TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 1, pp. 299-313, February 2013
DOI: 10.11650/tjm.17.2013.2023
This paper is available online at http://journal.taiwanmathsoc.org.tw

HARDY INEQUALITIES UNDER SOME NON-CONVEXITY MEASURES

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Abstract. Considering two different non-convexity measures, we obtain some new Hardy-type inequalities for non-convex domains $\Omega \subset \mathbb{R}^n$. We study the three-dimensional case and then generalise the approach to the $n-$dimensional case.

1. Introduction

We study high dimension variants of the classical integral Hardy-type inequality ([8])

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \mu \int_0^\infty f^p(x)dx, \tag{1}$$

where $p > 1$, $f(x) \ge 0$, and $F(x) = \int_0^x f(t)dt$ with constant μ. Inequality (1) with its improvements have played a fundamental role in the development of many mathematical branches such as spectral theory and PDE's, see for instance [2-5, 7] and [10]. We centre our attention on the multi-dimensional version of (1) for $p = 2$, which takes the following form (see for example [6]):

$$\mu \int_\Omega \frac{|f(x)|^2}{d(x)^2}\, dx \le \int_\Omega |\nabla f|^2 \, dx, \quad f \in \mathcal{C}_c^\infty(\Omega), \tag{2}$$

where

$$d(x) := \min\{|x-y| : y \notin \Omega\}. \tag{3}$$

For convex domains $\Omega \subset \mathbb{R}^n$, the sharp constant μ in (2) has been shown to equal $\frac{1}{4}$, see for instance [5] and [10]. However, the sharp constant for non-convex domains is unknown, although for arbitrary planar simply-connected domains $\Omega \subset \mathbb{R}^2$, A. Ancona

Received May 8, 2012, accepted July 3, 2012.
Communicated by Yong-Sheng Han.
2010 *Mathematics Subject Classification*: 26Dxx, 26D07, 26D10, 34A40.
Key words and phrases: Hardy inequality, Non-convex domains.

([1]) proved, using the Koebe one-quarter Theorem, that the constant μ in (2) is greater than or equal to $\frac{1}{16}$. Later A. Laptev and A. Sobolev ([9]) considered, under certain geometrical conditions, classes of domains for which there is a stronger version of the Koebe Theorem, this implied better estimates for the constant μ. Other specific examples of non-convex domains were presented by E. B. Davies ([6]).

Our main goal is to obtain new Hardy-type inequalities under some non-convexity measures for domains in $\mathbb{R}^n,\ n \geq 3$, focusing on obtaining upper bounds for μ. In fact we have two different conditions "measures" introduced in the following section.

2. Notations and Conditions

In this section we present two 'non-convexity measures' for domains $\Omega \subset \mathbb{R}^n; n \geq 3$. In order to do so let us introduce the following notations: Let w be a point in $\mathbb{R}^n$ and ν be a unit vector. For $\alpha \in \left(0, \frac{\pi}{2}\right)$ define

$$C_0(\nu, \alpha) = \{x \in \mathbb{R}^n : x \cdot \nu \geq |x| \cos \alpha\},$$

which is a cone in the Euclidean space $\mathbb{R}^n$ with vertex at 0 and symmetry axis in the ν direction. Denote by $C_w(\nu, \alpha) = C_0(\nu, \alpha) + w$, the transition of $C_0(\nu, \alpha)$ by $w \in \mathbb{R}^n$, i.e.

$$C_w(\nu, \alpha) = \{x \in \mathbb{R}^n : (x - w) \cdot \nu \geq |x - w| \cos \alpha\},$$

which can be seen as an n-dimensional cone with vertex at w and symmetry axis parallel to the ν direction with angle 2α at the vertex.
Now for $h \geq 0$, define the half-space $\mathbf{\Pi}_h(\nu)$ by

$$\mathbf{\Pi}_h(\nu) = \{x \in \mathbb{R}^n : x \cdot \nu \geq h\}.$$

Denote by $\mathbf{\Pi}_{h,w}(\nu) = \mathbf{\Pi}_h(\nu) + w$, the transition of $\mathbf{\Pi}_h(\nu)$ by $w \in \mathbb{R}^n$, i.e.

$$\mathbf{\Pi}_{h,w}(\nu) = \{x \in \mathbb{R}^n : (x - w) \cdot \nu \geq h\},$$

which is a half-space of 'height h' from the point w in the ν direction.
Define the region $K_{h,w}(\nu, \alpha)$ to be

$$K_{h,w}(\nu, \alpha) = C_w(\nu, \alpha) \cup \mathbf{\Pi}_{h,w}(\nu).$$

With the notations given above we now state the conditions or 'non-convexity measures' we use throughout the rest of this paper.

Condition 2.1. (Exterior Cone Condition).

We say that $\Omega \subset \mathbb{R}^n$ satisfies the Exterior Cone Condition if for each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset C^c_w(\nu, \alpha)$, with $(x - w) \cdot \nu = -|x|$.

Condition 2.1 means that for every point $x \in \Omega$ we can always find a cone $C_\omega(\nu, \alpha)$ such that x lies on its symmetry axis where Ω is completely outside that cone.

As a development of the above condition, we establish the following condition.

Condition 2.2. (Truncated Cone Region (TCR). Condition).

We say that $\Omega \subset \mathbb{R}^n$ satisfies the TCR Condition if for each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset K^c_{h,w}(\nu, \alpha)$, for some $h \geq 0$, with $(x - w) \cdot \nu = -|x|$.

Condition 2.2 means that for every point $x \in \Omega$ we can always find a truncated conical region $K_{h,\omega}(\nu, \alpha)$ such that x lies on its symmetry axis, which is the symmetry axis of $C_\omega(\nu, \alpha)$ where Ω is completely outside that truncated conical region.
Suppose that the domain Ω satisfies one of Conditions 2.1 and 2.2. For a fixed $x \in \Omega$, choose w, a mutual point of $\partial\Omega$ and $\partial\mathcal{B}$, to be such that $d(x) = |x - w|$. Denote by $\mathcal{B}$ the appropriate test domain, i.e. a cone (Condition 2.1) or truncated conical region (Condition 2.1). Furthermore, by $d_u(x)$ we mean the distance from $x \in \Omega$ to $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}, \tag{4}$$

and $\tilde{d}_u(x)$ the distance from $x \in \Omega$ to $\partial\mathcal{B}$, in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial\mathcal{B}\}.$$

Finally, denote by $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ the angle at which the line segment representing $\tilde{d}_u(x)$ leaves $\partial\mathcal{B}$ to infinity.

3. Main Results and Discussion

In this section we state and discuss our main theorems which will be proved in Section 4.

3.1. Results related to the Exterior Cone Condition

The following two theorems are related to the Exterior Cone Condition.

Theorem 3.1. *Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 2.1 with some $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then for any $f \in \mathcal{C}^\infty_c(\Omega)$ the following Hardy-type inequality holds:*

$$\mu(\alpha) \int\limits_\Omega \frac{|f(x)|^2}{d(x)^2} dx \leq \int\limits_\Omega |\nabla f(x)|^2 \, dx, \tag{5}$$

where

$$\mu(\alpha) = \frac{1}{4} \tan^2 \frac{\alpha}{2}. \tag{6}$$

Remark 3.2. If the domain Ω in Theorem 3.1 is convex, then $\alpha = \frac{\pi}{2}$. Thus the function $\mu(\alpha)$ defined by (6) will be $\mu(\frac{\pi}{2}) = \frac{1}{4}$ as known for convex domains.

Theorem 3.3. *Suppose that the domain $\Omega \subset \mathbb{R}^n$; $n \geq 3$, satisfies Condition 2.1. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:*

$$\mu(n, \alpha) \int\limits_\Omega \frac{|f(x)|^2}{d(x)^2} dx \leq \int\limits_\Omega |\nabla f(x)|^2 \, dx, \tag{7}$$

where

$$\begin{aligned} &\mu(n, \alpha) \\ &= \frac{1}{2\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left(\left((n-1)\cot^2\alpha + 1\right) \int\limits_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha \cos\alpha \right). \end{aligned} \tag{8}$$

Remark 3.4.

1. For convex domains we have $\alpha = \frac{\pi}{2}$. In this case, the function $\mu(n, \alpha)$, given by (8), becomes

$$\mu\left(n, \frac{\pi}{2}\right) = \frac{1}{2\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int\limits_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta = \frac{1}{4} \quad \text{for any } n, \tag{9}$$

as expected for a convex case.

2. For $n = 3$, the function $\mu(n, \alpha)$, given by (8), becomes

$$\begin{aligned} \mu(3, \alpha) &= \frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot \left[\left(2\cot^2\alpha + 1\right)(1 - \cos\alpha) - \cos\alpha\right] \\ &= \frac{1}{4}\left[\left(\frac{2\cos^2\alpha + 1 - \cos^2\alpha}{1 - \cos^2\alpha}\right)(1 - \cos\alpha) - \cos\alpha\right] \\ &= \frac{1}{4}\left[\frac{\cos^2\alpha + 1 - \cos\alpha - \cos^2\alpha}{1 + \cos\alpha}\right] = \frac{1}{4}\left[\frac{1 - \cos\alpha}{1 + \cos\alpha}\right] \\ &= \frac{1}{4}\tan^2\frac{\alpha}{2}, \end{aligned}$$

exactly as obtained in (6).

3.2. Results related to the TCR Condition

For the advantage of 'measuring how deep the dent' inside the domain is, let us consider domains $\Omega \subset \mathbb{R}^n$ that satisfy Condition 2.2.

Theorem 3.5. *Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 2.2. Then for any $f \in \mathcal{C}_c^\infty(\Omega)$ the following Hardy-type inequality holds:*

$$\int_\Omega \mu_1(x,\alpha,h)\frac{|f(x)|^2}{(h+d(x))^2}dx + \int_\Omega \mu_2(x,\alpha,h)\frac{|f(x)|^2}{d(x)^2}dx \le \int_\Omega |\nabla f(x)|^2\,dx, \tag{10}$$

where

$$\mu_1(x;\alpha,h) = \frac{1}{4}\cos^3\left(\tan^{-1}(a(x)\,\tan\alpha)\right), \tag{11}$$

and

$$\begin{aligned}
&\mu_2(x,\alpha,h)\\
&= \frac{1}{4\sin^2\alpha}\Big[3-\cos 2\left(\alpha-\left(\tan^{-1}(a(x)\,\tan\alpha)\right)\right)\\
&\quad -2\cos\left(2\alpha-\left(\tan^{-1}(a(x)\,\tan\alpha)\right)\right)\Big]\sin^2\frac{\left(\tan^{-1}(a(x)\,\tan\alpha)\right)}{2},
\end{aligned} \tag{12}$$

with $a(x) = \frac{1}{1+\frac{d(x)}{h}}$.

Remark 3.6.

1. If Ω is a convex domain then $\alpha = \frac{\pi}{2}$. Therefore, for convex domains with $a(x) \neq 0$, i.e. $h \not\to 0$, we have $\mu_1\left(x,\frac{\pi}{2},h\right) = 0$ and $\mu_2\left(x,\frac{\pi}{2},h\right) = \frac{1}{4}$, thus the Hardy-type inequality (10) reproduces the well-known bound (see for instance [5]):

$$\frac{1}{4}\int_\Omega \frac{|f(x)|^2}{d(x)^2}dx \le \int_\Omega |\nabla f(x)|^2\,dx. \tag{13}$$

2. As $\alpha \nearrow \frac{\pi}{2}$, the domain Ω approaches the convexity case, and hence it is natural to compare $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ given by (11) and (12) respectively, with their values for the convex case. To this end we use the Taylor expansion to expand $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ in powers of $\left(\frac{\pi}{2}-\alpha\right)$. Keeping in mind that for fixed h we have $\theta_0 = \tan^{-1}(a(x)\tan\alpha) = \frac{\pi}{2}$ where $\alpha = \frac{\pi}{2}$. Consequently, for $\mu_1(x,\alpha,h)$, we have

$$\mu_1\left(x,\frac{\pi}{2},h\right) = 0, \frac{\partial}{\partial\alpha}\mu_1\left(x,\frac{\pi}{2},h\right) = 0, \frac{\partial^2}{\partial\alpha^2}\mu_1\left(x,\frac{\pi}{2},h\right) = 0.$$

However,

$$\frac{\partial^3}{\partial\alpha^3}\mu_1\left(x,\frac{\pi}{2},h\right) = -\frac{3}{2a(x)^3} = -\frac{3\,(h+d(x))^3}{2h^3}, \cdots \text{ and so on.}$$

Thus $\mu_1(x,\alpha,h)$ can be written as follows:

$$\mu_1(x,\alpha,h)=\frac{(h+d(x))^3}{4h^3}\left(\frac{\pi}{2}-\alpha\right)^3+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{14}$$

Similarly, $\mu_2(x,\alpha,h)$ can be written as follows:

$$\mu_2(x,\alpha,h)=\frac{1}{4}+\frac{1}{2}\left(\alpha-\frac{\pi}{2}\right)+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^2\right). \tag{15}$$

For $\alpha=\frac{\pi}{2}$, we have $\mu_1(x,\alpha,h)=0$ and $\mu_2(x,\alpha,h)=\frac{1}{4}$, thus we obtain the same bound as in (13).
Relations (14) and (15) show that the second term in inequality (10) is the effective term when talking about the convex case, since $\mu_1(x,\alpha,h)$ decays rapidly to zero while $\mu_2(x,\alpha,h)$ tends to $\frac{1}{4}$, when α tends to $\frac{\pi}{2}$.

Theorem 3.7. *Suppose that the domain $\Omega\subset\mathbb{R}^n$; $n\geq 3$, satisfies Condition 2.2. Then for any function $f\in\mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:*

$$\begin{aligned}&\int_\Omega \mu_1(n,x,\alpha,h)\frac{|f(x)|^2}{(h+d(x))^2}dx+\int_\Omega \mu_2(n,x,\alpha,h)\frac{|f(x)|^2}{d(x)^2}dx\\ &\leq\int_\Omega |\nabla f(x)|^2\,dx,\end{aligned} \tag{16}$$

where

$$\mu_1(n,x,\alpha,h)=\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(-\sin^{n-1}\theta_0\cos\theta_0+\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right),\quad\text{and} \tag{17}$$

$$\begin{aligned}\mu_2(n,x,\alpha,h)=\;&\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{1}{\sin^2\alpha}\left(\left((n-1)\cot^2\alpha+1\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\ &\left.-\sin^{n-1}\theta_0\cos(2\alpha-\theta_0)\right),\end{aligned} \tag{18}$$

with θ_0 satisfies $\tan\theta_0=\frac{h}{h+d(x)}\tan\alpha$. In particular, when $\alpha=\frac{\pi}{2}$, we have $\mu_1(n,x,\alpha,h)=0$ and $\mu_2(n,x,\alpha,h)=\frac{1}{4}$.

Remark 3.8.

1. If Ω is a convex domain then $\alpha=\theta_0=\frac{\pi}{2}$. Consequently, the Hardy-type inequality (16) reproduces the well-known bound (13) for any convex domain $\Omega\subset\mathbb{R}^n$.
2. When $\alpha\nearrow\frac{\pi}{2}$, the domain Ω approaches the convexity case. Therefore, it is natural to compare $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$, given by (17) and (18) respectively, with their values for the convex case. Keeping in mind that when $\alpha=\frac{\pi}{2}$ we set $\theta_0=\theta_0(x,\alpha)=\tan^{-1}(a(x)\tan\alpha)=\frac{\pi}{2}$, and for fixed h, expressions for $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$ can be written as powers of $\left(\alpha-\frac{\pi}{2}\right)$. We find that, the function $\mu_1(n,x,\alpha,h)$ can be written as follows:

$$\mu_1(n,x,\alpha,h) = \frac{n\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \frac{(h+d(x))^3}{6h^3}\left(\frac{\pi}{2}-\alpha\right)^3 + \mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{19}$$

On the other hand, the function $\mu_2\left(n,x,\frac{\pi}{2},h\right)$ can be written as follows:

$$\mu_2(n,x,\alpha,h) = \frac{1}{4} + \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(\alpha-\frac{\pi}{2}\right) + \mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^2\right). \tag{20}$$

Relations (19) and (20) show that the second term in Hardy-type inequality (16) is the effective term when talking about the convex case, since $\mu_1(n,x,\alpha,h)$ tends to zero while $\mu_2(n,x,\alpha,h)$ tends to $\frac{1}{4}$ as α tends to $\frac{\pi}{2}$.

3. For fixed α, as h tends to ∞, $a(x)$ tends to 1, which means implicitly that θ_0 tends to α. Therefore, we obtain the following limit for $\mu_2(n,x,\alpha,h)$ as h tends to ∞:

$$\begin{aligned}\lim_{h\to\infty}\mu_2(n,x,\alpha,h) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\Bigg(&\left((n-1)\cot^2\alpha+1\right)\int_0^\alpha \sin^{n-2}\theta d\theta \\ &- \sin^{n-3}\alpha\cos(\alpha)\Bigg) = \mu(n,\alpha).\end{aligned} \tag{21}$$

Since all functions (f,μ_1,μ_2) are uniformly bounded, we can pass to the limit under the integral, thus the first term in Hardy-type inequality (16) tends to zero and we obtain the same result as in Theorem 3.3. On the other hand, as h tends to 0, $a(x)$ tends to 0, which leads to the tendency of θ_0 to 0 as well. This implies that $\mu_1(n,x,\alpha,h)\to\frac{1}{4}$ and $\mu_2(n,x,\alpha,h)\to 0$.

The key ingredient in proving Theorems 3.1, 3.3, 3.5 and 3.7 is the following proposition.

Proposition 3.9. (E. B. Davies, [4, 7]). *Let Ω be a domain in $\mathbb{R}^n$ and let $f\in\mathcal{C}_c^\infty(\Omega)$. Then*

$$\frac{n}{4}\int_\Omega \frac{|f(x)|^2}{m(x)^2}dx \le \int_\Omega |\nabla f(x)|^2\,dx,$$

where $m(x)$ is given by

$$\frac{1}{m(x)^2} := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}dS(u), \tag{22}$$

and

$$d_u(x) := \min\{|t| : x+tu\notin\Omega\},$$

for every unit vector $u\in\mathbb{S}^{n-1}$ and $x\in\Omega$. Here $\left|\mathbb{S}^{n-1}\right| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Our strategy to prove Theorems 3.1, 3.3, 3.5 and 3.7 is to obtain lower bounds for the function $\frac{1}{m(x)^2}$ given by (22), containing $d(x)$, then apply Proposition 3.9.

4. Proofs

Proof of Theorem 3.1. By (22) and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{23}$$

Since $\tilde{d}_u(x)$ is a symmetric function, with respect to the rotation about the symmetry axis of the cone $C_\omega(\nu, \alpha)$, then using spherical coordinates, (r, θ, ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, leads to $u = u(\theta, \phi)$, and $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x, \theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (23) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{4\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{1}{\tilde{d}(x, \theta)^2} \sin\theta \, d\theta d\phi = \int\limits_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x, \theta)^2} \sin\theta \, d\theta. \tag{24}$$

However, the angle θ can not exceed α, thus inequality (24) takes the following form:

$$\frac{1}{m(x)^2} \geq \int\limits_0^{\alpha} \frac{1}{\tilde{d}(x, \theta)^2} \sin\theta \, d\theta. \tag{25}$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition 2.1 and if we consider the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x, \theta)$, we conclude that

$$\tilde{d}(x, \theta) = \frac{d(x) \sin\alpha}{\sin(\alpha - \theta)}.$$

Thus, the lower bound (25), on the function $\frac{1}{m(x)^2}$, can be written as follows:

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{\int\limits_0^{\alpha} \sin^2(\alpha - \theta) \sin\theta d\theta}{d(x)^2 \sin^2\alpha} \\ &= \frac{\int\limits_0^{\alpha} \left(\sin\theta - \sin\theta \cos 2\left(\alpha - \theta\right)\right) d\theta}{2d(x)^2 \sin^2\alpha} \\ &= \frac{(1 - \cos\alpha)^2}{3\, d(x)^2 (1 - \cos^2\alpha)} = \frac{1}{3\, d(x)^2} \cdot \frac{1 - \cos\alpha}{1 + \cos\alpha} \\ &= \frac{\tan^2 \frac{\alpha}{2}}{3\, d(x)^2}. \end{aligned} \tag{26}$$

Apply Proposition 3.9 to this lower bound in (26) to obtain the Hardy-type inequality (5) with $\mu(\alpha)$ as given in (6), this completes the proof. ■

Proof of Theorem 3.3. By (22) and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{27}$$

Because of the definition of $\tilde{d}_u(x)$ and by using spherical coordinates, (r, θ, ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, we have $u = (\theta, \phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x, \theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (27) becomes

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta \int\limits_{\mathbb{S}^{n-2}} dw \\
&= 2\frac{\left|\mathbb{S}^{n-2}\right|}{|\mathbb{S}^{n-1}|} \int\limits_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta.
\end{aligned}$$

However, the angle θ can not exceed the value $\alpha < \frac{\pi}{2}$, hence

$$\frac{1}{m(x)^2} \geq 2\frac{\left|\mathbb{S}^{n-2}\right|}{|\mathbb{S}^{n-1}|} \int\limits_0^{\alpha} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta. \tag{28}$$

Since Ω satisfies Condition 2.1, i.e., we have symmetry with respect to the axis of $C_\omega(\nu, \alpha)$, we consider the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x, \theta)$, so we have

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha - \theta)}.$$

Thus inequality (28) can be rewritten as follows:

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{2\left|\mathbb{S}^{n-2}\right|}{|\mathbb{S}^{n-1}|\, d(x)^2 \sin^2\alpha} \int\limits_0^{\alpha} \sin^2(\alpha-\theta)\sin^{n-2}\theta d\theta \\
&= \frac{\left|\mathbb{S}^{n-2}\right|}{|\mathbb{S}^{n-1}|\, d(x)^2 \sin^2\alpha} \left(\int\limits_0^{\alpha} \sin^{n-2}\theta d\theta - I_1(\alpha) \right), \qquad (29)
\end{aligned}$$

where

$$I_1(\alpha) = \int\limits_0^{\alpha} \sin^{n-2}\theta \cos 2\,(\alpha - \theta)\, d\theta.$$

There are many ways to evaluate $I_1(\alpha)$. Rewrite $I_1(\alpha)$ as follows

$$
\begin{aligned}
I_1(\alpha) &= \cos 2\alpha \left[\int_0^\alpha \sin^{n-2}\theta d\theta - 2\int_0^\alpha \sin^n \theta d\theta \right] + \frac{2}{n}\sin 2\alpha \sin^n \alpha \\
&= \cos 2\alpha \left[\int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}\sin^{n-1}\alpha\cos\alpha - \frac{2n-2}{n}\int_0^\alpha \sin^{n-2}\theta d\theta \right] \\
&\quad + \frac{4}{n}\sin^{n+1}\alpha\cos\alpha \\
&= \frac{2-n}{n}\cos 2\alpha \int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}\sin^{n-1}\alpha\cos\alpha.
\end{aligned}
\tag{30}
$$

Thus using (30), inequality (29) produces the following lower bound on $\frac{1}{m(x)^2}$:

$$
\begin{aligned}
&\frac{1}{m(x)^2} \\
\geq\ & \frac{|\mathbb{S}^{n-2}|}{n\, d(x)^2\, |\mathbb{S}^{n-1}| \sin^2\alpha} \\
& \left[n\int_0^\alpha \sin^{n-2}\theta d\theta + (n-2)\cos 2\alpha \int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha \right] \\
=\ & \frac{|\mathbb{S}^{n-2}|}{n\, d(x)^2\, |\mathbb{S}^{n-1}| \sin^2\alpha} \\
& \left[(n+(n-2)\cos 2\alpha)\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha \right] \\
=\ & \frac{|\mathbb{S}^{n-2}|}{n\, d(x)^2\, |\mathbb{S}^{n-1}| \sin^2\alpha} \\
& \left[2((n-1)\cos^2\alpha + \sin^2\alpha)\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha \right] \\
=\ & \frac{2\,|\mathbb{S}^{n-2}|}{d(x)^2\, n\, |\mathbb{S}^{n-1}|} \left[((n-1)\cot^2\alpha + 1)\int_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha\cos\alpha \right].
\end{aligned}
\tag{31}
$$

Applying Proposition 3.9 to the lower bound (31) putting into account that

$$
\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)},
\tag{32}
$$

returns the Hardy-type inequality (7) with $\mu(n,\alpha)$ as in (8), this completes the proof. ∎

Proof of Theorem 3.5 By (22) and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{33}$$

Since the function $\tilde{d}_u(x)$ is symmetric, with respect to the rotation about the symmetry axis of the domain $K_{h,\omega}(\nu, \alpha)$, then using spherical coordinates, (r, θ, ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, leads to $u = u(\theta, \phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x, \theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (33) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta.$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition 2.2 and if we consider the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}_u(x)$, we can divide the above interval into two intervals considering the relation between $\tilde{d}(x, \theta)$ and $d(x)$. Thus, for $\theta \in (0, \theta_0)$, the function $\tilde{d}(x, \theta)$ can be expressed in the following form:

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha - \theta)}.$$

Besides, for $\theta \in (\theta_0, \frac{\pi}{2})$, the function $\tilde{d}(x, \theta)$ can be written as follows

$$\tilde{d}(x,\theta) = \frac{h + d(x)}{\cos\theta},$$

where θ_0 satisfies

$$\tan\theta_0 = \frac{1}{1 + \frac{d(x)}{h}} \tan\alpha. \tag{34}$$

Moreover, for $\alpha = \frac{\pi}{2}$ (for which Ω attains the convex case) we have

$$\tilde{d}(x,\theta) = \frac{d(x)}{\cos\theta}.$$

Thus, the function $\frac{1}{m(x)^2}$ is bounded from below as follows:

$$\frac{1}{m(x)^2} \geq \frac{\int_0^{\theta_0} \sin^2(\alpha - \theta)\sin\theta d\theta}{d(x)^2 \sin^2\alpha} + \frac{\int_{\theta_0}^{\frac{\pi}{2}} \cos^2\theta \sin\theta d\theta}{(h + d(x))^2}.$$

Using the substitution $u = \cos\theta$ in the second integral produces

$$
\begin{aligned}
&\frac{1}{m(x)^2} \\
&\geq \frac{-\cos\theta|_0^{\theta_0} - \frac{1}{2}\int_0^{\theta_0} \left(\sin\left(2\alpha-\theta\right) + \sin\left(3\theta - 2\alpha\right)\right) d\theta}{2d(x)^2 \sin^2\alpha} + \frac{\cos^3\theta_0}{3\left(h + d(x)\right)^2} \\
(35) \qquad &= \frac{1 - \cos\theta_0 - \frac{\cos(2\alpha-\theta_0)}{2} + \frac{\cos(3\theta_0 - 2\alpha)}{6} + \frac{\cos(2\alpha)}{3}}{2d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3\left(h+d(x)\right)^2} \\
&= \frac{\left(3 - \cos 2\left(\alpha - \theta_0\right) - 2\cos\left(2\alpha - \theta_0\right)\right)\sin^2\frac{\theta_0}{2}}{3d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3\left(h+d(x)\right)^2}.
\end{aligned}
$$

Applying Proposition 3.9 to this lower bound in (35) leads to

$$
\int_\Omega \mu_1^*\left(\theta_0\right) \frac{|f(x)|^2}{(h+d(x))^2} dx + \int_\Omega \mu_2^*\left(\theta_0, \alpha\right) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 \, dx,
$$

where

$$
\mu_1^*\left(\theta_0\right) = \frac{\cos^3\theta_0}{4},
$$

and

$$
\mu_2^*\left(\theta_0, \alpha\right) = \frac{\left(3 - \cos 2\left(\alpha-\theta_0\right) - 2\cos\left(2\alpha - \theta_0\right)\right)\sin^2\frac{\theta_0}{2}}{4\sin^2\alpha}.
$$

Now using (34), the relation between θ_0 and α, enables us to write $\mu_1^*\left(\theta_0\right)$ and $\mu_2^*\left(\theta_0, \alpha\right)$ as functions of x, α, and h as in (11) and (12) respectively. This completes the proof. ■

Proof of Theorem 3.7 As have been illustrated before, the function $\frac{1}{m(x)^2}$, has the following lower bound

$$
\frac{1}{m(x)^2} \geq 2\frac{\left|\mathbb{S}^{n-2}\right|}{\left|\mathbb{S}^{n-1}\right|}\int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta. \tag{36}
$$

Since Ω satisfies Condition 2.2, we consider the cross section containing the point $x \in \Omega$ and the line segments representing $d(x)$ and $\tilde{d}(x,\theta)$, then according to the relation between $\tilde{d}(x,\theta)$ and $d(x)$, we can rewrite inequality (36) as follows:

$$
\frac{1}{m(x)^2} \geq 2b\left[I_1(n,\theta_0) + I_2(n,\theta_0)\right]; \; b = \frac{\left|\mathbb{S}^{n-2}\right|}{\left|\mathbb{S}^{n-1}\right|} = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}, \tag{37}
$$

where

$$
I_1(n,\theta_0) = \int_0^{\theta_0} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta,
$$

$$I_2(n,\theta_0) = \int\limits_{\theta_0}^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta,$$

and $0 \le \theta_0 < \frac{\pi}{2}$ satisfies

$$\tan\theta_0 = \frac{h}{h+d(x)} \tan\alpha.$$

However, for all angles $\alpha < \frac{\pi}{2}$, we can easily find that: For $\theta \in [0,\theta_0)$, the relation between $\tilde{d}(x,\theta)$ and θ is

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)},$$

and for $\theta \in [\theta_0, \frac{\pi}{2})$, we have

$$\tilde{d}(x,\theta) = \frac{h+d(x)}{\cos\theta}.$$

On the other hand, for $\alpha = \frac{\pi}{2}$ the relation between $\tilde{d}(x,\theta)$ and θ is

$$\tilde{d}(x,\theta) = \frac{d(x)}{\cos\theta}.$$

Therefore, we can evaluate the first integral $I_1(n,\theta_0)$ as follows:

$$\begin{aligned} I_1(n,\theta_0) &= \frac{1}{d(x)^2\sin^2\alpha} \int\limits_0^{\theta_0} \sin^2(\alpha-\theta)\sin^{n-2}\theta d\theta \\ &= \frac{1}{2d(x)^2\sin^2\alpha} \left(\int\limits_0^{\theta_0} \sin^{n-2}\theta d\theta - I_3(n,\theta_0) \right), \end{aligned} \tag{38}$$

where

$$I_3(n,\theta_0) = \int\limits_0^{\theta_0} \sin^{n-2}\theta \cos 2\,(\alpha-\theta)\, d\theta.$$

On the other hand, we can rewrite $I_3(n,\theta_0)$ as

$$\begin{aligned} &I_3(n,\theta_0) \\ &= \cos 2\alpha \int\limits_0^{\theta_0} \sin^{n-2}\theta\cos 2\theta d\theta + \sin 2\alpha \int\limits_0^{\theta_0} \sin^{n-2}\theta \sin 2\theta d\theta \\ &= \cos 2\alpha \left[\int\limits_0^{\theta_0} \sin^{n-2}\theta\cos^2\theta d\theta - \int\limits_0^{\theta_0} \sin^n\theta d\theta \right] + 2\sin 2\alpha \int\limits_0^{\theta_0} \sin^{n-1}\theta\cos\theta d\theta \\ &= \cos 2\alpha \left[\frac{2}{n}\sin^{n-1}\theta_0\cos\theta_0 + \frac{2-n}{n} \int\limits_0^{\theta_0} \sin^{n-2}\theta d\theta \right] + \frac{2}{n}\sin 2\alpha \sin^n\theta_0\ . \end{aligned} \tag{39}$$

Substituting (39) into (38) produces

$$
\begin{aligned}
I_1(n,\theta_0) &= \frac{1}{nd(x)^2\sin^2\alpha}\left(\frac{1}{2}\left(n\left(1+\cos 2\alpha\right)-2\cos 2\alpha\right)\int\limits_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\
&\qquad\left.-\cos 2\alpha\sin^{n-1}\theta_0\cos\theta_0-\sin 2\alpha\sin^n\theta_0\right)\\
&= \frac{1}{nd(x)^2\sin^2\alpha}\left(\left(n\cos^2\alpha-\cos 2\alpha\right)\int\limits_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\
&\qquad\left.-\sin^{n-1}\theta_0(\cos 2\alpha\cos\theta_0+\sin 2\alpha\sin\theta_0)\right)\\
&= \frac{1}{nd(x)^2}\left(\left((n-1)\cot^2\alpha+1\right)\int\limits_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\
&\qquad\left.-\sin^{n-1}\theta_0\frac{\cos\left(2\alpha-\theta_0\right)}{\sin^2\alpha}\right).
\end{aligned}
\tag{40}
$$

Concerning $I_2(n,\theta_0)$, we have

$$
\begin{aligned}
I_2(n,\theta_0) &= \frac{1}{\left(h+d(x)\right)^2}\int\limits_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta\cos^2\theta d\theta\\
&= \frac{1}{\left(h+d(x)\right)^2}\left[\left.\frac{\sin^{n-1}\theta\cos\theta}{n}\right|_{\theta_0}^{\frac{\pi}{2}}+\frac{1}{n}\int\limits_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right]\\
&= \frac{1}{n\left(h+d(x)\right)^2}\left[-\sin^{n-1}\theta_0\cos\theta_0+\int\limits_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right].
\end{aligned}
\tag{41}
$$

Therefore, substituting (41) and (40) into (37) gives the following lower bound on the function $\frac{1}{m(x)^2}$:

$$
\begin{aligned}
&\frac{1}{m(x)^2}\\
&\geq \frac{2b}{n}\left[\frac{1}{d(x)^2}\left(\left((n-1)\cot^2\alpha+1\right)\int\limits_0^{\theta_0}\sin^{n-2}\theta d\theta\ -\sin^{n-1}\theta_0\frac{\cos\left(2\alpha-\theta_0\right)}{\sin^2\alpha}\right)\right.\\
&\qquad\left.+\frac{1}{\left(h+d(x)\right)^2}\left(\int\limits_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0\cos\theta_0\right)\right].
\end{aligned}
\tag{42}
$$

Apply Proposition 3.9 to the lower bound (42) to obtain the Hardy-type inequality (16) where $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$ as stated in (17) and (18) respectively. On the other hand, when $\alpha=\frac{\pi}{2}$, we have $\theta_0=\frac{\pi}{2}$ as well, this implies

$$\begin{aligned}\mu_1(n,x,\alpha,h) &= 0, \text{ and}\\ \mu_2(n,x,\alpha,h) &= \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\\ &= \frac{1}{4} \quad \text{for any } n.\end{aligned}$$

This completes the proof. ∎

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