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PROJECTIVE COVARIANT REPRESENTATIONS OF LOCALLY C^*-DYNAMICAL SYSTEMS

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Abstract. As a generalization of covariant completely positive maps, we consider (projective) covariant α-completely positive maps between locally C^*-algebras. We first study (projective) covariant J-representations of locally C^*-algebras on Krein modules over locally C^*-algebras. Secondly, we construct a covariant KS-GNS type representation associated with a covariant α-completely positive map on a locally C^*-algebra, and then study extensions to a locally C^*-crossed product of α-completely positive maps on a locally C^*-algebra. The results provide (projective) covariant representations of a locally C^*-crossed product on a Krein module over a locally C^*-algebra.

1. Introduction

One of most elegant approaches to quantum field theory is the algebraic approach, which works for massive fields as well as massless or gauge fields. In massless or gauge fields, the state space may be a space with indefinite metric. Motivated by this physical fact, many people extended the GNS construction to Krein spaces. In particular, Heo-Hong-Ji [5] introduced a notion of an α-completely positive (α-CP) map as a natural generalization of the notion of a completely positive map, and studied the KSGNS (Kasparov-Stinespring-Gelfand-Naimark-Segal) representation on a Krein C^*-module associated with an α-completely positive map. For more studies of α-completely positive maps with motivations, we refer to [6].

A *locally C^*-algebra* (or *pro-C^*-algebra*) is a complete Hausdorff (complex) topological $*$-algebra of which the topology is determined by the collection $\mathcal{S}(\mathcal{A})$ of all continuous C^*-seminorms on it. The notion of locally C^*-algebras was first systematically

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studied by Inoue [9] as a generalization of C^*-algebras, and then Phillips [14, 15] studied locally C^*-algebras with applications to representable K-theory of σ-C^*-algebras. A locally C^*-algebra is topologically $*$-isomorphic to an inverse limit of C^*-algebras. It is known that locally C^*-algebras are useful in the study of non-commutative algebraic topology, pseudodifferential operators and quantum field theory [2, 3, 14].

On the other hand, the crossed products of C^*-algebras have been studied extensively with rich applications to operator algebras, noncommutative geometry and mathematical physics, see [16, 13, 10, 11] and the references cited therein. The covariant algebra in statistical mechanics and group measure space construction by Murray and von Neumann lead crossed products. The canonical representation $\pi \rtimes u$, constructed by a covariant representation (π, u) of a C^*-dynamical system $(\mathcal{A}, G, \theta)$, of the crossed product $\mathcal{A} \rtimes G$ has been studied by many authors [10, 11, 13, 16] (and the references cited therein).

Main purpose of this paper is to study a covariant KSGNS type J-representation on a Krein locally C^*-module associated with a covariant α-CP map of a locally C^*-dynamical system. For our purpose, we first study a (projective) covariant representation of a locally C^*-crossed product and a covariant KSGNS type representation of a locally C^*-algebra associated with a covariant α-CP map on the algebra. Then, by studying the canonical extension to a locally C^*-crossed product of an α-CP map on a locally C^*-algebra, we construct a covariant representation of a crossed product of a locally C^*-dynamical system on a Krein module over a locally C^*-algebra.

This paper is organized as follows. In Section 2, we recall some basic notions of Hilbert modules over locally C^*-algebras which are necessary for our study. In Section 3, we study a covariant representation of a locally C^*-algebra. We consider an α-completely positive map on a locally C^*-algebra and construct a KSGNS type representation of a locally C^*-algebra $\mathcal{A}$ associated with an α-completely positive map ρ (see [7]). The construction leads to a J_ρ-representation of the locally C^*-algebra $\mathcal{A}$ on a Krein $\mathcal{A}$-module. Then we construct a (projective) covariant version of KSGNS type representation of a locally C^*-algebra associated with a (projective) covariant α-completely positive map. Finally, in Section 4, by studying the canonical extension to a crossed product of an α-completely positive map on a locally C^*-algebra, we construct a covariant version of KSGNS type representation of a locally C^*-crossed product on a Krein module.

2. Preliminaries and Notations

Let $\mathcal{A}$ be a locally C^*-algebra of which topology is understood as following. We denote by $\mathcal{S}(\mathcal{A})$ the directed set of all continuous C^*-seminorms on $\mathcal{A}$. For each $p \in \mathcal{S}(\mathcal{A})$, the kernel $\ker(p) = \{a \in \mathcal{A} : p(a) = 0\}$ is a closed $*$-ideal in $\mathcal{A}$, so that $\mathcal{A}_p = \mathcal{A}/\ker(p)$ is a C^*-algebra with the norm induced by p. We denote by $\mathbf{q}_p$ the canonical map from $\mathcal{A}$ onto $\mathcal{A}_p$ and by $a_p = \mathbf{q}_p(a)$ the image of a in $\mathcal{A}_p$. For any

$p, q \in \mathcal{S}(\mathcal{A})$ with $p \geq q$, there is a canonical surjective map $\mathbf{q}_{pq} : \mathcal{A}_p \to \mathcal{A}_q$ such that $\mathbf{q}_{pq}(a_p) = a_q$ for every $a_p \in \mathcal{A}_p$. Then the set $\{\mathcal{A}_p, \mathbf{q}_{pq} : \mathcal{A}_p \to \mathcal{A}_q, p \geq q\}$ becomes an inverse system of C^*-algebras and the inverse limit $\varprojlim_p \mathcal{A}_p$ is isomorphic to the locally C^*-algebra $\mathcal{A}$. Let $M_n(\mathcal{A})$ denote the $*$-algebra of all $n \times n$ matrices over $\mathcal{A}$ with the usual algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of $\mathcal{A}$. Then $M_n(\mathcal{A})$ becomes a locally C^*-algebra and it is isomorphic to

$$\varprojlim_p M_n(\mathcal{A}_p),$$

where p runs through $\mathcal{S}(\mathcal{A})$. The topology on the locally C^*-algebra $M_n(\mathcal{A})$ is determined by the family of C^*-seminorms $\{p_n : p \in \mathcal{S}(\mathcal{A})\}$, where $p_n([a_{ij}]) = \big\|[\mathbf{q}_p(a_{ij})]\big\|_{M_n(\mathcal{A}_p)}$. We refer to [9, 14] for more detailed information about locally C^*-algebras.

Let $\mathcal{A}$ be a locally C^*-algebra, and let $\mathcal{E}$ be a (complex) vector space which is a right $\mathcal{A}$-module, compatibly with the algebra structure. Then $\mathcal{E}$ is called a *pre-Hilbert $\mathcal{A}$-module* if it is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ which is linear in the second variable and satisfies the following properties: for any $\xi, \eta \in \mathcal{E}$ and $a \in \mathcal{A}$,

(i) $\langle \xi, \xi \rangle \geq 0$, and the equality holds only if $\xi = 0$,

(ii) $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,

(iii) $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$.

We say that $\mathcal{E}$ is a *Hilbert $\mathcal{A}$-module* if $\mathcal{E}$ is complete with respect to the seminorms $\|\xi\|_p = p(\langle \xi, \xi \rangle)^{1/2}$ for any $p \in \mathcal{S}(\mathcal{A})$.

Throughout this paper, $\mathcal{A}$ and $\mathcal{E}$ denote a locally C^*-algebra and a Hilbert $\mathcal{A}$-module, respectively, unless specified otherwise.

For any $p \in \mathcal{S}(\mathcal{A})$ and $\mathcal{N}_p = \{\xi \in \mathcal{E} : p(\langle \xi, \xi \rangle) = 0\}$, we write $\mathcal{E}_p$ for the Hilbert $\mathcal{A}_p$-module $\mathcal{E}/\mathcal{N}_p$ with $(\xi + \mathcal{N}_p)\mathbf{q}_p(a) = \xi a + \mathcal{N}_p$ and $\langle \xi + \mathcal{N}_p, \eta + \mathcal{N}_p \rangle = \mathbf{q}_p(\langle \xi, \eta \rangle)$. We denote by $\mathbf{Q}_p$ the canonical map from $\mathcal{E}$ onto $\mathcal{E}_p$ and ξ_p denotes the image $\mathbf{Q}_p(\xi)$. For $p, q \in \mathcal{S}(\mathcal{A})$ with $p \geq q$, there is a canonical surjective map $\mathbf{Q}_{pq} : \mathcal{E}_p \to \mathcal{E}_q$ such that $\mathbf{Q}_{pq}(\xi_p) = \xi_q$ for $\xi_p \in \mathcal{E}_p$. Then $\{\mathcal{E}_p, \mathbf{Q}_{pq} : \mathcal{E}_p \to \mathcal{E}_q, p \geq q\}$ is an inverse system of Hilbert C^*-modules in the sense that

$$\begin{aligned} \mathbf{Q}_{pq}(\xi_p a_p) &= \mathbf{Q}_{pq}(\xi_p)\mathbf{q}_{pq}(a_p) \quad \text{for } \xi_p \in \mathcal{E}_p,\ a_p \in \mathcal{A}_p, \\ \langle \mathbf{Q}_{pq}(\xi_p), \mathbf{Q}_{pq}(\eta_p) \rangle &= \mathbf{q}_{pq}(\langle \xi_p, \eta_p \rangle) \quad \text{for } \xi_p, \eta_p \in \mathcal{E}_p, \\ \mathbf{Q}_{qr} \circ \mathbf{Q}_{pq} &= \mathbf{Q}_{pr} \quad \text{for } p \geq q \geq r. \end{aligned}$$

Then the inverse limit $\varprojlim_{p \in \mathcal{S}(\mathcal{A})} \mathcal{E}_p$ is a Hilbert $\mathcal{A}$-module with

$$(\xi_p)_{p \in \mathcal{S}(\mathcal{A})}(a_p)_{p \in \mathcal{S}(\mathcal{A})} = (\xi_p a_p)_{p \in \mathcal{S}(\mathcal{A})} \text{ and } \langle (\xi_p)_{p \in \mathcal{S}(\mathcal{A})}, (\eta_p)_{p \in \mathcal{S}(\mathcal{A})} \rangle = (\langle \xi_p, \eta_p \rangle)_{p \in \mathcal{S}(\mathcal{A})}$$

and it is isomorphic to the Hilbert $\mathcal{A}$-module $\mathcal{E}$.

Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{E} \to \mathcal{F}$ is said to be *adjointable* if there is a map $T^* : \mathcal{F} \to \mathcal{E}$ such that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for all $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$. Note that an adjointable map T is a continuous $\mathcal{A}$-module map. We denote by $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ the set of all adjointable maps from $\mathcal{E}$ into $\mathcal{F}$ and write $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ for $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$. The *strict topology* on $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is defined by the family of seminorms $\{\| \cdot \|_{p,\xi} : p \in \mathcal{S}(\mathcal{A}), \xi \in \mathcal{E}\}$, where

$$\|T\|_{p,\xi} = \|T\xi\|_p + \|T^*\xi\|_p.$$

For any $p \in \mathcal{S}(\mathcal{A})$ and $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, we have $T(\mathcal{N}_p) \subset \mathcal{N}_p^{\mathcal{F}} = \{\eta \in \mathcal{F} : p(\langle \eta, \eta\rangle) = 0\}$ and define a map $\boldsymbol{\ell}_p : \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p)$ by

$$\boldsymbol{\ell}_p(T)(\mathbf{Q}_p(\xi)) = \mathbf{Q}_p^{\mathcal{F}}(T(\xi)), \quad (T \in \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}), \quad \xi \in \mathcal{E}), \tag{2.1}$$

where $\mathbf{Q}_p^{\mathcal{F}}$ is the canonical map from $\mathcal{F}$ onto $\mathcal{F}_p$. We denote by T_p the operator $\boldsymbol{\ell}_p(T)$. The topology on $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is given by the family of seminorms $\{\tilde{p}\}_{p \in \mathcal{S}(\mathcal{A})}$, where

$$\tilde{p}(T) = \|\boldsymbol{\ell}_p(T)\| = \|T_p\|.$$

Then $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ becomes a locally C^*-algebra. The connecting maps of the inverse system $\{\mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p) : p \in \mathcal{S}(\mathcal{A})\}$ are denoted by $\boldsymbol{\ell}_{pq} : \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p) \to \mathcal{L}_{\mathcal{A}_q}(\mathcal{E}_q, \mathcal{F}_q)$ and the connecting maps are defined as follows:

$$\boldsymbol{\ell}_{pq}(T_p)(\mathbf{Q}_q(\xi)) = \mathbf{Q}_{pq}^{\mathcal{F}}(T_p(\mathbf{Q}_p(\xi))) \quad \text{for } p \geq q,$$

where $\mathbf{Q}_{pq}^{\mathcal{F}} : \mathcal{F}_p \to \mathcal{F}_q$ $(p \geq q)$ are the connecting maps of a family $\{\mathcal{F}_p\}$ of Hilbert C^*-modules. Then the family $\{\mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p), \boldsymbol{\ell}_{pq}, p \geq q\}$ is an inverse system of Banach spaces and the inverse limit $\varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p)$ is isomorphic to $\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. For the study of inverse limits of Hilbert $\mathcal{A}$-modules and Banach spaces, we refer to [14].

An *approximate unit* for a locally C^*-algebra $\mathcal{A}$ is an increasing net $\{e_\lambda\}$ of positive elements of $\mathcal{A}$ such that

$$\|e_\lambda\|_\infty = \sup\{p(e_\lambda) : p \in \mathcal{S}(\mathcal{A})\} \leq 1 \quad \text{for all } \lambda,$$

and $p(e_\lambda a - a) \to 0$, $p(ae_\lambda - a) \to 0$ for any $a \in \mathcal{A}$, $p \in \mathcal{S}(\mathcal{A})$. Let $\mathcal{B}$ be a locally C^*-algebra and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module. A continuous completely positive map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is *strict* if for some approximate unit $\{e_\lambda\}$ of $\mathcal{A}$, $\{\rho(e_\lambda)\}$ is strictly Cauchy in $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$. A *multiplier algebra* $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is the set of all multipliers on $\mathcal{A}$ which is a pair $(\mathbf{l}, \mathbf{r})$, where $\mathbf{l}, \mathbf{r} : \mathcal{A} \to \mathcal{A}$ are morphisms of left, right $\mathcal{A}$-modules, respectively, such that $\mathbf{l}(ab) = \mathbf{l}(a)b$, $\mathbf{r}(ab) = a\mathbf{r}(b)$ and $a\mathbf{l}(b) = \mathbf{r}(a)b$ for all $a, b \in \mathcal{A}$. The *strict topology* on $\mathcal{M}(\mathcal{A})$ is the topology generated by the seminorms $\| \cdot \|_{p,a}$ for $p \in \mathcal{S}(\mathcal{A})$ and $a \in \mathcal{A}$, where

$$\|(\mathbf{l}, \mathbf{r})\|_{p,a} = p(\mathbf{l}(a)) + p(\mathbf{r}(a)).$$

The map $a \in \mathcal{A} \mapsto (\mathbf{l}_a, \mathbf{r}_a) \in \mathcal{M}(\mathcal{A})$ is a homeomorphism of $\mathcal{A}$ onto the closed ideal of $\mathcal{A}$, where $\mathbf{l}_a(b) = ab$ and $\mathbf{r}_a(b) = ba$ and the image of $\mathcal{A}$ under this map is dense in $\mathcal{M}(\mathcal{A})$ in the strict topology. For each $p \in \mathcal{S}(\mathcal{A})$, we define a C^*-seminorm on $\mathcal{M}(\mathcal{A})$ by $\|(\mathbf{l}, \mathbf{r})\|_p = \sup\{p(\mathbf{l}(a)) : p(a) \leq 1\}$. Then $\mathcal{M}(\mathcal{A})$ becomes a locally C^*-algebra with the topology determined by the C^*-seminorms $\{\| \cdot \|_p\}$.

Let $\mathcal{B}$ be a locally C^*-algebra and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module with the inner product $\langle \cdot, \cdot \rangle$. Suppose that a (fundamental) symmetry J on $\mathcal{F}$, i.e., $J = J^* = J^{-1}$, is given to produce a $\mathcal{B}$-valued indefinite inner product

$$\langle \xi, \eta \rangle_J = \langle \xi, J\eta \rangle, \qquad (\xi, \eta \in \mathcal{F}).$$

In this case, the pair $(\mathcal{F}, J)$ is called a *Krein $\mathcal{B}$-module*. A representation $\pi : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is called a *J-representation* on a Krein $\mathcal{B}$-module $(\mathcal{F}, J)$ if π is a homomorphism of $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ such that

$$\pi(a^*) = \pi(a)^J := J\pi(a)^*J \quad \text{for all } a \in \mathcal{A}.$$

Let G be a locally compact group. An *action* of G on $\mathcal{A}$ is a homomorphism θ from G into Aut($\mathcal{A}$), where Aut($\mathcal{A}$) is the group of $*$-automorphisms of $\mathcal{A}$. By [14, Theorem 5.2], these $*$-automorphisms on $\mathcal{A}$ are continuous and have continuous inverses. The action θ is *continuous* if the function $(t, a) \in G \times \mathcal{A} \mapsto \theta_t(a) \in \mathcal{A}$ is jointly continuous. The action θ is called an *inverse limit action* if we can write $\mathcal{A}$ as an inverse limit $\varprojlim \mathcal{A}_p$ of C^*-algebras in such a way that there are actions $\theta^{(p)}$ of G on $\mathcal{A}_p$ such that

$$\theta_t = \varprojlim \theta_t^{(p)} \quad \text{for all } t \in G.$$

A *locally C^*-dynamical system* is the triple $(\mathcal{A}, G, \theta)$, where $\mathcal{A}$ is a locally C^*-algebra, G is a locally compact group and θ is a continuous inverse limit action of G on $\mathcal{A}$.

Let $\mathcal{B}$ be a locally C^*-algebra and let $(\mathcal{F}, J)$ be a Krein $\mathcal{B}$-module. A *unitary representation* u of G on $\mathcal{F}$ is a map from G into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ such that each u_t is a unitary, $u_{st} = u_s u_t$ and the map $t \mapsto u_t(\xi)$ is continuous for every $\xi \in \mathcal{F}$. A linear map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is said to be *(θ, u)-covariant* if $\rho(\theta_t(a)) = u_t \rho(a) u_t^*$ for all $t \in G$ and $a \in \mathcal{A}$. A *(non-degenerate) covariant J-representation* of a locally C^*-dynamical system $(\mathcal{A}, G, \theta)$ on $(\mathcal{F}, J)$ is a pair (π, v) satisfying

$$\pi(\theta_t(a)) = v_t \pi(a) v_t^J \quad \text{for all } a \in \mathcal{A} \text{ and } t \in G,$$

where π is a (non-degenerate) J-representation of $\mathcal{A}$ on $\mathcal{F}$ and v is a J-unitary representation of G on $\mathcal{F}$, i.e. $v_t^J v_t = v_t v_t^J = 1$.

Let G be a locally compact group with a left Haar measure μ. We use the customary notation dt instead of $d\mu(t)$. Let θ be a continuous inverse limit action of G on $\mathcal{A}$ and let $C_c(G, \mathcal{A})$ be the linear space of continuous functions from G into $\mathcal{A}$ with compact supports. We define an involution and an convolution on $C_c(G, \mathcal{A})$ by

$$f^*(s) = \Delta(s)^{-1}\theta_s\big(f(s^{-1})^*\big) \quad \text{and} \quad (f * g)(s) = \int f(t)\theta_t\big(g(t^{-1}s)\big)dt,$$

respectively, where Δ is the modular function on G. Then $C_c(G, \mathcal{A})$ becomes a $*$-algebra with the convolution as product and the involution as $*$-operation. We denote by $L^1(G, \mathcal{A})$ the Hausdorff completion of $C_c(G, \mathcal{A})$ with respect to the topology defined by the family of submultiplicative seminorms $\{m_p : p \in \mathcal{S}(\mathcal{A})\}$, where the seminorms m_p on $C_c(G, \mathcal{A})$ is defined by

$$m_p(f) = \int_G p(f(t))dt.$$

The *crossed product* $\mathcal{A} \rtimes_\theta G$ of a locally C^*-algebra $\mathcal{A}$ by G is the enveloping locally C^*-algebra of $L^1(G, \mathcal{A})$.

Remark 2.1. Since θ is an inverse limit action, $\mathcal{A}$ can be written by the inverse limit $\varprojlim \mathcal{A}_p$ of C^*-algebras $\mathcal{A}_p$ and $\theta_t = \varprojlim \theta_t^{(p)}$ for all $t \in G$, where $\theta^{(p)}$ are actions of G on the C^*-algebras $\mathcal{A}_p$. For each $p \in \mathcal{S}(\mathcal{A})$, the C^*-algebra $(\mathcal{A} \rtimes_\theta G)_p$ can be identified with the C^*-crossed product $\mathcal{A}_p \rtimes_{\theta^{(p)}} G$ associated with the C^*-dynamical system $(\mathcal{A}_p, G, \theta^{(p)})$. Hence, $\mathcal{A} \rtimes_\theta G$ is identified with the inverse limit $\varprojlim \mathcal{A}_p \rtimes_{\theta^{(p)}} G$. See [11] for more information of locally C^*-crossed products.

From now on, let $(\mathcal{A}, G, \theta)$ be a unital locally C^*-dynamical system, $\mathcal{B}$ a locally C^*-algebra and $\mathcal{F}$ a Hilbert $\mathcal{B}$-module. Note that if v is a unitary representation of G on $\mathcal{F}$, then, for each $p \in S(\mathcal{B})$ the map $\boldsymbol{\ell}_p \circ v$ is a representation of G on $\mathcal{F}_p$ where $\boldsymbol{\ell}_p$ is a map from $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$ into $\mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p)$ given by (2.1). Moreover, we have that $v_t = \varprojlim_p v_t^{(p)}$, where $v_t^{(p)} = \boldsymbol{\ell}_p(v_t)$ for all $t \in G$.

The following proposition is known in [12] for the case of locally C^*-algebras $\mathcal{A}$, $\mathcal{B}$ and a covariant representation of $(\mathcal{A}, G, \theta)$ on $\mathcal{F}$. The proof is similar to that of [12, Proposition 3.4], so that we omit it.

Proposition 2.2. *Let $(\mathcal{A}, G, \theta)$ be a locally C^*-dynamical system such that θ is an inverse limit action and let $(\mathcal{F}, J)$ be a Krein module over a locally C^*-algebra $\mathcal{B}$. If (π, v) is a non-degenerate covariant J-representation of $(\mathcal{A}, G, \theta)$ on $(\mathcal{F}, J)$, then there exists a J-representation $\pi \times v$ of $\mathcal{A} \rtimes_\theta G$ on $(\mathcal{F}, J)$ such that*

$$(\pi \times v)(f) = \int_G \pi\big(f(t)\big)v_t dt, \quad (f \in C_c(G, \mathcal{A})).$$

Conversely, if ϕ is a non-degenerate J-representation of $\mathcal{A} \rtimes_\theta G$ on $(\mathcal{F}, J)$, then there is a covariant J-representation (π, v) of $(\mathcal{A}, G, \theta)$ on $(\mathcal{F}, J)$.

3. Covariant KSGNS Representation of Locally C^*-Algebra

Bhatt and Karia [1] gave the Stinespring's construction for locally C^*-algebras and Joiţa [10] generalized the Kasparov-Stinespring-Gelfand-Naimark-Segal (KSGNS) construction in the context of Hilbert modules over locally C^*-algebras.

A Hermitian linear map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is said to be *α-completely positive (α-CP)* [5] if there is a continuous linear Hermitian map $\alpha : \mathcal{A} \to \mathcal{A}$ such that

(ρ1) $\alpha^2 = \mathrm{id}_{\mathcal{A}}$, where $\mathrm{id}_{\mathcal{A}}$ is the identity map on $\mathcal{A}$,

(ρ2) for any approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{A}$, $\{\alpha(e_\lambda)\}_{\lambda\in\Lambda}$ is also an approximate unit,

(ρ3) $\rho(ab) = \rho(\alpha(a)\alpha(b)) = \rho(\alpha(ab))$ for any $a, b \in \mathcal{A}$,

(ρ4) for any $n \geq 1$, $a_1, \ldots, a_n \in \mathcal{A}$ and $\xi_1, \ldots, \xi_n \in \mathcal{F}$,

$$\sum_{i,j=1}^{n} \left\langle \xi_i, \rho(\alpha(a_i)^* a_j)\xi_j \right\rangle \geq 0,$$

(ρ5) for any $a, a_1, \ldots, a_n \in \mathcal{A}$, there exists a constant $M(a) > 0$ such that

$$\left(\rho(\alpha(aa_i)^* aa_j)\right)_{n\times n} \leq M(a) \left(\rho\left(\alpha(a_i)^* a_j\right)\right)_{n\times n},$$

where $(\cdot)_{n\times n}$ denotes a $n \times n$ operator matrix,

(ρ6) there exist a strictly continuous positive linear map $\rho' : \mathcal{M}(\mathcal{A}) \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ and a constant $K > 0$ such that

$$\rho(\alpha(a)^* a) \leq K\rho'(a^* a), \qquad a \in \mathcal{A}.$$

For more detailed study of α-CP map, we refer to [5]–[8].

A linear map β on $\mathcal{A}$ is said to be *continuous in the inverse system* if for any $p \in \mathcal{S}(\mathcal{A})$ there exists a constant $C_p > 0$ such that $p(\beta(a)) \leq C_p\, p(a)$ for all $a \in \mathcal{A}$.

In the following theorem, we review a representation associated with an α-CP map between locally C^*-algebras, which is a generalization of KSGNS type representation associated to an α-CP map on a Krein C^*-module [5].

Theorem 3.1 ([7]). *Let $\mathcal{A}, \mathcal{B}$ be locally C^*-algebras and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module. If $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is a strictly continuous α-CP map, where α is continuous in the inverse system, then there exist a Krein $\mathcal{B}$-module $(\mathcal{F}_\rho, J_\rho)$, a J_ρ-representation $\pi_\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ and an operator $V_\rho \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}_\rho)$ such that*

(*i*) *$\rho(a) = V_\rho^* \pi_\rho(a) V_\rho$ (so, $\rho(a^*) = V_\rho^* \pi_\rho(a)^{J_\rho} V_\rho$) for all $a \in \mathcal{A}$,*

(*ii*) *$\pi_\rho(\mathcal{A}) V_\rho(\mathcal{F})$ is dense in $\mathcal{F}_\rho$,*

(*iii*) *$V_\rho^* \pi_\rho(a)^* \pi_\rho(b) V_\rho = V_\rho^* \pi_\rho(\alpha(a)^* b) V_\rho$ for all $a, b \in \mathcal{A}$.*

The quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ satisfying (i) and (iii) in Theorem 3.1 is called the *Krein quadruple* associated with the α-CP map ρ. If, in addition, (ii) is satisfied, then such a quadruple is said to be *minimal*. Such a minimal quadruple is unique up to unitary equivalence in the sense that for a minimal Krein quadruple $(\mathcal{F}', J_\rho, \pi, W)$ associated with ρ, there exist a unitary operator $U \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho, \mathcal{F}')$ such that

$$W = UV_\rho \text{ and } \pi(a) = U\pi_\rho(a)U^* \quad \text{for all } a \in \mathcal{A}.$$

The following theorem is regarded as the covariant version of Theorem 3.1.

Theorem 3.2. *Let $(\mathcal{A}, G, \theta)$ be a locally C^*-dynamical system and let $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ be a (θ, u)-covariant and strictly continuous α-CP map, where α is continuous in the inverse system. If α and θ are equivariant in the sense that $\alpha \circ \theta_g = \theta_g \circ \alpha$ for all $g \in G$, then there exist a Krein $\mathcal{B}$-module $(\mathcal{F}_\rho, J_\rho)$, a J_ρ-representation π_ρ of $\mathcal{A}$ on $\mathcal{F}_\rho$, an operator $V_\rho \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}_\rho)$ and a J_ρ-unitary representation $v : G \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ such that*

(i) $\rho(a) = V_\rho^* \pi_\rho(a) V_\rho$ *(so, $\rho(a^*) = V_\rho^* \pi_\rho(a)^{J_\rho} V_\rho$) for all $a \in \mathcal{A}$,*

(ii) $\pi_\rho(\mathcal{A})\big[V_\rho(\mathcal{F})\big]$ *is dense in $\mathcal{F}_\rho$,*

(iii) $V_\rho^* \pi_\rho(a)^* \pi_\rho(b) V_\rho = V_\rho^* \pi_\rho\big(\alpha(a)^* b\big) V_\rho$ *for all $a, b \in \mathcal{A}$,*

(iv) $\pi_\rho\big(\theta_g(a)\big) = v_g \pi_\rho(a) v_g^{J_\rho}$ *and $v_g V_\rho = V_\rho u_g$ for all $g \in G$.*

Proof. By Theorem 3.1, there exist a Krein $\mathcal{B}$-module $(\mathcal{F}_\rho, J_\rho)$, a J_ρ-representation $\pi_\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ and an operator $V_\rho \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}, \mathcal{F}_\rho)$ such that (i), (ii) and (iii) hold. It follows from the construction of $\mathcal{F}_\rho$ that

$$\mathcal{F}_\rho = \varprojlim_p \mathcal{F}_{\rho_p},$$

where $\mathcal{F}_{\rho_p}$ is the completion of $\mathcal{A}_{q_p} \otimes \mathcal{F}_p / \ker(\langle \cdot, \cdot \rangle_p)$ (see the proof of Theorem 3.1 in [7]). Hence we may assume that $\mathcal{F}_\rho$ is the completion of $\mathcal{A}_p \otimes \mathcal{F} / \ker(\langle \cdot, \cdot \rangle_p)$. For each $g \in G$, we define a linear map $v_g : \mathcal{F}_\rho \to \mathcal{F}_\rho$ by

$$v_g\big(\mathbf{q}_p(a) \otimes \xi + \ker(\langle \cdot, \cdot \rangle_p)\big) = \mathbf{q}_p\big(\theta_g(a)\big) \otimes u_g(\xi) + \ker(\langle \cdot, \cdot \rangle_p), \quad (a \in \mathcal{A},\ \xi \in \mathcal{F}).$$

We obtain from the equivariance of α and θ that v_t is in $\mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$. We need only to show the property (iv). Let $a, b \in \mathcal{A}$, $\xi \in \mathcal{F}$ and $t \in G$. Then we obtain that

$$\begin{aligned}
&\big(v_g \pi_\rho(a) v_g^{J_\rho}\big)\big(\mathbf{q}_p(b) \otimes \xi + \ker(\langle \cdot, \cdot \rangle_p)\big) \\
&= \big(v_g \pi_\rho(a)\big)\big(\mathbf{q}_p(\theta_{g^{-1}}(b)) \otimes u_{g^{-1}}(\xi) + \ker(\langle \cdot, \cdot \rangle_p)\big) \\
&= v_g\big(\mathbf{q}_p(a\theta_{g^{-1}}(b)) \otimes u_{g^{-1}}(\xi) + \ker(\langle \cdot, \cdot \rangle_p)\big) \\
&= \mathbf{q}_p\big(\theta_g(a) b\big) \otimes \xi + \ker(\langle \cdot, \cdot \rangle_p) \\
&= \pi_\rho\big(\theta_g(a)\big)\big(\mathbf{q}_p(b) \otimes \xi + \ker(\langle \cdot, \cdot \rangle_p)\big),
\end{aligned}$$

which implies that $v_g\pi_\rho(a)v_g^{J_\rho} = \pi_\rho(\theta_g(a))$ for all $g \in G$ and $a \in \mathcal{A}$. It follows immediately from the definitions of v_g and V_ρ that $v_g V_\rho = V_\rho u_g$ for any $g \in G$. ∎

Let $\mathcal{S}$ be a unital semigroup. We denote by ϑ an action of $\mathcal{S}$ on $\mathcal{A}$, which means that $\vartheta(s,\vartheta(t,a)) = \vartheta(st,a)$ and $\vartheta(e,a) = a$, where e is a unit element of $\mathcal{S}$. A *multiplier on* $\mathcal{S}$ is a function $\sigma : \mathcal{S}\times\mathcal{S} \to \mathbb{T}$ satisfying the equations

$$\sigma(r,s)\sigma(rs,t) = \sigma(r,st)\sigma(s,t) \quad \text{and} \quad \sigma(s,e) = \sigma(e,s) = 1$$

for all $r, s, t \in \mathcal{S}$.

Definition 3.3. Let $\mathcal{B}$ be a locally C^*-algebra and let $\mathcal{F}$ be a Hilbert $\mathcal{B}$-module.

(i) A *projective isometric σ-representation* of $\mathcal{S}$ on $\mathcal{F}$ is a map $W : \mathcal{S} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ which has the following properties;
  (a) W_s is an isometry for each $s \in \mathcal{S}$,
  (b) $W_{st} = \sigma(s,t)W_sW_t$ for all $s, t \in \mathcal{S}$.

(ii) Let W be a projective isometric σ-representation of $\mathcal{S}$ on $\mathcal{F}$. A linear map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is *projective (ϑ, W)-covariant* if

$$\rho(\vartheta_s(a))W_s = W_s\rho(a) \quad \text{for all } s \in \mathcal{S} \text{ and } a \in \mathcal{A}.$$

Let $\mathcal{S}$ be a left-cancellative discrete semigroup with unit and let σ be a multiplier on $\mathcal{S}$. We denote by $\widetilde{\mathcal{F}}$ the Hilbert $\mathcal{B}$-module of all square summable $\mathcal{F}$-valued functions defined on $\mathcal{S}$ with the obvious operations and a $\mathcal{B}$-valued inner product. Let $\mathcal{S}$ act on $\mathcal{A}$ by ϑ as an automorphism of $\mathcal{A}$ in the sense that $s \mapsto \vartheta_s$ is a homomorphism of $\mathcal{S}$ into the automorphism group Aut($\mathcal{A}$) of $\mathcal{A}$. Assume that α and ϑ are equivariant. If ρ is an α-CP map of $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\mathcal{F})$, then $\widetilde{\rho} : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\widetilde{\mathcal{F}})$ defined by

$$\big[\widetilde{\rho}(a)\xi\big](s) = \rho(\vartheta_s^{-1}(a))\xi(s), \quad (a \in \mathcal{A}, \xi \in \widetilde{\mathcal{F}}, s \in \mathcal{S}) \tag{3.1}$$

is also α-CP. Indeed, for any $a_1, \ldots, a_n \in \mathcal{A}$ and $\xi_1, \ldots, \xi_n \in \widetilde{\mathcal{F}}$ we have that

$$\begin{aligned}
\sum_{i,j=1}^{n} \big\langle \xi_i, \widetilde{\rho}(\alpha(a_i)^*a_j)\xi_j \big\rangle &= \sum_{i,j=1}^{n}\sum_{s\in\mathcal{S}} \big\langle \xi_i(s), \rho\left(\vartheta_s^{-1}(\alpha(a_i)^*a_j)\right)\xi_j(s)\big\rangle \\
&= \sum_{i,j=1}^{n}\sum_{s\in\mathcal{S}} \big\langle \xi_i(s), \rho\left(\alpha(\vartheta_s^{-1}(a_i)^*)\vartheta_s^{-1}(a_j)\right)\xi_j(s)\big\rangle \\
&= \sum_{s\in\mathcal{S}}\sum_{i,j=1}^{n} \big\langle \xi_i(s), \rho\left(\alpha(\vartheta_s^{-1}(a_i)^*)\vartheta_s^{-1}(a_j)\right)\xi_j(s)\big\rangle \\
&\geq 0,
\end{aligned}$$

where the second equality follows from the equivariance of α and ϑ. It is routine to check other conditions, so that we omit them.

Example 3.4. For each $s \in \mathcal{S}$, we define a map W_s on $\widetilde{\mathcal{F}}$ as follows;

$$\big[W_s\xi\big](r) = \begin{cases} \overline{\sigma(s,t)}\xi(t), & \text{if } r = st \text{ for some } t \in \mathcal{S}, \\ 0, & \text{if } r \notin s\,\mathcal{S}. \end{cases}$$

Then W is a projective σ-isometric representation of $\mathcal{S}$ on $\widetilde{\mathcal{F}}$. The map $\widetilde{\rho}$ defined by (3.1) is projective (ϑ, W)-covariant since

$$\begin{aligned} \big[\widetilde{\rho}(\vartheta_s(a))W_s\xi\big](r) &= \rho\big(\vartheta_r^{-1}(\vartheta_s(a))\big)\big[W_s\xi\big](r) \\ &= \begin{cases} \overline{\sigma(s,t)}\rho\big(\vartheta_{st}^{-1}(\vartheta_s(a))\big)\xi(t), & \text{if } r = st \text{ for some } t \in \mathcal{S} \\ 0, & \text{if } r \notin s\,\mathcal{S} \end{cases} \\ &= \begin{cases} \overline{\sigma(s,t)}\rho\big(\vartheta_t^{-1}(a)\big)\xi(t), & \text{if } r = st \text{ for some } t \in \mathcal{S} \\ 0, & \text{if } r \notin s\,\mathcal{S} \end{cases} \\ &= \big[W_s\widetilde{\rho}(a)\xi\big](r). \end{aligned}$$

If ρ is α-CP, then $\widetilde{\rho}$ is a projective (ϑ, W)-covariant α-CP map of $\mathcal{A}$ into $\mathcal{L}_{\mathcal{B}}(\widetilde{\mathcal{F}})$. ■

The following theorem says that a projective covariant α-CP map induces a projective covariant representation.

Theorem 3.5. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$ be as above and let a left-cancellative semigroup $\mathcal{S}$ with an involution act on $\mathcal{A}$ by ϑ. Suppose that an α-CP map $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ is projective (ϑ, W)-covariant and strictly continuous, where α is continuous in the inverse system. If α and ϑ are equivariant, then there exist a minimal Krein quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ and a projective isometric σ-representation $v : \mathcal{S} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ such that π_ρ is projective (ϑ, v)-covariant.*

Proof. By Theorem 3.1, then there exist a minimal Krein quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$. From the construction of $\mathcal{F}_\rho$, we may assume that $\mathcal{F}_\rho$ is the completion of $\mathcal{A}_p \otimes \mathcal{F}/\mathcal{N}_p$, where $\mathcal{N}_p = \ker(\langle\cdot,\cdot\rangle_p)$. For each $s \in \mathcal{S}$, we define a linear map v_s in $\mathcal{L}_{\mathcal{B}}(\mathcal{F}_\rho)$ by

$$v_s\big(\mathbf{q}_p(a) \otimes \xi + \mathcal{N}_p)\big) = \mathbf{q}_p\big(\vartheta_s(a)\big) \otimes W_s\xi + \mathcal{N}_p, \quad (a \in \mathcal{A},\ \xi \in \mathcal{F}).$$

To show that v is a projective isometric σ-representation, let $a, b \in \mathcal{A}$, $\xi, \eta \in \mathcal{F}$ and $s, t \in \mathcal{S}$. Then we have that

$$\begin{aligned} \big\langle v_s\big(\mathbf{q}_p(a) \otimes \xi + \mathcal{N}_p\big), &v_s\big(\mathbf{q}_p(b) \otimes \eta + \mathcal{N}_p\big)\big\rangle \\ &= \big\langle \mathbf{q}_p\big(\vartheta_s(a)\big) \otimes W_s\xi + \mathcal{N}_p, \mathbf{q}_p\big(\vartheta_s(b)\big) \otimes W_s\eta + \mathcal{N}_p\big\rangle \\ &= \big\langle \mathbf{q}_p\big(\rho\big(\vartheta_s(\alpha(a)^*b)\big)\big)W_s\xi, W_s\eta\big\rangle \\ &= \big\langle \mathbf{q}_p(a) \otimes \xi + \mathcal{N}_p, \mathbf{q}_p(b) \otimes \eta + \mathcal{N}_p\big\rangle, \end{aligned}$$

where the second equality follows from the equivariance of α and ϑ and the third equality follows from the projective (ϑ, W)-covariance of ρ. Moreover, we obtain that

$$\begin{aligned} v_{st}\big(\mathbf{q}_p(a)\otimes\xi+\mathcal{N}_p)\big) &= \mathbf{q}_p\big(\vartheta_{st}(a)\big)\otimes W_{st}\xi+\mathcal{N}_p \\ &= \sigma(s,t)v_s v_t\big(\mathbf{q}_p(a)\otimes\xi+\mathcal{N}_p)\big), \end{aligned}$$

which implies that v is a projective isometric σ-representation. It follows immediately from the definition of v that π is projective (ϑ, v)-covariant and that $v_s V_\rho = V_\rho W_s$ for any $s \in \mathcal{S}$. ■

4. α-Completely Positive Maps on Locally Crossed Product

In this section we discuss an extension of an α-CP map to a locally C^*-crossed product. In general, unlike the case of a CP map, it is not easy to prove the α-complete positivity of a linear map.

Let $\rho : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ be a strictly continuous α-CP map where α is continuous in the inverse system. By Theorem 3.1, there exists a minimal Krein quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ associated with ρ. If, in addition, ρ is (θ, u)-covariant, by Theorem 3.2 there is a covariant J_ρ-representation (π_ρ, v) of $(\mathcal{A}, G, \theta)$ on $(\mathcal{F}_\rho, J_\rho)$. It follows from Proposition 2.2 that the map $(\pi_\rho \times v)$ defined by

$$(\pi_\rho \times v)(f) = \int_G \pi_\rho\big(f(t)\big)v_t dt$$

is a J_ρ-representation of $\mathcal{A} \rtimes_\theta G$ on $(\mathcal{F}_\rho, J_\rho)$. That is, for any $f, g \in L^1(G, \mathcal{A})$,

$$\begin{aligned} (\pi_\rho \times v)(f*g) &= \Big((\boldsymbol{\ell}_p\circ\pi)\times v^{(p)}(f*g)\Big) \\ &= \Big((\boldsymbol{\ell}_p\circ\pi)\times v^{(p)}(f)\Big)\cdot\Big((\boldsymbol{\ell}_p\circ\pi)\times v^{(p)}(g)\Big) \\ &= (\pi_\rho\times v)(f)\cdot(\pi_\rho\times v)(g) \end{aligned} \tag{4.1}$$

and, similarly, we have that

$$(\pi_\rho \times v)(f^*) = (\pi_\rho \times v)(f)^{J_\rho}. \tag{4.2}$$

We define a map $\widetilde{\rho} : \mathcal{A} \rtimes_\theta G \to \mathcal{L}_{\mathcal{B}}(\mathcal{F})$ by

$$\widetilde{\rho}(f) = V_\rho^*(\pi_\rho \times v)(f)V_\rho, \quad (f \in \mathcal{A} \rtimes_\theta G). \tag{4.3}$$

Then by (i) and (iv) in Theorem 3.2, we have that

$$\widetilde{\rho}(f) = \int_G \rho\big(f(t)\big)u_t dt, \quad (f \in C_c(G, \mathcal{A})).$$

Lemma 4.1. *If α and θ are equivariant, that is, $\alpha \circ \theta_t = \theta_t \circ \alpha$ for any $t \in G$, and α is homomorphic, then there exists a continuous (in the inverse system) linear map $\widetilde{\alpha}$ on $\mathcal{A} \rtimes_\theta G$ such that*

$$\widetilde{\alpha}(f)(t) = \alpha\big(f(t)\big) \quad \textit{for } f \in C_c(G, \mathcal{A}) \textit{ and } t \in G. \tag{4.4}$$

Proof. For the given continuous (in the inverse system) Hermitian linear map α and each $p \in \mathcal{S}(\mathcal{A})$, it is clear that $\alpha(a) \in \ker(p)$ for all $a \in \ker(p)$, and so we denote by α_p the induced map on $\mathcal{A}_p$ from α. Then $\alpha_p : \mathcal{A}_p \to \mathcal{A}_p$ is a hermitian map such that

$$\alpha_p \circ \mathbf{q}_p = \mathbf{q}_p \circ \alpha \quad \text{and} \quad \alpha_p^2 = \mathrm{id}_{\mathcal{A}_p}. \tag{4.5}$$

Hence, for any $f \in L^1(G, \mathcal{A}_p)$ we have that

$$\int_G p\big(\alpha_p(f(t))\big)dt \le C_{\alpha,p} \int_G p\big(f(t)\big)dt$$

for some constant $C_{\alpha,p} \ge 0$, which implies that there exists a continuous induced map $\widetilde{\alpha}_p : L^1(G, \mathcal{A}_p) \to L^1(G, \mathcal{A}_p)$ such that

$$\widetilde{\alpha}_p(f)(t) = \alpha_p(f(t)), \qquad f \in C_c(G, \mathcal{A}_p). \tag{4.6}$$

Note that $\|f\|_{\mathcal{A}_p \rtimes_{\theta(p)} G} = \sup_\sigma \|\sigma(f)\|$ for any $f \in L^1(G, \mathcal{A}_p)$ where the supremum is taken over all non-degenerate $*$-representations of $L^1(G, \mathcal{A}_p)$. Since α and θ are equivariant and α is homomorphic, we see that $\sigma(\widetilde{\alpha}_p)$ is also a $*$-representation of $L^1(G, \mathcal{A}_p)$ for any $*$-representation σ of $L^1(G, \mathcal{A}_p)$. Hence, for any $f \in L^1(G, \mathcal{A}_p)$ we have that

$$\big\|\widetilde{\alpha}_p(f)\big\|_{\mathcal{A}_p \rtimes_{\theta(p)} G} = \sup_\sigma \big\|\sigma\big(\widetilde{\alpha}_p(f)\big)\big\| \le \|f\|_{\mathcal{A}_p \rtimes_{\theta(p)} G},$$

where σ ranges over all non-degenerate $*$-representations of $L^1(G, \mathcal{A}_p)$, which implies that the map $\widetilde{\alpha}_p : \mathcal{A}_p \rtimes_{\theta(p)} G \to \mathcal{A}_p \rtimes_{\theta(p)} G$ is continuous. Then there exists a continuous linear map $\widetilde{\alpha}$ on $\mathcal{A} \rtimes_\theta G$ such that for any $p \in \mathcal{S}(\mathcal{A})$,

$$\widetilde{\mathbf{q}}_p \circ \widetilde{\alpha} = \widetilde{\alpha}_p \circ \widetilde{\mathbf{q}}_p, \tag{4.7}$$

and then from (4.5) and (4.6), we can easily see that (4.4) holds. Also, (4.7) gives the continuity in the inverse system of $\widetilde{\alpha}$. ■

If α and θ are equivariant, then $\widetilde{\alpha}$ is also a Hermitian map on $C_c(G, \mathcal{A})$ and it can be extended to a Hermitian map on $\mathcal{A} \rtimes_\theta G$. It follows from the definition that $\widetilde{\alpha}^2 = \mathrm{id}_{\mathcal{A} \rtimes_\theta G}$.

From now on we assume that α is homomorphic, and α and θ are equivariant.

Lemma 4.2. *If* $\{e_\lambda\}$ *is an approximate unit for* $\mathcal{A} \rtimes_\theta G$*, then* $\{\widetilde{\alpha}(e_\lambda)\}$ *is also an approximate unit for* $\mathcal{A} \rtimes_\theta G$*.*

Proof. Since θ is an inverse limit action, we see from [4, 11] that

$$\mathcal{A} \rtimes_\theta G = \varprojlim_p \mathcal{A}_p \rtimes_{\theta(p)} G.$$

Hence we can see that $\{e_\lambda\}$ is an approximate unit for $\mathcal{A} \rtimes_\theta G$ if and only if $\{\widetilde{\mathbf{q}}_p(e_\lambda)\}$ is an approximate unit for $\mathcal{A}_p \rtimes_{\theta(p)} G$ for every $p \in \mathcal{S}(\mathcal{A})$. Since α is homomorphic, $\widetilde{\alpha}$ is also homomorphic. Hence by [8, Lemma 5.3], the proof is straightforward. ■

Theorem 4.3. *The map* $\widetilde{\rho}$ *given by* (4.3) *is* $\widetilde{\alpha}$*-completely positive.*

Proof. We only have to prove the conditions (ρ3)-(ρ6) for the α-CP map with $\alpha = \widetilde{\alpha}$ and $\rho = \widetilde{\rho}$. For any $f, g \in L^1(G, \mathcal{A})$, we obtain that

$$\begin{aligned}
\widetilde{\rho}\big[\widetilde{\alpha}(f) * \widetilde{\alpha}(g)\big] &= \int_G \int_G \rho\left[\alpha(f(t))\theta_t\big(\alpha(g(t^{-1}s))\big)\right] u_s dt ds \\
&= \int_G \int_G \rho\left[f(t)\theta_t\big(g(t^{-1}s)\big)\right] u_s dt ds \\
&= \int_G \rho\big[(f * g)(s)\big] u_s ds \\
&= \widetilde{\rho}[f * g],
\end{aligned}$$

where we used the condition that α and θ are equivariant for the second equality. Hence, the condition (ρ3) is satisfied. From the equations (4.1) and (4.2), we obtain that

$$\begin{aligned}
\sum_{i,j=1}^{n} \big\langle \xi_i,\, \widetilde{\rho}\big([\widetilde{\alpha}(f_i)]^* * f_j\big)\xi_j \big\rangle &= \sum_{i,j=1}^{n} \big\langle V_\rho \xi_i,\, \big[(\pi_\rho \times v)(f_i)\big]^* (\pi_\rho \times v)(f_j) V_\rho \xi_j \big\rangle \\
&\geq 0
\end{aligned}$$

for any $\xi_i, \xi_j \in \mathcal{F}$ and $f_i, f_j \in L^1(G, \mathcal{A})$ $(i, j = 1, \ldots, n)$, which implies the condition (ρ4).

Let $f, f_i \in L^1(G, \mathcal{A})$ $(i = 1, \ldots, n)$. By the equations (4.1) and (4.2), we have that

$$\begin{aligned}
&\big(\widetilde{\rho}\big([\widetilde{\alpha}(f * f_i)]^* * (f * f_j)\big)\big)_{n \times n} \\
&= \big(V_\rho^* (\pi_\rho \times v)(\widetilde{\alpha}(f * f_i)^*)(\pi_\rho \times v)(f * f_j) V_\rho\big)_{n \times n} \\
&= \big(V_\rho^* \big[(\pi_\rho \times v)(f * f_i)\big]^* (\pi_\rho \times v)(f * f_j) V_\rho\big)_{n \times n} \qquad (4.8)\\
&= \big(V_\rho^* \big[(\pi_\rho \times v)(f_i)\big]^* \big[(\pi_\rho \times v)(f)\big]^* (\pi_\rho \times v)(f)(\pi_\rho \times v)(f_j) V_\rho\big)_{n \times n} \\
&\leq C(f) \big(V_\rho^* \big[(\pi_\rho \times v)(f_i)\big]^* (\pi_\rho \times v)(f_j) V_\rho\big)_{n \times n},
\end{aligned}$$

where the constant $C(f)$ is given by

$$C(f) = \sup_{p\in\mathcal{S}(\mathcal{A})} \left\| \left((\boldsymbol{\ell}_p \circ \pi_\rho) \times v^{(p)}\right)(f) \right\|^2 \le \sup_{p\in\mathcal{S}(\mathcal{A})} \int_G p\big(f(t)\big)dt < \infty.$$

Therefore, we obtain that

$$\big(\widetilde{\rho}\big([\widetilde{\alpha}(f * f_i)]^* * (f * f_j)\big)\big)_{n\times n} \le C(f)\,\big(\widetilde{\rho}\big([\widetilde{\alpha}(f_i)]^* * f_j\big)\big)_{n\times n},$$

which implies the condition (ρ5). By the equation (4.8), we have that

$$\widetilde{\rho}\big([\widetilde{\alpha}(f)]^* * f\big) = V_\rho^*[(\pi_\rho \times v)(f)]^*(\pi_\rho \times v)(f)V_\rho \ge 0,$$

which implies the condition (ρ6). Hence the map $\widetilde{\rho}$ is $\widetilde{\alpha}$-completely positive. ■

Let G be unimodular. Since θ is an inverse limit action, there exist a countable family $\{\mathcal{A}_p\}_{p\in\Lambda}$ of C^*-algebras and a family $\{\theta^{(p)}\}_{p\in\Lambda}$ of actions $\theta^{(p)}$ of G on $\mathcal{A}_p$ such that

$$\mathcal{A} = \varprojlim \mathcal{A}_p \quad \text{and} \quad \theta_t = \varprojlim \theta_t^{(p)}.$$

We define a group action $\widetilde{\theta}^{(p)}$ ($p \in \Lambda$) of G on $\mathcal{A}_p \rtimes_\theta G$ by

$$\big(\widetilde{\theta}_t^{(p)}(f)\big)(s) = \theta_t^{(p)}\big(f(t^{-1}st)\big), \quad (t \in G,\ f \in L^1(G, \mathcal{A}_p)),$$

and define a group action $\widetilde{\theta}$ of G on $\mathcal{A} \rtimes_\theta G$ by

$$\widetilde{\theta}_t(f) = \left(\widetilde{\theta}_t^{(p)}\big(\widetilde{\mathbf{q}}_p(f)\big)\right), \quad (t \in G,\ f = (\widetilde{\mathbf{q}}_p(f)) \in \mathcal{A} \rtimes_\theta G).$$

Then $\widetilde{\theta}$ becomes a continuous inverse limit action of G on $\mathcal{A} \rtimes_\theta G$.

Proposition 4.4. *The map $\widetilde{\rho}$ given by* (4.3) *is* $(\widetilde{\theta}, u)$*-covariant.*

Proof. We only need to prove the covariant property. In fact, for any $f \in C_c(G, \mathcal{A})$ and $t \in G$, we obtain that

$$\begin{aligned}\widetilde{\rho}\big(\widetilde{\theta}_t(f)\big) &= \int_G \rho\big(\theta_t(f(t^{-1}st))\big)u_s ds\\ &= \int_G u_t\rho\big(f(t^{-1}st)\big)u_{t^{-1}st}u_t^* ds = u_t\widetilde{\rho}(f)u_t^*,\end{aligned}$$

which implies the proof. ■

On the other hand, for any $t \in G$, $f \in C_c(G, \mathcal{A})$, we obtain that

$$\begin{aligned}(\pi_\rho \times v)\big(\widetilde{\theta}_t(f)\big) &= \int_G \pi_\rho\big(\theta_t(f(t^{-1}st))\big)v_s ds\\ &= \int_G v_t\pi_\rho\big(f(s)\big)v_s v_t^* ds = v_t(\pi_\rho \times v)(f)v_t^*.\end{aligned}$$

Therefore, we have the following theorem.

Theorem 4.5. *The pair* $(\pi_\rho \times v, v)$ *is a* $(\widetilde{\theta}, v)$*-covariant* J_ρ*-representation of* $(\mathcal{A} \rtimes_\theta G, G, \widetilde{\theta})$ *on* $(\mathcal{F}_\rho, J_\rho)$.

Proof. The proof is straightforward from the above argument and Theorem 3.2. ■

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