha l y}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots x_r]_{q^\alpha}^l\\
&\qquad q^{-2x_1-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r).
\end{aligned}$$

From (28) and (29), we obtain the following corollary.

Corollary 8. *For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, one has*

$$\beta_{n,q}^{(0,\alpha,r)}(x)=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}q^{\alpha l x}\beta_{l,q}^{(0,\alpha,r)},$$

and

$$\beta_{n,q}^{(0,\alpha,r)}(x+y)=\sum_{l=0}^{n}\binom{n}{l}[y]_{q^\alpha}^{n-l}q^{\alpha l y}\beta_{l,q}^{(0,\alpha,r)}(x).$$

Now, we also consider the polynomial of $\beta_{n,q}^{(h,\alpha,1)}$. From the integral equation on $\mathbb{Z}_p$, we see that

(30)
$$\begin{aligned}
&\beta_{n,q}^{(h,\alpha,1)}(x)\\
&=\int_{Z_p}[x+x_1]_{q^\alpha}^n q^{x_1(h-2)}d\mu_q(x_1)\\
&=\Big(\frac{1}{1-q^\alpha}\Big)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{\alpha l+h-1}{[\alpha l+h-1]_q}
\end{aligned}$$

From (30), it is easy to show that

$$\begin{aligned}\beta_{n,q}^{(h,\alpha,1)}(x) &= \Big(\frac{1}{1-q^{\alpha}}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l + h - 1}{[\alpha + h - 1]_q} \\ &= \Big(\frac{1}{1-q^{\alpha}}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} (\alpha l) \frac{1-q}{1-q^{\alpha l + h - 1}} \\ &\quad + \Big(\frac{1}{1-q^{\alpha}}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} (h - l) \frac{1-q}{1-q^{\alpha l + h - 1}} \\ &= -n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{(h+\alpha-1)m+\alpha x} [x+m]_{q^{\alpha}}^{n-1} \\ &\quad + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} [x+m]_{q^{\alpha}}^{n}.\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 9. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$\beta_{n,q}^{(h,\alpha,1)}(x) = -n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{(h+\alpha-1)m+\alpha x} [x+m]_{q^{\alpha}}^{n-1} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} [x+m]_{q^{\alpha}}^{n}.$$

From the definition of p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_{q^{\alpha}}^n d\mu_q(x_1) \\ &= \lim_{N\to\infty} \frac{1}{[fp^N]_q} \sum_{x_1=0}^{fp^N-1} q^{(h-2)x_1} [x+x_1]_{q^{\alpha}}^n q^{x_1} \\ &= \frac{[f]_{q^{\alpha}}^n}{[f]_q} \sum_{a_1=0}^{f-1} q^{(h-1)a_1} \int_{\mathbb{Z}_p} q^{f(h-2)x_1} \Big[\frac{x+a_1}{f} + x_1\Big]_{q^{\alpha f}}^n d\mu_{q^f}(x_1) \\ &= \frac{[f]_{q^{\alpha}}^n}{[f]_q} \sum_{a_1=0}^{f-1} q^{(h-1)a_1} \beta_{n,q^f}^{(h,\alpha,1)}\Big(\frac{x+a_1}{f}\Big).\end{aligned}$$

By (30), we easily get

$$\begin{aligned}&\int_{\mathbb{Z}_p} [x+x_1]_{q^{\alpha}}^n q^{x_1(h-2)} d\mu_q(x_1) \\ &= q^{-\alpha x} \int_{\mathbb{Z}_p} [x+x_1]_{q^{\alpha}}^n \big((q^{\alpha}-1)[x+x_1]_{q^{\alpha}} + 1\big) q^{(h-\alpha-2)x_1} d\mu_q(x_1) \qquad (31) \\ &= q^{-\alpha x} \Big((q^{\alpha}-1)\beta_{n,q}^{(h-\alpha,\alpha,1)}(x) + \beta_{n,q}^{(h-\alpha,\alpha,1)(x)}\Big).\end{aligned}$$

From (31), one has

$$\beta_{n,q}^{(h,\alpha,1)}(x) = q^{-\alpha x}\Big((q^{\alpha}-1)\beta_{n,q}^{(h-\alpha,\alpha,1)}(x) + \beta_{n,q}^{(h-\alpha,\alpha,1)(x)}\Big).$$

That is,

$$q^{\alpha x}\beta_{n,q}^{(h,\alpha,1)}(x) = \Big((q^{\alpha}-1)\beta_{n,q}^{(h-\alpha,\alpha,1)}(x) + \beta_{n,q}^{(h-\alpha,\alpha,1)(x)}\Big).$$

By (30) and (31), we easily see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n} q^{x_1(h-2)} d\mu_q(x_1)\\
&= \int_{\mathbb{Z}_p} q^{(h-2)x_1}\big([x]_{q^{\alpha}} + q^{\alpha x}[x]_{q^{\alpha}}\big)^{n} d\mu_q(x_1) \qquad (32)\\
&= \sum_{l=0}^{n}\binom{n}{l}[x]_{q^{\alpha}}^{n-l} q^{\alpha l x}\int_{\mathbb{Z}_p} q^{(h-2)x_1}[x_1]_{q^{\alpha}}^{l} d\mu_q(x_1),
\end{aligned}$$

By Theorem (9) we easily see that

$$\begin{aligned}
&q^{h-1}\int_{\mathbb{Z}_p} q^{(h-2)x_1}[x+x_1+1]_{q^{\alpha}}^{n} d\mu_q(x_1) - \int_{\mathbb{Z}_p} q^{(h-2)x_1}[x+x_1]_{q^{\alpha}}^{n} d\mu_q(x_1)\\
&= n\frac{\alpha}{[\alpha]_q} q^{\alpha}[x]_{q^{\alpha}}^{n-1} + (1-h)(1-q)[x]_{q^{\alpha}}^{n}.
\end{aligned}$$

For $x = 0$, one has as below.

$$\begin{aligned}
&q^{h-1}\int_{\mathbb{Z}_p} q^{(h-2)x_1}[x_1+1]_{q^{\alpha}}^{n} d\mu_q(x_1) - \int_{\mathbb{Z}_p} q^{(h-2)x_1}[x_1]_{q^{\alpha}}^{n} d\mu_q(x_1) \qquad (33)\\
&= \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n = 1,\\ 0 & \text{if } n > 1. \end{cases}
\end{aligned}$$

and

$$\beta_{0,q}^{(h,\alpha,1)} = \int_{\mathbb{Z}_p} q^{(h-2)x_1} d\mu_q(x_1) = \frac{h-1}{[h-1]_q}.$$

From (32) and (33), we can derive the recurrence relation for $\beta_{n,q}^{(h,\alpha,1)}$ as follows;

$$q^{h-1}\beta_{n,q}^{(h,\alpha,1)}(1) - \beta_{n,q}^{(h,\alpha,1)} = \delta_{n,1} \qquad (34)$$

where $\delta_{n,1}$ is Kronecker symbol.

By (32), (33) and (34), we obtain the following theorem.

Theorem 10. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$\beta_{n,q}^{(h,\alpha,1)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \beta_{n,q}^{(h,\alpha,1)}$$

and

$$\begin{aligned} &q^{h-1}\beta_{n,q}^{(h,\alpha,1)}(x+1) - \beta_{n,q}^{(h,\alpha,1)}(x) \\ &= n\frac{\alpha}{[\alpha]_q} q^{\alpha x}[x]_{q^\alpha}^{n-1} + (1-h)(1-q)[x]_{q^\alpha}^{n}. \end{aligned}$$

Furthermore, by (34) and (31)

$$q^{h-\alpha-1}(q^\alpha - 1)\beta_{n,q}^{h-\alpha,\alpha,1}(1) + q^{h-\alpha-1}\beta_{n,q}^{(h-\alpha,\alpha,1)}(1) - \beta_{n,q}^{(h,\alpha,1)} = \delta_{n,1}$$

where $\delta_{n,1}$ is Kronecker symbol.

From the definition of p-adic q-integral on Z_p, we note that

$$\begin{aligned} &\int_{\mathbb{Z}_p} q^{-(h-2)x_1}[1-x+x_1]_{q^{-\alpha}}^{n} d\mu_{q^{-1}}(x_1) \\ &= (-1)^n q^{\alpha n+h-2}\Big(\frac{1}{1-q^\alpha}\Big)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{\alpha+h-1}{[\alpha l+h-l]_q} \\ &= (-1)^n q^{\alpha n+h-2}\beta_{n,q}^{(h,\alpha,1)}(x). \end{aligned} \tag{35}$$

Therefore, from (35) one has as below.

$$\beta_{n,q^{-1}}^{(h,\alpha,1)}(1-x) = (-1)^n q^{\alpha n+h-2}\beta_{n,q}^{(h,\alpha,1)}(x).$$

In particular, in case $\alpha = 1$, note that

$$B_n(1-x) = \lim_{q\to1}\beta_{n,q^{-1}}^{(h,\alpha,1)}(1-x) = \lim_{q\to1}(-1)^n q^{\alpha n+h-2}\beta_{n,q}^{(h,\alpha,1)}(x) = (-1)^n B_n(x)$$

where $B_n(x)$ are the n-th ordinary Bernoulli polynomials. In the special case, $x = 1$, for $n > 1$ we get the following;

$$\beta_{n,q^{-1}}^{(h,\alpha,1)} = (-1)^n q^{\alpha n+h-2}\beta_{n,q}^{(h,\alpha,1)}(1) = (-1)^n q^{\alpha n-1}\beta_{n,q}^{(h,\alpha,1)}.$$

For $f \in \mathbb{N}$, it is easy to show that

$$\begin{aligned} \beta_{n,q}^{(h,\alpha,1)}(fx) &= \int_{Z_p}[fx+x_1]_{q^\alpha}^{n} q^{(h-2)x_1} d\mu_q(x_1) \\ &= \frac{[f]_{q^\alpha}^{n}}{[\alpha]_q}\sum_{l=0}^{f-1} q^{l(h-1)}\beta_{n,q^f}^{h,\alpha,1}\Big(x+\frac{l}{f}\Big) \end{aligned}$$

Now, we consider Barnes' type multiple q-Bernoulli polynomials. For $w_1, w_2, \cdots, w^r \in \mathbb{Z}_p$ and $\delta_1, \delta_2, \cdots, \delta_r \in \mathbb{Z}_p$, we define Barnes' type multiple q-Bernoulli polynomials as follows;

$$\begin{aligned} &\beta_{n,q}^{\alpha,r}(x|w_1, w_2, \cdots, w_r : \delta_1, \delta_2, \cdots, \delta_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{Z_p} [w_1x_1 + \cdots + w_rx_r + x]_{q^\alpha}^n q^{\sum_{j=1}^r (\delta_j - 1)x_j} d\mu_q(x_1) \cdots d\mu_q(x_r). \end{aligned} \tag{36}$$

From (36), we easily derive the following equation.

$$\begin{aligned} &\beta_{n,q}^{\alpha,r}(x|w_1, w_2, \cdots, w_r : \delta_1, \delta_2, \cdots, \delta_r) \\ &= \Big(\frac{1}{1-q^\alpha}\Big) \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\prod_{j=1}^r (\alpha l w_j + \delta_j)}{\prod_{j=1}^r [\alpha l w_j + \delta_j]_q}. \end{aligned}$$

Let $\delta_r = \delta_1 + r - 1$. Then, one has as below.

$$\begin{aligned} &\beta_{n,q}^{\alpha,r}(x|w_1, w_1, \cdots, w_1 : \delta_1, \delta_1 + 1, \cdots, \delta_1 + r - 1) \\ &= \Big(\frac{1}{1-q^\alpha}\Big) \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l w_1 + \delta_1 + r - 1}{r}}{\binom{\alpha l w_1 + \delta_1 + r - 1}{r}_q} \frac{r!}{[r]_q!}. \end{aligned} \tag{37}$$

Hence, from (37) we get the following theorem.

Theorem 11. *For $w_1 \in \mathbb{Z}_p, r \in \mathbb{N}$ and $\delta_1 \in \mathbb{Z}$, one has*

$$\begin{aligned} &\beta_{n,q}^{\alpha,r}(x|w_1, w_1, \cdots, w_1 : \delta_1, \delta_1 + 1, \cdots, \delta_1 + r - 1) \\ &= \Big(\frac{1}{1-q^\alpha}\Big) \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l w_1 + \delta_1 + r - 1}{r}}{\binom{\alpha l w_1 + \delta_1 + r - 1}{r}_q} \frac{r!}{[r]_q!}. \end{aligned} \tag{37}$$

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