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MOLECULAR DECOMPOSITION OF WEIGHTED ANISOTROPIC HARDY SPACES

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Abstract. In this paper, the weighted anisotropic Hardy spaces associated with the general discrete group dilations, the weighted anisotropic atoms and molecules are introduced. Then the molecular decomposition of the weighted anisotropic Hardy spaces is obtained. As an application, the boundedness of Calderón-Zygmund operators on the weighted anisotropic Hardy spaces is discussed.

1. Introduction

Harmonic analysis plays an important role in partial differential equations. The theory of Hardy spaces constitute the important part of harmonic analysis. As we know, the atomic decomposition and molecular decomposition of Hardy spaces make the linear operators acting on Hardy spaces very simple. Indeed, many problems in analysis have natural formulations as questions of continuity of linear operators defined on spaces of functions or distributions. Such questions can often be answered by rather straightforward techniques if they can first be reduced to the study of the operator on an appropriate class of simple functions which generate the entire space in some appropriate sense. This fundamental principle was applied by many authors to problems where atomic or molecular decomposition exists (see [1-5] etc.). Thus, the decompositions of function spaces are very critical in harmonic analysis.

In 2003, Bownik introduced the anisotropic Hardy spaces associated with very general discrete group dilations [6]. Such anisotropic Hardy spaces include the classical isotropic Hardy spaces and the parabolic Hardy spaces introduced by Fefferman and Stein [7], and Calderón and Torchinsky [8], respectively. Then, some authors discussed more about anisotropic function spaces, such as anisotropic Besov spaces and

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Triebel-Lizorkin spaces, weighted anisotropic Hardy spaces, and anisotropic Herz type Hardy spaces (see [9-12] etc.). So far, we haven't seen the results about molecular decompositions of the weighted anisotropic Hardy spaces. Since the anisotropic Hardy spaces, classical Hardy spaces and weighted Hardy spaces have the atomic decomposition and molecular decomposition (see [6], [13-18]), we can discuss the molecular decomposition of the weighted anisotropic Hardy spaces.

It is well known that one purpose of the atomic and molecular decompositions of spaces of functions or distributions is to prove the boundedness of linear operators on these spaces becoming simple. Therefore, as an application, we will discuss the boundedness of singular integral operators on the weighted anisotropic spaces, obtain that the Calderón-Zygmund operators are bounded on the weighted anisotropic Hardy spaces.

First of all, let us recall some basic knowledge for the anisotropic dilations (see [6]).

An $n\times n$ real matrix A is called an expansive matrix, sometimes called a dilation, if all eigenvalues λ of A satisfy $|\lambda|>1$. Suppose $\lambda_1,\cdots,\lambda_n$ are eigenvalues of A (taken according to the multiplicity) so that $1<|\lambda_1|\leqslant\cdots\leqslant|\lambda_n|$. Let λ_-,λ_+ be numbers such that $1<\lambda_-<|\lambda_1|\leqslant\cdots\leqslant|\lambda_n|<\lambda_+$. A set $\Delta\subset\mathbb{R}^n$ is said to be an ellipsoid if $\Delta=\{x\in\mathbb{R}^n:|Px|<1\}$ for some non degenerate $n\times n$ real matrix P, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. For a dilation A, there exist an ellipsoid Δ and $r>1$ such that $\Delta\subset r\Delta\subset A\Delta$. By a scaling we can assume that $|\Delta|=1$. Let $B_k=A^k\Delta$, $k\in\mathbb{Z}$. Then we have $B_k\subset rB_k\subset B_{k+1}$, and $|B_k|=b^k$, where $b=|\text{det}A|>1$. Let $\mathcal{B}$ denote the collection of dilated balls associated with the dilation A, i.e., $\mathcal{B}=\{x+B_k:x\in\mathbb{R}^n,k\in\mathbb{Z}\}$. Suppose μ is the smallest integer so that $2B_0\subset A^\mu B_0=B_\mu$. Obviously, $\mu\geqslant 1$. A homogeneous quasi-norm associated with an expansive matrix A is a measurable mapping $\rho_A:\mathbb{R}^n\to[0,\infty)$ satisfying

$$\begin{aligned}&\rho_A(x)>0, \qquad \text{for } x\neq 0,\\ &\rho_A(Ax)=|\text{det}A|\rho_A(x), \qquad \text{for } x\in\mathbb{R}^n,\\ &\rho_A(x+y)\leqslant C_A(\rho_A(x)+\rho_A(y)), \qquad \text{for } x,y\in\mathbb{R}^n,\end{aligned}$$

where $C_A>0$ is a constant. We know that all homogeneous quasi-norms associated with a fixed dilation A are equivalent (see [6]). In this paper, the step homogeneous quasi-norm ρ associated with the dilation A on $\mathbb{R}^n$ is defined by

$$\rho(x)=\begin{cases} b^j, & \text{for } x\in B_{j+1}\backslash B_j,\\ 0, & \text{for } x=0.\end{cases}$$

Then for any $x,y\in\mathbb{R}^n$, there is $\rho(x+y)\leqslant b^\mu(\rho(x)+\rho(y))$.

Suppose $S_N=\{\varphi\in\mathcal{S}(\mathbb{R}^n):\|\varphi\|_{\tau,m}\leqslant 1,|\tau|\leqslant N,m\leqslant N\}$, where $\|\varphi\|_{\tau,m}=\sup_{x\in\mathbb{R}^n}\rho(x)^m|\partial^\tau\varphi(x)|$, $N\in\mathbb{N}$. For $\varphi\in\mathcal{S}(\mathbb{R}^n)$, $k\in\mathbb{Z}$, define the dilation of φ to the scale k by $\varphi_k(x) = b^{-k}\varphi(A^{-k}x)$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The nontangential maximal function of f with respect to φ is defined by $M_\varphi f(x) = \sup\{|f * \varphi_k(y)| : x - y \in B_k, k \in \mathbb{Z}\}$. The tangential maximal function of f with respect to φ is defined as $M_\varphi^0 f(x) = \sup_{k\in\mathbb{Z}} |f * \varphi_k(x)|$. For a given non-negative integer $N \in \mathbb{N}$, the nontangential grand maximal function and the radial grand maximal function of f are defined respectively as

$$M_N f(x) = \sup_{\varphi \in S_N} M_\varphi f(x) \quad \text{and} \quad M_N^0 f(x) = \sup_{\varphi \in S_N} M_\varphi^0 f(x).$$

The radial and nontangential grand maximal functions are pointwise equivalent, i.e., for every $N \geqslant 0$, there is a constant $C = C(N)$ so that for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $M_N^0 f(x) \leqslant M_N f(x) \leqslant C M_N^0 f(x),\ x \in \mathbb{R}^n$.

Suppose

$$N_p = \begin{cases} [(1/p - 1)\ln b/\ln \lambda_-] + 2, & 0 < p \leqslant 1, \\ 2, & p > 1, \end{cases}$$

where $[t]$ denotes the biggest integer which doesn't exceed the real number t. In [6], Bownik introduced the anisotropic Hardy spaces, and established the atomic decomposition of anisotropic Hardy space. The anisotropic Hardy space associated with the dilation A is defined by

$$H_A^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : M_N f \in L^p(\mathbb{R}^n)\}, \quad \text{for } N \geqslant N_p,$$

with the quasi-norm $\|f\|_{H_A^p} = \|M_N f\|_p$, and it is independent of the choice of N as long as $N \geqslant N_p$.

Definition 1.1. [6]. Let $0 < p \leqslant 1$, $1 \leqslant q \leqslant \infty$, $p < q$, $s \in \mathbb{N}$ and $s \geqslant [(1/p - 1)\ln b/\ln \lambda_-]$. A function $a(x)$ is called a (p, q, s)-atom associated with the dilation A, if it satisfies

(1) $\operatorname{supp}(a) \subset x_0 + B_j$, for some $x_0 \in \mathbb{R}^n$, and $j \in \mathbb{Z}$;

(2) $\|a\|_q \leqslant |B_j|^{1/q - 1/p}$;

(3) $\int a(x)x^\alpha\, dx = 0$, for all $|\alpha| \leqslant s$.

Proposition 1.2. [6]. *A function $f \in H_A^p(\mathbb{R}^n)$, $0 < p \leqslant 1$, if and only if the series $f(x) = \sum_{j=1}^\infty \lambda_j a_j$ is convergence in distribution sense, where every a_j is a (p, q, s)-atom, and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Furthermore, $\|f\|_{H_A^p}^p \sim \inf \sum_{j=1}^\infty |\lambda_j|^p$, where the infimum is taken over all the decompositions of f.*

The definition of molecule and molecular decomposition of anisotropic Hardy spaces are given by Definition 1.3 and Proposition 1.4, respectively.

Definition 1.3. [13]. Let $\frac{\ln b}{\ln b + \ln \lambda_-} < p \leqslant 1 < q < \infty$, $\varepsilon > 1/p - 1$, $a = 1 - 1/p + \varepsilon$, $d = 1 - 1/q + \varepsilon$. A function $M(x) \in L^q(\mathbb{R}^n)$ is called a (p, q, ε)-molecule centered at x_0 associated with A, if it satisfies

(i) $(\rho(\cdot - x_0))^d M(\cdot) \in L^q(\mathbb{R}^n)$, for some $x_0 \in \mathbb{R}^n$;

(ii) $\mathcal{R}_q(M) = \|M\|_q^{a/d} \|(\rho(\cdot - x_0))^d M(\cdot)\|_q^{1-a/d} < +\infty$;

(iii) $\int_{\mathbb{R}^n} M(x)\, dx = 0$.

Proposition 1.4. [13]. *Let* $\frac{\ln b}{\ln b + \ln \lambda_-} < p \leqslant 1 < q < \infty$, $\varepsilon > 1/p - 1$. *A function* $f \in H^p_A(\mathbb{R}^n)$ *if and only if the series* $f(x) = \sum_{j=1}^{\infty} \lambda_j M_j(x)$ *is convergence in distribution sense, where every* M_j *is a* (p, q, ε)*-molecule, and* $\mathcal{R}_q(M_j) \leq C_0$, $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. *Moreover,* $\|f\|^p_{H^p_A} \sim \inf \sum_{j=1}^{\infty} |\lambda_j|^p$, *where the infimum is taken over all the decompositions of* f.

2. Decompositions of the Weighted Anisotropic Hardy Space

First, we introduce the Muckenhoupt class A_p on anisotropic spaces, the weighted anisotropic Hardy spaces and the atomic decomposition of weighted anisotropic Hardy space.

Definition 2.1. [10]. We call that a function $\omega : \mathbb{R}^n \to (0, \infty)$ is in the Muckenhoupt class on anisotropic space $A_q(\mathbb{R}^n, \rho)$ $(1 < q < +\infty)$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B_k|} \int_{x_0 + B_k} \omega(y) dy\right) \left(\frac{1}{|B_k|} \int_{x_0 + B_k} \omega^{-\frac{1}{q-1}}(y) dy\right)^{q-1} \leqslant C, \quad \text{for all } x_0 + B_k \in \mathcal{B}.$$

For $q = 1$, we say $\omega \in A_1(\mathbb{R}^n, \rho)$, if

$$\left(\frac{1}{|B_k|} \int_{x_0 + B_k} \omega(y) dy\right) \left(\sup_{x_0 + B_k} \omega^{-1}\right) \leqslant C, \quad \text{for all } x_0 + B_k \in \mathcal{B}.$$

And $A_\infty(\mathbb{R}^n, \rho) = \bigcup_q A_q(\mathbb{R}^n, \rho)$.

The basic properties of $A_q(\mathbb{R}^n, \rho)$ are the same as the classical Muckenhoupt A_p spaces on $\mathbb{R}^n$ (we refer to [1] and [4] etc.). For example, we can prove the doubling condition of ω as follows. Let $\omega \in A_1(\mathbb{R}^n, \rho)$, $m \in \mathbb{Z}_+, m > 0$, $B_k = A^k B_0$. Then

$$\begin{aligned}\omega(B_{k+m}) &= \int_{B_{k+m}} \omega(x)\, dx = \frac{|B_{k+m}|}{|B_{k+m}|} \int_{B_{k+m}} \omega(x)\, dx \leqslant C|B_{k+m}| \inf_{B_{k+m}} \omega(x) \\ &\leqslant C|B_{k+m}| \inf_{B_k} \omega(x) = Cb^m |B_k| \inf_{B_k} \omega(x) \leqslant Cb^m \int_{B_k} \omega(x)\, dx = C\omega(B_k).\end{aligned}$$

For convenience, in the following, we also use A_q to denote $A_q(\mathbb{R}^n, \rho)$.

Definition 2.2. [12]. Suppose $0 < p \leqslant 1$, $\omega \in A_1$, the weighted anisotropic Hardy space associated with the dilation A is defined by

$$H^p_\omega(\mathbb{R}^n) = H^p_\omega(\mathbb{R}^n, \rho) = \{f \in \mathcal{S}'(\mathbb{R}^n) : M_N f \in L^p_\omega(\mathbb{R}^n)\}, \quad N \geqslant N_p.$$

And the quasi-norm of $f \in H^p_\omega(\mathbb{R}^n)$ is $\|f\|_{H^p_\omega} = \|M_N f\|_{L^p_\omega}$.

Definition 2.3. [12]. Suppose $\omega \in A_1$, $0 < p \leqslant 1 < q \leqslant \infty$, $s \geqslant [(1/p - 1)\ln b/\ln \lambda_-]$ is a non-negative integer. A function $a(x)$ is said to be a (p, q, s, ω)-atom with center $x_0 \in \mathbb{R}^n$, if it satisfies

(i) $\operatorname{supp} a(x) \subset x_0 + B_k$, for some $k \in \mathbb{Z}$;

(ii) $\|a\|_{L^q_\omega} \leqslant [\omega(x_0 + B_k)]^{1/q - 1/p}$;

(iii) $\int a(x) x^\nu dx = 0$, for all ν with $|\nu| \leqslant s$.

The atomic decomposition of the weighted anisotropic Hardy space is the following theorem.

Theorem 2.4. [12]. *Let $\omega \in A_1$, $0 < p \leqslant 1 < q \leqslant \infty$. Suppose that $s \geqslant [(1/p - 1)\ln b/\ln \lambda_-]$ is a non-negative integer. Then $f \in H^p_\omega(\mathbb{R}^n)$ if and only if $f(x) = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x)$ is convergence in distribution sense, where every a_k is a (p, q, s, ω)-atom, and $\sum_{k=-\infty}^{+\infty} |\lambda_k|^p < \infty$. Moreover, $\|f\|^p_{H^p_\omega} \sim \inf \sum_{k=-\infty}^{\infty} |\lambda_k|^p$, where the infimun is taken over all the decompositions of f.*

Now, we discuss the molecular decomposition of the weighted anisotropic Hardy space. For this purpose, we should introduce the molecule on the weighted anisotropic spaces.

Definition 2.5. Let ω be in the anisotropic Muckenhoupt class A_1. Suppose $\frac{\ln b}{\ln b + \ln \lambda_-} < p \leqslant 1 < q < +\infty$, $\varepsilon > 1/p - 1/q$, $a_0 = 1 - 1/p + \varepsilon$, $d = 1 - 1/q + \varepsilon$. A function $M \in L^q_\omega(\mathbb{R}^n)$ is called a $(p, q, \varepsilon, \omega)$-molecule with center x_0, if it satisfies

(i) $\omega(B_{x_0}(x))^d M(x) \in L^q_\omega(\mathbb{R}^n)$;

(ii) $\Re_{q,\omega}(M) = \|M\|^{a_0/d}_{L^q_\omega} \|\omega(B_{x_0}(x))^d M(x)\|^{1 - a_0/d}_{L^q_\omega} < \infty$;

(iii) $\int_{\mathbb{R}^n} M(x) dx = 0$;

where $B_{x_0}(x) = x_0 + B_{[\log_b \rho(x - x_0)]} = x_0 + A^{[\log_b \rho(x - x_0)]} B_0$.

Similar to the classical situation, we can prove that every weighted atom is also a weighted molecule.

Proposition 2.6. *Suppose that p, q, ε and ω are the same as in Definition 2.5, $s \geqslant [(1/p - 1)\ln b/\ln \lambda_-]$. If $a(x)$ is a (p, q, s, ω)-atom with center x_0, then $a(x)$ is a $(p, q, \varepsilon, \omega)$-molecule with center x_0, and $\Re_{q,\omega}(a) \leqslant C$.*

Proof. By the definitions, we only need to check the condition (ii) in Definition 2.5 for a. Let $\operatorname{supp} a \subset x_0 + B_k$. It is easy to see that $B_{x_0}(x) \subset x_0 + B_k$, for any $x \in x_0 + B_k$. Thus

$$\begin{aligned}\Re_{L^q_\omega}(a) &\leqslant \|a\|_{L^q_\omega}^{a_0/d}\Big(\int_{x_0+B_k} \omega(B_{x_0}(x))^{dq}|a(x)|^q\omega(x)dx\Big)^{1/q(1-a_0/d)}\\ &\leqslant C\|a\|_{L^q_\omega}^{a_0/d}\omega(x_0+B_k)^{d(1-a_0/d)}\|a\|_{L^q_\omega}^{1-a_0/d} \leqslant C,\end{aligned}$$

where C is independent of a. ■

The molecular decomposition of weighted anisotropic Hardy space is as follows.

Theorem 2.7. *Suppose $\omega \in A_1$, $\frac{\ln b}{\ln b+\ln \lambda_-} < p \leqslant 1 < q < +\infty$, $\varepsilon > 1/p - 1/q$. Then $f \in H^p_\omega(\mathbb{R}^n)$ if and only if $f = \sum_{k=-\infty}^{\infty} \mu_k M_k$ is convergence in distribution sense, where every M_k is a $(p, q, \varepsilon, \omega)$-molecule, $\Re_q(M_k) \leqslant C_0$, C_0 is a constant independent of M_k, and $\sum_{k=-\infty}^{\infty} |\mu_k|^p < \infty$. Furthermore, $\|f\|^p_{H^p_\omega} \sim \inf \sum_{k=-\infty}^{\infty} |\mu_k|^p$, where the infimun is taken over all the decompositions of f.*

According to Proposition 2.6, the necessity of Theorem 2.7 is included in the atomic decomposition Theorem 2.4. Thus, to prove Theorem 2.7, we only need to prove the sufficiency. Obviously, the result of the following proposition is enough for the sufficiency of Theorem 2.7.

Proposition 2.8. *Let p, q, ε, ω be the same as in Definition 2.5. If M is a $(p, q, \varepsilon, \omega)$-molecule, then $M \in H^p_\omega(\mathbb{R}^n)$ and $\|M\|_{H^p_\omega(\mathbb{R}^n)} \leqslant C$, where C is independent of M.*

To prove Proposition 2.8, we need to introduce the weighted Campanato spaces.

Definition 2.9. Suppose $\omega \in A_1$, $0 < p \leqslant 1 < q < \infty$, the weighted Campanato space is the collection of all local L^q functions g on $\mathbb{R}^n$ satisfying

$$\begin{aligned}&\|g\|_{C^{1/p-1,q',0}_\omega}\\ &= \sup_{x_0+B_k\in\mathcal{B}} \omega(x_0+B_k)^{1/q-1/p}\Big[\int_{x_0+B_k} |g(x) - \pi^0_{B_k} g(x)|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'} < \infty,\end{aligned}$$

where $\pi^0_{B_k} : L^1(x_0 + B_k) \to \mathcal{P}_0$ (where $f \in L^1(x_0+B_k)$ means $\int_{x_0+B_k} |f(x)|dx < \infty$ and $\mathcal{P}_0$ is the linear space of constant functions) is the natural mapping in $\mathcal{P}_0$ determined by the Riesz lemma

$$\begin{aligned}&\int_{x_0+B_k} (\pi^0_{B_k} f(x))Q(x)\,dx\\ &= \int_{x_0+B_k} f(x)Q(x)\,dx,\ f \in L^1(x_0 + B_k), Q \in \mathcal{P}_0,\end{aligned}$$

and $|\pi^0_{B_k} f| \leqslant C|B_k|^{-1} \int_{x_0+B_k} |f(x)|dx$.

In fact, there are other two equivalent norms of f in $C^{1/p-1,q',0}_{\omega}$.

Proposition 2.10. *Suppose* $0 < p \leqslant 1 < q < \infty$, $\varepsilon > 1/p - 1/q$, $\omega \in A_1$. *Then the following three results are equivalent.*

(i) $\|g\|_{C^{1/p-1,q',0}_{\omega}} < \infty$.

(ii) $\|g\|^{*}_{C^{1/p-1,q',0}_{\omega}}$

$$= \sup_{x_0+B_k\in\mathcal{B}} \omega(x_0+B_k)^{\frac{1}{q}-\frac{1}{p}}[\inf_{P\in\mathcal{P}_0} \int_{x_0+B_k} |g(x)-P|^{q'}\omega^{-q'/q}(x)dx]^{1/q'} < \infty.$$

(iii) $\|g\|^{**}_{C^{1/p-1,q',0}_{\omega}} = \sup_{x_0+B_k\in\mathcal{B}} \omega(x_0+B_k)^{1-\frac{1}{p}}$

$$\left\{ \inf_{c\in\mathbb{R}} \int_{\mathbb{R}^n} \left(\frac{\omega(x_0+B_k)^{\varepsilon}|g(x)-c|}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1/q'+\varepsilon}} \right)^{q'} \omega(x)^{-q'/q}dx \right\}^{1/q'} < \infty.$$

To prove Proposition 2.10, we should point out that: if $\omega \in A_1$, then $\sigma = \omega^{1-q'} = \omega^{-q'/q} \in A_{q'}$, $1/q + 1/q' = 1$, $1 < q < \infty$, and for any non-negative integer m, there exists a constant δ, $0 < \delta < 1$, such that $\omega(x_0+B_k)/\omega(x_0+B_{k+m}) \leqslant Cb^{-m}$ and $\omega(x_0+B_{k+m})^{-q'/q} \geqslant Cb^{m\delta}\omega(x_0+B_k)^{-q'/q}$. In fact, $\dfrac{1}{q-1} = q'-1$, and $\omega \in A_1 \subset A_q$, there is

$$\left(\frac{1}{|B_k|}\int_{x_0+B_k} \omega^{1-q'}dx\right)\left(\frac{1}{|B_k|}\int_{x_0+B_k} (\omega^{1-q'}\right)^{\frac{-1}{q'-1}} dx)^{q'-1}$$

$$= \left(\frac{1}{|B_k|}\int_{x_0+B_k} \omega^{\frac{-1}{q-1}}dx\right)\left(\frac{1}{|B_k|}\int_{x_0+B_k} \omega dx\right)^{q'-1}$$

$$= \left(\left(\frac{1}{|B_k|}\int_{x_0+B_k} \omega^{-\frac{1}{q-1}}dx\right)^{q-1}\left(\frac{1}{|B_k|}\int_{x_0+B_k} \omega dx\right)\right)^{q'-1} \leqslant C.$$

Thus $\sigma = \omega^{1-q'} \in A_{q'}$. Similar to the classical result, there exists a constant δ, $0 < \delta < 1$, such that $\omega(x_0+B_k)/\omega(x_0+B_{k+m}) \leqslant C|B_k|/|B_{k+m}| \leqslant Cb^{-m}$ and $\sigma(x_0+B_k)/\sigma(x_0+B_{k+m}) \leqslant C(|B_k|/|B_{k+m}|)^{\delta} \leqslant Cb^{-m\delta}$, hence, $\omega(x_0+B_{k+m})^{1-q'} \geqslant Cb^{m\delta}\omega(x_0+B_k)^{1-q'}$.

Proof of Proposition 2.10. Obviously, $\|g\|^{*}_{C^{1/p-1,q',0}_{\omega}} \leqslant \|g\|_{C^{1/p-1,q',0}_{\omega}}$, that is $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. For any $P \in \mathcal{P}_0$, there is

$$\begin{aligned}
&\Big[\int_{x_0+B_k}|g(x)-\pi^0_{B_k}g(x)|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\leqslant\Big[\int_{x_0+B_k}|g(x)-P|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\quad+\Big[\int_{x_0+B_k}|P-\pi^0_{B_k}g(x)|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\leqslant\Big[\int_{x_0+B_k}|g(x)-P|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\quad+|\pi^0_{B_k}(P-g)|\Big[\int_{x_0+B_k}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\leqslant\Big[\int_{x_0+B_k}|g(x)-P|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\quad+C\frac{1}{|B_k|}\int_{x_0+B_k}|P-g|dx\Big[\int_{x_0+B_k}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\leqslant\Big[\int_{x_0+B_k}|g(x)-P|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\quad+C\omega(x_0+B_k)^{-1/q}\int_{x_0+B_k}|P-g|\omega^{-1/q}(x)\omega^{1/q}(x)dx\\
&\leqslant\Big[\int_{x_0+B_k}|g(x)-P|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\quad+C\Big[\int_{x_0+B_k}|P-g|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}\\
&\leqslant C\Big[\int_{x_0+B_k}|g(x)-P|^{q'}\omega^{-q'/q}(x)dx\Big]^{1/q'}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big[\int_{x_0+B_k}|g(x)-\pi^0_{B_k}g(x)|^{q'}\omega^{-q'/q}(x)dx]^{1/q'}\\
&\leqslant C\inf_{P\in\mathcal{P}_0}\Big[\int_{x_0+B_k}|g\,(x)-P|^{q'}\omega^{-q'/q}\,(x)\,dx]^{1/q'}.
\end{aligned}$$

Therefore

$$\|g\|_{C^{1/p-1,q',0}_{\omega}}\leqslant C\|g\|^*_{C^{1/p-1,q',0}_{\omega}}.$$

Next to prove $(iii)\ \Rightarrow (ii)$. Since

$$\begin{aligned}
&\omega(x_0+B_k)^{1-\frac{1}{p}}\{\inf_{c\in\mathbb{R}}\int_{\mathbb{R}^n}(\frac{\omega(x_0+B_k)^{\varepsilon}|g(x)-c|}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1/q'+\varepsilon}})^{q'}\omega(x)^{-q'/q}dx\}^{1/q'}\\
&=\inf_{P\in\mathcal{P}_0}\{\int_{\mathbb{R}^n}\frac{\omega(x_0+B_k)^{\varepsilon q'+(1-\frac{1}{p})q'}|g(x)-P|^{q'}}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1+\varepsilon q'}}\omega(x)^{-q'/q}dx\}^{1/q'}\\
&\geqslant\inf_{P\in\mathcal{P}_0}\{\int_{x_0+B_k}\frac{\omega(x_0+B_k)^{\varepsilon q'+(1-\frac{1}{p})q'}|g(x)-P|^{q'}}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1+\varepsilon q'}}\omega(x)^{-q'/q}dx\}^{1/q'}
\end{aligned}$$

$$\geqslant C\inf_{P\in\mathcal{P}_0}\{\int_{x_0+B_k}\frac{\omega(x_0+B_k)^{(1-\frac{1}{p})q'}|g(x)-P|^{q'}}{\omega(x_0+B_k)}\omega(x)^{-q'/q}dx\}^{1/q'}$$
$$\geqslant C\omega(x_0+B_k)^{(\frac{1}{q}-\frac{1}{p})}\{\inf_{P\in\mathcal{P}_0}\int_{x_0+B_k}|g(x)-P|^{q'}\omega(x)^{-q'/q}dx\}^{1/q'},$$

we obtain

$$\|g\|^{**}_{C_\omega^{1/p-1,q',0}}$$
$$\geqslant C\sup_{x_0+B_k\in\mathcal{B}}\omega(x_0+B_k)^{(\frac{1}{q}-\frac{1}{p})}\{\inf_{P\in\mathcal{P}_0}\int_{x_0+B_k}|g(x)-P|^{q'}\omega(x)^{-q'/q}dx\}^{1/q'}$$
$$=C\|g\|^{*}_{C_\omega^{1/p-1,q',0}}.$$

Finally, we prove $(i)\ \Rightarrow (iii)$. Let $\|g\|_{C_\omega^{1/p-1,q',0}}<\infty$. For any $x_0+B_k\in\mathcal{B}$, by using the conclusion mentioned above, there is

$$\omega(x_0+B_k)^{1-\frac{1}{p}}\{\inf_{c\in\mathbb{R}}\int_{\mathbb{R}^n}(\frac{\omega(x_0+B_k)^{\varepsilon}|g(x)-c|}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1/q'+\varepsilon}})^{q'}\omega^{-q'/q}(x)dx\}^{1/q'}$$
$$\leqslant\omega(x_0+B_k)^{1-\frac{1}{p}}\{\int_{\mathbb{R}^n}(\frac{\omega(x_0+B_k)^{\varepsilon}|g(x)-\pi^0_{B_k}g|}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1/q'+\varepsilon}})^{q'}\omega^{-q'/q}(x)dx\}^{1/q'}$$
$$\leqslant C\omega(x_0+B_k)^{1-\frac{1}{p}}\{[\omega(x_0+B_k)^{-1}\int_{x_0+B_k}|g(x)-\pi^0_{B_k}g|^{q'}\omega^{-q'/q}(x)dx]^{1/q'}$$
$$+\sum_{m=1}^{\infty}[\int_{x_0+B_{k+m}\backslash B_{k+m-1}}|g(x)-\pi^0_{B_k}g|^{q'}\frac{\omega(x_0+B_k)^{\varepsilon q'}\omega^{-q'/q}(x)}{(\omega(x_0+B_k)+\omega(B_{x_0}(x)))^{1+\varepsilon q'}}dx]^{1/q'}\}$$
$$\leqslant C\omega(x_0+B_k)^{1-\frac{1}{p}}\{[\omega(x_0+B_k)^{-1}\int_{x_0+B_k}|g(x)-\pi^0_{B_k}g|^{q'}\omega^{-q'/q}(x)dx]^{1/q'}$$
$$+\omega(x_0+B_k)^{\varepsilon}\sum_{m=1}^{\infty}[\omega(x_0+B_{k+m-1})^{-1-\varepsilon q'}$$
$$\int_{x_0+B_{k+m}}|g(x)-\pi^0_{B_k}g|^{q'}\omega^{-q'|/q}(x)dx]^{1/q'}\}$$
$$\leqslant C\omega(x_0+B_k)^{1-\frac{1}{p}}\sum_{m=0}^{\infty}b^{-m\varepsilon}[\omega(x_0+B_{k+m})^{-1}$$
$$\int_{x_0+B_{k+m}}|g(x)-\pi^0_{B_k}g|^{q'}\omega^{-q'/q}(x)dx]^{1/q'}$$
$$\leqslant C\omega(x_0+B_k)^{1-\frac{1}{p}}\sum_{m=0}^{\infty}b^{-m\varepsilon}[\omega(x_0+B_{k+m})^{-1}$$
$$\int_{x_0+B_{k+m}}|g(x)-\pi^0_{B_{k+m}}g|^{q'}\omega^{-q'/q}(x)dx]^{1/q'}$$

$$+C\omega(x_0+B_k)^{1-\frac{1}{p}}\sum_{m=0}^{\infty}b^{-m\varepsilon}[\omega(x_0+B_{k+m})^{-1}$$
$$\int_{x_0+B_{k+m}}|\pi^0_{B_{k+m}}g-\pi^0_{B_k}g|^{q'}\omega^{-q'/q}(x)dx]^{1/q'}$$
$$=\mathrm{I}+\mathrm{II}.$$

Notice that $\varepsilon>1/p-1/q>1/p-1$, then

$$\mathrm{I}\leqslant C\|g\|_{C^{1/p-1,q',0}_\omega}\sum_{m=0}^{\infty}b^{-m\varepsilon}\omega(x_0+B_k)^{1-1/p}b^{m(1/p-1)}\omega(x_0+B_k)^{1/p-1}$$
$$\leqslant C\|g\|_{C^{1/p-1,q',0}_\omega}\sum_{m=0}^{\infty}b^{m[-\varepsilon+(1/p-1)]}\leqslant C\|g\|_{C^{1/p-1,q',0}_\omega}.$$

Also, by using the conclusion mentioned above, we obtain

$$|\pi^0_{B_{k+m}}g-\pi^0_{B_k}g|\leqslant\sum_{j=0}^{m-1}|\pi^0_{B_{k+j+1}}g-\pi^0_{B_{k+j}}g|$$
$$\leqslant\sum_{j=0}^{m-1}[(\int_{x_0+B_{k+j}}\omega(x)^{-q'/q}dx)^{-1}(\int_{x_0+B_{k+j}}|\pi^0_{B_{k+j+1}}g-\pi^0_{B_{k+j}}g|^{q'}\omega(x)^{-q'/q}dx)]^{1/q'}$$
$$\leqslant C\sum_{j=0}^{m-1}\{[(\int_{x_0+B_{k+j}}\omega^{-q'/q}dx)^{-1}(\int_{x_0+B_{k+j}}|\pi^0_{B_{k+j+1}}g-g|^{q'}\omega(x)^{-q'/q}dx)]^{1/q'}$$
$$+[(\int_{x_0+B_{k+j}}\omega(x)^{-q'/q}dx)^{-1}(\int_{x_0+B_{k+j}}|\pi^0_{B_{k+j}}g-g|^{q'}\omega(x)^{-q'/q}dx)]^{1/q'}\}$$
$$\leqslant C\sum_{j=0}^{m-1}(\int_{x_0+B_{k+j}}\omega(x)^{-q'/q}dx)^{-1/q'}\|g\|_{C^{1/p-1,q',0}_\omega}$$
$$[\omega(x_0+B_{k+j+1})^{1/p-1/q}+\omega(x_0+B_{k+j})^{1/p-1/q}]$$
$$\leqslant C\sum_{j=0}^{m-1}[\omega(x_0+B_k)^{-q'/q}]^{-1/q'}b^{-j\delta/q'}\|g\|_{C^{1/p-1,q',0}_\omega}b^{(j+1)(1/p-1/q)}\omega(x_0+B_k)^{1/p-1/q}$$
$$=C\|g\|_{C^{1/p-1,q',0}_\omega}\omega(x_0+B_k)^{1/p}\sum_{j=0}^{m-1}b^{j(1/p-1/q-\delta/q')}.$$

Hence

$$\mathrm{II}\leqslant C\omega(x_0+B_k)^{1-1/p}\sum_{m=0}^{\infty}b^{-m\varepsilon}\omega(x_0+B_{k+m})^{-1/q'}|\pi^0_{B_{k+m}}g-\pi^0_{B_k}g|$$
$$(\int_{x_0+B_{k+m}}\omega^{-q'/q}(x)dx)^{1/q'}$$
$$\leqslant C\|g\|_{C^{1/p-1,q',0}_\omega}\sum_{m=0}^{\infty}b^{-m\varepsilon-m\delta}\sum_{j=0}^{m-1}b^{j(1/p-1/q)-j\delta/q'}.$$

Since $\varepsilon > 1/p - 1/q$, there is

$$\mathrm{II} \leqslant C\|g\|_{C_\omega^{1/p-1,q',0}} \sum_{m=0}^{\infty} b^{-m\varepsilon - m\delta + m(1/p-1/q-\delta/q')} \leqslant C\|g\|_{C_\omega^{1/p-1,q',0}}.$$

Consequently,

$$\|g\|^{**}_{C_\omega^{1/p-1,q',0}} \leqslant C\|g\|_{C_\omega^{1/p-1,q',0}}.$$

The proof of Proposition 2.10 is completed. ■

Similar to [6] and [18], we can prove that the dual of weighted anisotropic Hardy space is characterized by weighted Campanato functions. In other words, the dual of $H^p_\omega(\mathbb{R}^n,\rho)$ is contained in $C_\omega^{1/p-1,q',0}$. Here, we omit the detail.

Proof of Proposition 2.8. For the molecule M, let $r = \|M\|_{L^q_\omega}^{\frac{1}{1/q-1/p}}$. Suppose that l_r is the only one integer satisfying $\omega(x_0 + B_{l_r}) < r \leqslant \omega(x_0 + B_{l_r+1})$. Denote $E_0 = x_0 + B_{l_r}$, $E_k = x_0 + B_{l_r+k}\backslash B_{l_r+k-1}$, where $k \in \mathbb{N}$. Set

$$\varphi_k(x) = M(x)\chi_{E_k}(x) - \frac{\chi_{E_k}(x)}{|E_k|}\int_{\mathbb{R}^n} M(y)\chi_{E_k}(y)dy,\ k \in \mathbb{N}\cup\{0\}.$$

Then

$$M(x) = \sum_{k=0}^{\infty}\varphi_k(x) + \sum_{k=0}^{\infty}\frac{\chi_{E_k}(x)}{|E_k|}\int_{\mathbb{R}^n} M(y)\chi_{E_k}(y)dy. \tag{2.1}$$

Obviously, $\mathrm{supp}\varphi_k \subset x_0 + B_{l_r+k}$, $\int_{\mathbb{R}^n}\varphi_k(x)dx = 0$. Moreover,

$$\begin{aligned}
\|\varphi_0\|_{L^q_\omega} &\leqslant C\{\|M\chi_{E_0}\|_{L^q_\omega} + \frac{1}{|B_{l_r}|}\int_{\mathbb{R}^n}|M(y)\chi_{E_0}(y)|dy\|\chi_{E_0}(x)\|_{L^q_\omega}\}\\
&\leqslant C\{\|M\|_{L^q_\omega} + \frac{1}{\omega(x_0+B_{l_r})}\int_{\mathbb{R}^n}|M(y)\chi_{E_0}(y)|\omega(y)dy\omega(x_0+B_{l_r})^{1/q}\}\\
&\leqslant C\{\|M\|_{L^q_\omega} + \omega(x_0+B_{l_r})^{1/q-1}\int_{x_0+B_{l_r}}|M(y)|\omega(y)dy\}\\
&\leqslant C\{\|M\|_{L^q_\omega} + \omega(x_0+B_{l_r})^{1/q-1}\|M\|_{L^q_\omega}\omega(x_0+B_{l_r})^{1-1/q}\}\\
&\leqslant C\|M\|_{L^q_\omega} = Cr^{1/q-1/p} \leqslant \omega(x_0+B_{l_r})^{1/q-1/p}.
\end{aligned}$$

For $k > 0$, there is

$$\begin{aligned}
\|\varphi_k\|_{L^q_\omega} &\leqslant C\{\int_{\mathbb{R}^n}|M(x)\chi_{E_k}(x)|^q\omega(x)dx\\
&\quad + \int_{\mathbb{R}^n}\chi_{E_k}{}^q(x)(\frac{1}{|E_k|}\int_{\mathbb{R}^n}|M(y)\chi_{E_k}(y)|dy)^q\omega(x)dx\}^{1/q}\\
&\leqslant C\{\int_{E_k}|M(x)|^q\omega(x)dx + \omega(E_k)^{1-q}(\int_{E_k}|M(y)|\omega(y)dy)^q\}^{1/q}
\end{aligned}$$

$$\leqslant C\{\int_{E_k}|M(x)|^q\omega(x)dx+\int_{E_k}|M(y)|^q\omega(y)dy\}^{1/q}$$
$$\leqslant C\{\int_{\mathbb{R}^n}|M(x)|^q\omega(B_{x_0}(x))^{dq}\omega(B_{x_0}(x))^{-dq}\chi_{E_k}(x)\omega(x)dx\}^{1/q}$$
$$\leqslant C\omega(x_0+B_{l_r+k-1})^{-d}\{\int_{\mathbb{R}^n}|M(x)|^q\omega(B_{x_0}(x))^{dq}\omega(x)dx\}^{1/q}$$
$$\leqslant C\omega(x_0+B_{l_r+k-1})^{-d}\|M(x)\omega(B_{x_0}(x))^d\|_{L^q_\omega}.$$

Since $\Re_{L^q_\omega}(M)\leq C_0$, there is $\|\omega(B_{x_0}(x))^dM(x)\|_{L^q_\omega}\leqslant C_0{}^{\frac{1}{1-a_0/d}}\|M\|_{L^q_\omega}^{\frac{-a_0/d}{1-a_0/d}}=C_0{}^{\frac{1}{1-a_0/d}}\|M\|_{L^q_\omega}^{\frac{a_0}{1/q-1/p}}$. Thus

$$\|\varphi_k\|_{L^q_\omega}\leqslant C\omega(x_0+B_{l_r+k-1})^{-d}C_0{}^{\frac{1}{1-a_0/d}}\|M\|_{L^q_\omega}^{\frac{a_0}{1/q-1/p}}$$
$$\leqslant C\omega(x_0+B_{l_r+k-1})^{-d}C_0{}^{\frac{1}{1-a_0/d}}r^{a_0}$$
$$\leqslant C\omega(x_0+B_{l_r+k})^{-d}b^dC_0{}^{\frac{1}{1-a_0/d}}\omega(x_0+B_{l_r+1})^{a_0}$$
$$\leqslant C\omega(x_0+B_{l_r+k})^{-d}b^dC_0{}^{\frac{1}{1-a_0/d}}b^{-(k-1)a_0\delta}\omega(x_0+B_{l_r+k})^{a_0}$$
$$\leqslant Cb^{-ka_0\delta}\omega(x_0+B_{l_r+k})^{1/q-1/p}.$$

Therefore, if we denote $\lambda_{1,k}=Cb^{-ka_0\delta}$, $a_{1,k}=\varphi_k/\lambda_{1,k}$, then $a_{1,k}$ is a $(p,q,0,\omega)$-atom with center x_0, moreover,

$$\sum_{k=0}^\infty\varphi_k(x)=\sum_{k=0}^\infty\lambda_{1,k}a_{1,k}(x),\quad\text{and}\quad\sum_{k=0}^\infty|\lambda_{1,k}|^p\leqslant C\sum_{k=0}^\infty b^{-ka_0\delta p}\leqslant C,$$

where C is a constant independent of M.

For the other part in (2.1), suppose $m_k=\sum_{i=k}^\infty\int_{\mathbb{R}^n}M(x)\chi_{E_i}(x)dx$, $\psi_k(x)=|E_k|^{-1}\chi_{E_k}(x)$. Notice that $m_0=\sum_{i=0}^\infty\int_{\mathbb{R}^n}M(x)\chi_{E_i}(x)dx=\int_{\mathbb{R}^n}M(x)dx=0$, we can rewrite

$$\sum_{k=0}^\infty\frac{\chi_{E_k}(x)}{|E_k|}\int_{\mathbb{R}^n}M(y)\chi_{E_k}(y)dy=\sum_{k=0}^\infty\psi_k(x)(m_k-m_{k+1})$$
$$=\sum_{k=0}^\infty m_{k+1}(\psi_{k+1}(x)-\psi_k(x)).$$

Obviously,

$$\int_{\mathbb{R}^n}m_{k+1}(\psi_{k+1}(x)-\psi_k(x))dx=m_{k+1}(\int_{\mathbb{R}^n}\frac{\chi_{E_{k+1}}(x)}{|E_{k+1}|}dx-\int_{\mathbb{R}^n}\frac{\chi_{E_k}(x)}{|E_k|}dx)=0,$$

and $\operatorname{supp}\{m_{k+1}(\psi_{k+1}(x)-\psi_k(x))\}\subset x_0+B_{l_r+k+1}$. Moreover,

$$
\begin{aligned}
&\|m_{k+1}(\psi_{k+1}-\psi_k)\|_{L^q_\omega} \leqslant C|m_{k+1}|(\|\psi_{k+1}\|_{L^q_\omega}+\|\psi_k\|_{L^q_\omega})\\
&\leqslant C|m_{k+1}|(\frac{\omega(x_0+B_{l_r+k+1})^{1/q}}{|E_{k+1}|}+\frac{\omega(x_0+B_{l_r+k})^{1/q}}{|E_k|})\\
&\leqslant C(1+Cb^{1-\delta/q})\frac{\omega(x_0+B_{l_r+k+1})^{1/q}}{|E_{k+1}|}\int_{(x_0+B_{l_r+k})^c}|M(x)|dx\\
&\leqslant C\frac{\omega(x_0+B_{l_r+k+1})^{1/q}}{|E_{k+1}|}\|\omega(B_{x_0}(x))^d M(x)\|_{L^q_\omega}\\
&\quad(\int_{(x_0+B_{l_r+k})^c}\omega(B_{x_0}(x))^{-dq'}\omega(x)^{-q'/q}dx)^{1/q'}\\
&\leqslant Cb^{-k}\omega(x_0+B_{l_r+k+1})^{1/q}C_0^{\frac{1}{1-a_0/d}}\|M\|_{L^q_\omega}^{\frac{a_0}{1/q-1/p}}\omega(x_0+B_{l_r+k+1})^{-d}b^d\\
&\quad(\sum_{i=k+1}^{\infty}\int_{E_i}\omega(x)^{-q'/q}dx)^{1/q'}\\
&\leqslant C\omega(x_0+B_{l_r+k+1})^{1/q}\omega(x_0+B_{l_r+k+1})^{a_0}b^{-k\delta a_0}\omega(x_0+B_{l_r+k+1})^{-d}\\
&\quad\sum_{i=k+1}^{\infty}\omega(E_{k+1})^{-1/q}b^{-(i-k-1)\delta/q}\\
&\leqslant Cb^{-ka_0\delta}\omega(x_0+B_{l_r+k+1})^{1/q-1/p},
\end{aligned}
$$

where C is independent of k. Thus, if we write $\lambda_{2,k}=Cb^{-ka_0\delta}$, $a_{2,k}=m_{k+1}(\psi_{k+1}-\psi_k)/\lambda_{2,k}$, then

$$\sum_{k=0}^{\infty}\frac{\chi_{E_k}(x)}{|E_k|}\int_{\mathbb{R}^n}M(y)\chi_{E_k}(y)dy=\sum_{k=0}^{\infty}\lambda_{2,k}a_{2,k}(x),$$

and every $a_{2,k}$ is a $(p,q,0,\omega)$-atom. Furthermore, $\sum_{k=0}^{\infty}|\lambda_{2,k}|^p\leqslant C\sum_{k=0}^{\infty}b^{-ka_0\delta p}\leqslant C$. Hence, we obtain the decomposition of M as follows,

$$M(x)=\sum_{k=0}^{\infty}[\varphi_k(x)+\frac{\chi_{E_k}(x)}{|E_k|}\int_{\mathbb{R}^n}M(y)\chi_{E_k}(y)dy]=\sum_{k=0}^{\infty}\lambda_{1,k}a_{1,k}(x)+\sum_{k=0}^{\infty}\lambda_{2,k}a_{2,k}(x).$$

To prove $M\in H^p_\omega(\mathbb{R}^n,\rho)$, we only need to prove that for any function $g\in C_\omega^{1/p-1,q',0}$, there is

$$\int_{\mathbb{R}^n}M(x)g(x)dx=\lim_{m\to\infty}\sum_{k=0}^{m}\int_{\mathbb{R}^n}(\lambda_{1,k}a_{1,k}(x)+\lambda_{2,k}a_{2,k}(x))g(x)dx. \tag{2.2}$$

By Proposition 2.10, if $g\in C_\omega^{1/p-1,q',0}$, then there exists $c\in\mathbb{R}$, so that $\frac{g(x)-c}{\omega(x)}[1+\omega(B_{x_0}(x))]^{-d}\in L^{q'}_\omega$, where $d=1/q'+\varepsilon$. Obviously $\frac{c}{\omega(x)}[1+\omega(B_{x_0}(x))]^{-d}\in$

$L^{q'}_\omega$. Thus $\frac{g(x)}{\omega(x)}[1+\omega(B_{x_0}(x))]^{-d} \in L^{q'}_\omega$. Therefore, by the scale condition of molecule, we have

$$\begin{aligned}|\int_{\mathbb{R}^n} M(x)g(x)dx| &\leqslant \int_{\mathbb{R}^n} |M(x)|[1+\omega(B_{x_0}(x))]^d \frac{|g(x)|/\omega(x)}{[1+\omega(B_{x_0}(x))]^d}\omega(x)dx \\ &\leqslant (\int_{\mathbb{R}^n}|M(x)[1+\omega(B_{x_0}(x))]^d|^q\omega(x)dx)^{1/q}(\int_{\mathbb{R}^n}|\frac{|g(x)|/\omega(x)}{[1+\omega(B_{x_0}(x))]^d}|^{q'}\omega(x)dx)^{1/q'}<\infty.\end{aligned}$$

That means that the left hand integral of (2.2) makes sense. By Proposition 2.6, the right hand of (2.2) is also meaningful. Then, for $x \in x_0 + B_{l_r+m+1}$,

$$M(x) = \sum_{k=0}^{m} (\lambda_{1,k}a_{1,k}(x) + \lambda_{2,k}a_{2,k}(x)).$$

Hence

$$\int_{x_0+B_{l_r+m+1}} M(x)g(x)dx = \sum_{k=0}^{m} \int_{x_0+B_{l_r+m+1}} (\lambda_{1,k}a_{1,k}(x) + \lambda_{2,k}a_{2,k}(x))g(x)dx.$$

So, if $m \to \infty$, (2.2) is proved. It is the end of proof of Proposition 2.8. ∎

3. Boundedness of the Calderón-Zygmund Operators

In this section, we will give an application of the atomic and molecular decompositions of weighted anisotropic Hardy spaces. The boundedness of the Calderón-Zygmund operators on weighted anisotropic Hardy spaces is obtained.

Bownik defined the Calderón-Zygmund operators associated with dilation A and the quasi-norm ρ in [6] as follows.

Definition 3.1. [6]. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear operator. We call that T is a Calderón-Zygmund operator (associated with dilation A and quasi-norm ρ), if there exist $C > 0$ and $\gamma > 0$, such that

(i) K, the kernel of T, is defined on $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ satisfying

(a) $|K(x, y)| \leqslant C\rho(x-y)^{-1}$;

(b) if $(x, y) \in \Omega$, and $\rho(x'-y) \leqslant \rho(x-y)/b^{2\mu}$, then $|K(x', y) - K(x, y)| \leqslant C\rho(x-x')^\gamma \rho(x-y)^{-1-\gamma}$;

(c) if $(x, y) \in \Omega$, and $\rho(y-y') \leqslant \rho(x-y)/b^{2\mu}$, then $|K(x, y') - K(x, y)| \leqslant C\rho(y'-y)^\gamma \rho(x-y)^{-1-\gamma}$.

(ii) T can be extended to a continuous linear operator on L^2, and $\|T\| \leqslant C$.

Bownik proved that the Calderón-Zygmund operators associated with the dilation A are bounded on $L^q(\mathbb{R}^n)$ $(1 < q < +\infty)$. By increasing the smoothness on the kernel K, he obtained the boundedness of the Calderón-Zygmund operators on the anisotropic Hardy spaces[6]. Similar to the proof of Bownik used in [6], or as a special case of homogenous space, we can obtain the boundedness of Calderón-Zygmund operators associated with the dilation A on weighted anisotropic spaces $L^q_\omega(\mathbb{R}^n, \rho)$ $(1 < q < +\infty)$. Here, we omit the detail. By the result in [12], the Calderón-Zygmund operator associated with the dilation is bounded from $H^1_\omega(\mathbb{R}^n)$ to $L^1_\omega(\mathbb{R}^n)$. Now we focus on proving the boundedness of the Calderón-Zygmund operators on weighted anisotropic Hardy spaces $H^p_\omega(\mathbb{R}^n, \rho)$.

Theorem 3.2. *Suppose* $\frac{\ln b}{\ln b + \ln \lambda_-} < p \leqslant 1$, $\omega \in A_1$. *If T is a Calderón-Zygmund operator associated with the dilation A, and* $T^*(1) = 0$ *(T^* is the dual of T), then T is a bounded linear operator from* $H^p_\omega(\mathbb{R}^n)$ *to itself.*

Obviously, the conclusion of Theorem 3.2 is the instant corollary of the next theorem.

Theorem 3.3. *Suppose* $\frac{\ln b}{\ln b + \ln \lambda_-} < p \leqslant 1 < q < +\infty$, $\varepsilon > 1/p - 1/q$. *Let T be a Calderón-Zygmund operators associated with the dilation A. For any* $(p, q, 0, \omega)$*-atom* $a(x)$ *with center* x_0, *if* $\int_{\mathbb{R}^n} Ta(x)dx = 0$, *then, Ta is a* $(p, q, \varepsilon, \omega)$*-molecule with center* x_0, *moreover,*

$$\Re_{q,\omega}(Ta) = \|Ta\|_{L^q_\omega}^{a_0/d} \|\omega(B_{x_0}(x))^d Ta(x)\|_{L^q_\omega}^{1-a_0/d} \leqslant C, \tag{3.1}$$

where $a_0 = 1 - 1/p + \varepsilon$, $d = 1 - 1/q + \varepsilon$, *and C is a constant independent of a.*

Proof. Suppose a is a $(p, q, 0, \omega)$-atom with support B_k, $k \in \mathbb{Z}$. Obviously, for the conclusion of Theorem 3.3, we only need to prove (3.1). In order to do this, we have the following estimates

$$\begin{aligned} \|\omega(B_0(x))^d Ta(x)\|_{L^q_\omega}^q &\leqslant \int_{B_{k+4\mu}} |Ta(x)|^q \omega(B_0(x))^{dq} \omega(x)dx \\ &\quad + \int_{(B_{k+4\mu})^c} |Ta(x)|^q \omega(B_0(x))^{dq} \omega(x)dx \\ &= J_1 + J_2. \end{aligned}$$

For J_1, by using the L^q_ω boundedness of T, there is

$$\begin{aligned} J_1 &\leqslant C\omega(B_{k+4\mu})^{dq} \|Ta\|_{L^q_\omega}^q \leqslant C\omega(B_{k+4\mu})^{dq} \omega(B_k)^{(1/q-1/p)q} \\ &\leqslant Cb^{4\mu dq} \omega(B_k)^{dq} \omega(B_k)^{(a_0-d)q} \leqslant C\omega(B_k)^{a_0 q}. \end{aligned}$$

For J_2, since $x \in (B_{k+4\mu})^c$, by using the vanishing moment condition of a and the condition of the kernel K in Definition 3.1, we obtain

$$
\begin{aligned}
J_2 &\leqslant \int_{(B_{k+4\mu})^c} [\int_{B_k} |K(x,y)-K(x,0)||a(y)|dy]^q \omega(B_0(x))^{qd}\omega(x)dx \\
&\leqslant C\int_{(B_{k+4\mu})^c} [\int_{B_k} \frac{\rho(y)^\gamma}{\rho(x)^{1+\gamma}}|a(y)|dy]^q \omega(B_0(x))^{qd}\omega(x)dx \\
&\leqslant C|B_k|^{q-1}\int_{B_k}|a(y)|^q\int_{(B_{k+4\mu})^c}\frac{\rho(y)^{q\gamma}}{\rho(x)^{q+q\gamma}}\omega(B_0(x))^{qd}\omega(x)dxdy \\
&\leqslant C\frac{1}{|B_k|^{\varepsilon q}}\omega(B_k)^{qd}\int_{B_k}|a(y)|^q\int_{(B_{k+4\mu})^c}\frac{\rho(y)^{q\gamma}}{\rho(x)^{q+q\gamma-qd}}\omega(x)dxdy \\
&\leqslant C\frac{1}{|B_k|^{\varepsilon q}}\omega(B_k)^{qd}\int_{B_k}|a(y)|^q\sum_{j=1}^{+\infty}\int_{B_{k+4\mu+j}\setminus B_{k+4\mu+j-1}}\frac{\rho(y)^{q\gamma}}{\rho(x)^{q+q\gamma-qd}}\omega(x)dxdy \\
&\leqslant C\omega(B_k)^{qd}\int_{B_k}|a(y)|^q M(\omega(y))dxdy \\
&\leqslant C\omega(B_k)^{qd}\int_{B_k}|a(y)|^q\omega(y)dy \leqslant C\omega(B_k)^{a_0 q},
\end{aligned}
$$

where $M(\omega(y))$ is the Hardy-Littlewood maximal operator of ω, i.e., $M(\omega(y)) = \sup_{k\in\mathbb{Z}}\sup_{x\in y+B_k}\frac{1}{|B_k|}\int_{x+B_k}|\omega(z)|dz$ (see [12]). Thus

$$
\Re_{q,\omega}(Ta) = \|Ta\|_{L^q_\omega}^{a_0/d}\|\omega(B_0(x))^d Ta(x)\|_{L^q_\omega}^{1-a_0/d} \leqslant C\|a\|_{L^q_\omega}^{a_0/d}\omega(B_k)^{a_0(1-a_0/d)} \leqslant C,
$$

where C is independent of a. This completes the proof of Theorem 3.3. ■

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