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Analysis of a Stochastic Lotka-Volterra Competitive Model with Infinite Delay and Impulsive Perturbations

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Abstract. This paper considers a stochastic Lotka-Volterra competitive model with infinite delay and impulsive perturbations. This model is new, more feasible and more accordance with the actual. The aim is to analyze what happens under the impulsive perturbations. With space C_g as phase space, sufficient conditions for permanence in time average are established as well as extinction, stability in time average and global attractivity of each population. Numerical simulations are also exhibited to illustrate the validity of the results in this paper. In addition, a knowledge is given to illustrate that the statement in [21] is incorrect by choosing space C_g as phase space. Our results demonstrate that impulsive perturbations which may represent human factor play a key role in protecting the population when environmental noise and interaction rates are disadvantageous to population survival.

1. Introduction

Recently, functional differential equations with infinite delay have long played important roles in the history of population dynamics, and they will no doubt continue to serve as in dispensable tools in future investigations (see e.g., [4,5,7,11,14]). A classic Lotka-Volterra competitive model with infinite delay can be expressed by

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)\left(b_1 - a_{11}x_1(t) - a_{12}\int_{-\infty}^{0} x_2(t+\theta)\,d\mu_2(\theta)\right), \\ \frac{dx_2(t)}{dt} &= x_2(t)\left(b_2 - a_{21}\int_{-\infty}^{0} x_1(t+\theta)\,d\mu_1(\theta) - a_{22}x_2(t)\right), \end{aligned} \tag{1.1}$$

where x_i denotes the size of the ith population; $b_i > 0$, $a_{ij} > 0$ and $\mu_i(\theta)$ is a probability measure on $(-\infty, 0]$. A further and extensive feature is considered in the model (1.1)

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or models similar to (1.1) towards persistence, extinction or other dynamical behavior. Here, we only refer to Kuang and Smith [14], Gopalsamy [4, 5], He and Gopalsamy [9], Lisena [19] and Kuang [13]. In particular, Kuang (see [13, p. 231]) claimed that if $\Psi_1 > 0$ and $\Psi_2 > 0$, then model (1.1) has a positive equilibrium $x^* = (x_1^*, x_2^*) = (\Psi_1/\Psi, \Psi_2/\Psi)$ which is globally asymptotically stable, where $\Psi = a_{11}a_{22} - a_{12}a_{21}$, $\Psi_1 = b_1a_{22} - b_2a_{12}$, $\Psi_2 = b_2a_{11} - b_1a_{21}$. It is important to point out that if $\Psi_1 > 0$ and $\Psi_2 > 0$, then $\Psi > 0$.

In the real world, the intrinsic growth rates of many species are always disturbed by environmental noise (see e.g., [2,10,16,18,22,28,29,32,34,36,41,42]). In particular, May [36] has pointed out that due to environmental noise, the birth rates, carrying capacity and other parameters involved in the model should be stochastic. In this paper, we assume that b_1 and b_2 are stochastic, then by the central limit theorem, we can replace b_1 and b_2 by

$$b_1 \to b_1 + \sigma_1 \dot{B}_1(t), \quad b_2 \to b_2 + \sigma_2 \dot{B}_2(t),$$

where, for $i = 1, 2$, $B_i(t)$ represents a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, σ_i^2 is the intensity of the environmental noise. Then we obtain the following stochastic Lotka-Volterra competitive model with infinite delay:

$$\begin{aligned} dx_1(t) &= x_1(t)\left(b_1 - a_{11}x_1(t) - a_{12}\int_{-\infty}^{0} x_2(t+\theta)\,d\mu_2(\theta)\right)dt + \sigma_1 x_1(t)\,dB_1(t), \\ dx_2(t) &= x_2(t)\left(b_2 - a_{21}\int_{-\infty}^{0} x_1(t+\theta)\,d\mu_1(\theta) - a_{22}x_2(t)\right)dt + \sigma_2 x_2(t)\,dB_2(t). \end{aligned}$$

Anything else, affected by a variety of factors manly, such as crop-dusting, planting, hunting and harvesting, the inner discipline of species or environment often suffers some dispersed changes over a relatively short time interval at the fixed times. In mathematics perspective, such sudden changes could be described by impulses (see e.g., [1,15,17,20,23–25]). In particularly, Liu and Wang incorporated the impulsive perturbation into stochastic population model at first time (see e.g., [23–25]), to the best of our knowledge. Motivated by these, we will study the following stochastic Lotka-Volterra competitive model with infinite delay and impulsive perturbations

$$\begin{aligned} dx_1(t) &= x_1(t)\left(b_1 - a_{11}x_1(t) - a_{12}\int_{-\infty}^{0} x_2(t+\theta)\,d\mu_2(\theta)\right)dt \\ &\qquad + \sigma_1 x_1(t)\,dB_1(t), \quad t \neq t_k,\ k \in N, \\ x_1(t_k^+) - x_1(t_k) &= I_k x_1(t_k), \quad k \in N, \\ dx_2(t) &= x_2(t)\left(b_2 - a_{21}\int_{-\infty}^{0} x_1(t+\theta)\,d\mu_1(\theta) - a_{22}x_2(t)\right)dt \\ &\qquad + \sigma_2 x_2(t)\,dB_2(t), \quad t \neq t_k,\ k \in N, \\ x_2(t_k^+) - x_2(t_k) &= H_k x_2(t_k), \quad k \in N. \end{aligned} \tag{1.2}$$

Here N denotes the set of positive integers, $0 < t_1 < t_2 \cdots$, $\lim_{k\to+\infty} t_k = +\infty$. As is known to all, permanence, extinction and stability is one of the most important questions in biomathematics. In this paper, we establish the sufficient conditions for permanence in time average, extinction, stability in time average and global attractivity of model (1.2) and investigate how impulsive perturbations affect on permanence in time average, extinction, stability in time average and global attractivity. Our results show that the impulse perturbations have no effect on permanence in time average, extinction and stability in time average if the impulsive perturbations are bounded. However, permanence in time average, extinction and stability in time average could be changed significantly when the impulsive perturbations are unbounded. Moreover, under the assumption that the impulsive perturbations are bounded, we find that the global attractivity have closer relationships with the impulsive perturbations.

In model (1.2), we let the initial value $\xi = (\xi_1, \xi_2)$ be positive and belong to the phase space C_g (see [6, 7]) which is defined by

$$C_g = \left\{\varphi \in C((-\infty, 0]; \mathbb{R}^2) : \|\varphi\|_{c_g} = \sup_{-\infty<s\le 0} e^{\mathbf{r}s} |\varphi(s)| < +\infty\right\},$$

where we choose $g(s) = e^{-\mathbf{r}s}$, $\mathbf{r} > 0$ and let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^2$.

For model (1.2) we always assume:

(A1) From biological meanings, we consider $1+I_k > 0$, $1+H_k > 0$, $I_k \neq 0$, $H_k \neq 0$, $k \in N$. When $I_k > 0$ or $H_k > 0$, is satisfied, the perturbation turn to be the description process of planting of species and harvesting if not.

(A2) For $i = 1, 2$, μ_i is the probability measure on $(-\infty, 0]$ satisfying that

$$\mu_{\mathbf{r}} = \int_{-\infty}^{0} e^{-2\mathbf{r}\theta}\, d\mu_i(\theta) < +\infty.$$

Clearly, the above assumption (A2) holds when $\mu_i(\theta) = e^{k\mathbf{r}\theta}$ $(k > 2)$ for $\theta \le 0$, thus there is a large number of these probability measures.

For convenience, we introduce the following notations

$$\mathbb{R}^2_+ = \left\{g = (g_1, g_2) \in \mathbb{R}^2 \mid g_i > 0, i = 1, 2\right\}, \quad \langle f(t)\rangle = t^{-1}\int_0^t f(s)\, ds.$$

Throughout this paper, K stands for a generic positive constant whose values may be different at different places.

The rest of the paper is arranged as follows. In Section 2, we established the sufficient condition for permanence in time average and extinction for model (1.2). In Section 3, we obtained the sufficient condition for global attractivity of model (1.2). In Section 4, we introduced an example to illustrate the main results. Finally, we close the paper with conclusions and remarks in Section 5.

2. Permanence, stability and extinction

In the initial stage, we draw into a new concept raised by Lu and Wu [30]. Now let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W(t)$ denote a m-dimension standard Brownian motion defined on this probability space.

Definition 2.1. Considering the following impulsive stochastic functional differential equation with infinite delay:

$$\begin{aligned} &dX(t) = F_1(t, X(t-\tau), X_t)\,dt + F_2(t, X(t-\tau), X_t)\,dW(t), \quad t \neq t_k,\ k \in N, \\ &X(t_k^+) - X(t_k) = I_k X(t_k), \quad k \in N, \end{aligned} \tag{2.1}$$

where $X(t) = (X_1(t), \ldots, X_d(t))^T$, $t \in \mathbb{R}_+$. Since phase space $BC((-\infty, 0]; \mathbb{R}^d)$ may cause the usual well-posedness questions related to functional equations of unbounded delay (see e.g., [14, 37, 40]), we choose space C_g (see [7, 37]) as phase place, which is defined by

$$C_g = \left\{ \varphi \in C((-\infty, 0]; \mathbb{R}^d) : \|\varphi\|_{c_g} = \sup_{-\infty < s \leq 0} e^{\mathbf{r}s} |\varphi(s)| < +\infty \right\},$$

where we choose $g(s) = e^{-\mathbf{r}s}$, $\mathbf{r} > 0$ and let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d$. Furthermore, C_g is an admissible Banach space (see [4]).

In (2.1), $X_t = \{X(t+\theta) : -\infty < \theta \leq 0\}$ can be regarded as C_g-value stochastic process. The initial value $X(t)$, $t \leq 0$ is nonrandom and positive, and belongs to the phase space C_g above. An $\mathbb{R}^d$-value stochastic process $X(t)$ defined on $\mathbb{R}$ is called a solution of the equation (2.1) with initial data above, if $X(t)$ has the following properties:

(i) $X(t)$ is $\mathcal{F}_t$-adapted and continuous on $(0, t_1)$ and (t_k, t_{k+1}), $k \in N$; $t \to F_1(t, X_t) \in \mathcal{L}^1(\overline{\mathbb{R}}_+; \mathbb{R}^d)$ and $t \to F_2(t, X_t) \in \mathcal{L}^2(\overline{\mathbb{R}}_+; \mathbb{R}^{d\times m})$, where $\mathcal{L}^k(\overline{\mathbb{R}}_+, \mathbb{R}^d)$ is all $\mathbb{R}^d$ valued $\mathcal{F}_t$ adapted processes $f(t)$ such that $\int_0^T |f(t)|\,dt < +\infty$ a.s. (almost surely) for all $T > 0$. $\mathcal{L}^k(\overline{\mathbb{R}}_+, \mathbb{R}^{d\times m})$ is defined similarly.

(ii) for each t_k, $k \in N$, $X(t_k^+) = \lim_{t \to t_k^+} X(t)$ and $X(t_k^-) = \lim_{t\to t_k^-} X(t)$ exist and $X(t_k^-) = X(t_k)$ with probability one.

(iii) $X(t) = \xi(t)$ for $t \leq 0$, for almost all $t \in [0, t_1]$, $X(t)$ obeys the integral equation

$$X(t) = \xi(0) + \int_0^t F_1(s, X(s-\tau), X_s)\,ds + \int_0^t F_2(s, X(s-\tau), X_s)\,dW(s).$$

And for almost all $t \in (t_k, t_{k+1}]$, $k \in N$, $X(t)$ obeys the integral equation

$$X(t) = X(t_k^+) + \int_{t_k}^t F_1(s, X(s-\tau), X_s)\,ds + \int_{t_k}^t F_2(s, X(s-\tau), X_s)\,dW(s).$$

Moreover, $X(t)$ satisfies the impulsive conditions at each $t = t_k$, $k \in N$ with probability one.

Lemma 2.2. (Liu et al. [27]) *Suppose that* $z(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$.

(i) *If there exist two positive constants* T *and* ρ_0 *such that* $\ln z(t) \le \rho t - \rho_0 \int_0^t z(s)\, ds + \sum_{i=1}^2 \alpha_i B_i(t)$ *for all* $t \ge T$*, where* α_i*,* $i = 1, 2$*, are constants, then*

$$\limsup_{t\to+\infty} \langle z(t)\rangle \le \rho/\rho_0 \quad a.s.\ \text{if } \rho \ge 0,$$
$$\lim_{t\to+\infty} z(t) = 0 \quad a.s.\ \text{if } \rho < 0.$$

(ii) *If there exist three positive constants* T*,* ρ *and* ρ_0 *such that* $\ln z(t) \ge \rho t - \rho_0 \int_0^t z(s)\, ds + \sum_{i=1}^2 \alpha_i B_i(t)$ *for all* $t \ge T$*, then* $\liminf_{t\to+\infty} \langle z(t)\rangle \ge \rho/\rho_0$ *a.s.*

Lemma 2.3. *Let the assumptions* (A1)–(A2) *hold. For any given initial value* $\xi \in C_g$*, then system* (1.2) *has a unique positive solution* $(x_1(t), x_2(t))$ *on* $t \in \mathbb{R}$ *and the solution will remain in* $\mathbb{R}_+^2$ *with probability* 1*, namely* $(x_1(t), x_2(t)) \in \mathbb{R}_+^2$ *for all* $t \in \mathbb{R}$ *almost surely.*

Proof. The proof of existence and unique of positive solution to model (1.2) is motivated by Liu and Wang [25]. Consider the following stochastic delay differential equation without impulsive perturbations:

$$(2.2)\quad \begin{aligned} dy_1(t) &= y(t)\left[b_1 - \prod_{0<t_k<t}(1+I_k)a_{11}y_1(t) - a_{12}\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+H_k)y_2(t+\theta)\,d\mu_2(\theta)\right]dt \\ &\quad + \sigma_1 y_1(t)\,dB_1(t), \\ dy_2(t) &= y(t)\left[b_2 - a_{21}\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+I_k)y_1(t+\theta)\,d\mu_1(\theta) - \prod_{0<t_k<t}(1+H_k)a_{22}y_2(t)\right]dt \\ &\quad + \sigma_2 y_2(t)\,dB_2(t) \end{aligned}$$

with the same initial value as model (1.2). Now let us prove model (2.2) has a unique positive solution $(y_1(t), y_2(t))$ on $t \in \mathbb{R}$ and the solution will remain in $\mathbb{R}_+$ with probability 1. The proof is standard and hence is omitted (see e.g., [8, 28]).

Let

$$x_1(t) = \prod_{0<t_k<t}(1+I_k)y_1(t), \quad x_2(t) = \prod_{0<t_k<t}(1+H_k)y_2(t),$$

where $(y_1(t), y_2(t))$ is the solution of the model (2.2). We need only to clarify that $(x_1(t), x_2(t))$ is the solution of (1.2). As a matter of fact, $x_1(t)$ is continuous on $(t_k, t_{k+1}) \subset$

$(0, +\infty)$, $k \in N$ and for every $t \neq t_k$,

$$\begin{aligned}
dx_1(t) &= d\left[\prod_{0<t_k<t}(1+I_k)y_1(t)\right] = \prod_{0<t_k<t}(1+I_k)\,dy_1(t)\\
&= \prod_{0<t_k<t}(1+I_k)y_1(t)\\
&\quad\times\left[b_1 - \prod_{0<t_k<t}(1+I_k)a_{11}y_1(t) - a_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+H_k)y_2(t+\theta)\,d\mu_2(\theta)\right]dt\\
&\quad+\sigma_1\left(\prod_{0<t_k<t}(1+I_k)\right)y_1(t)\,dB_1(t)\\
&= x_1(t)\left[b_1 - a_{11}x_1(t) - a_{12}\int_{-\infty}^{0}x_2(t+\theta)\,d\mu_2(\theta)\right]dt + \sigma_1 x_1(t)\,dB_1(t).
\end{aligned}$$

Similarly, we have

$$dx_2(t) = x_2(t)\left[b_2 - a_{21}\int_{-\infty}^{0}x_1(t+\theta)\,d\mu_1(\theta) - a_{22}x_2(t)\right]dt + \sigma_2 x_2(t)\,dB_2(t).$$

In addition, for every $k \in N$ and $t_k \in [0, +\infty)$,

$$\begin{aligned}
x_1(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+I_j)y_1(t) = \prod_{0<t_j\leq t_k}(1+I_j)y_1(t_k^+)\\
&= (1+I_k)\prod_{0<t_j<t_k}(1+I_j)y_1(t_k) = (1+I_k)x_1(t_k).
\end{aligned}$$

At the same time,

$$\begin{aligned}
x_1(t_k^-) &= \lim_{t\to t_k^-}\prod_{0<t_j<t}(1+I_j)y_1(t) = \prod_{0<t_j<t_k}(1+I_j)y_1(t_k^-)\\
&= \prod_{0<t_j<t_k}(1+I_j)y_1(t_k) = x_1(t_k).
\end{aligned}$$

Similarly, for every $k \in N$ and $t_k \in [0, +\infty)$,

$$\begin{aligned}
x_2(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+H_j)y_2(t) = \prod_{0<t_j\leq t_k}(1+H_j)y_2(t_k^+)\\
&= (1+H_k)\prod_{0<t_j<t_k}(1+H_j)y_2(t_k) = (1+H_k)x_2(t_k)
\end{aligned}$$

and

$$\begin{aligned}
x_2(t_k^-) &= \lim_{t\to t_k^-}\prod_{0<t_j<t}(1+H_j)y_2(t) = \prod_{0<t_j<t_k}(1+H_j)y_2(t_k^-)\\
&= \prod_{0<t_j<t_k}(1+H_j)y_2(t_k) = x_2(t_k).
\end{aligned}$$

Our next job is to prove the uniqueness of the solution for model (1.2). For $t \in [0, t_1]$, the model (1.2) becomes the following equation:

$$\begin{aligned} dx_1(t) &= x_1(t)\left[b_1 - a_{11}x_1(t) - a_{12}\int_{-\infty}^{0} x_2(t+\theta)\,d\mu_2(\theta)\right]dt + \sigma_1 x_1(t)\,dB_1(t), \\ dx_2(t) &= x_2(t)\left[b_2 - a_{21}\int_{-\infty}^{0} x_1(t+\theta)\,d\mu_1(\theta) - a_{22}x_2(t)\right]dt + \sigma_2 x_2(t)\,dB_2(t). \end{aligned} \tag{2.3}$$

Owing to the coefficients of (2.3) are locally Lipschitz continuous, by the theory of stochastic differential equation [40,41], the solution of (2.3) is unique. For $t \in (t_k, t_{k+1}]$, $k \in N$, the model (1.2) becomes

$$\begin{aligned} dx_1(t) &= x_1(t)\left[b_1 - a_{11}x_1(t) - a_{12}\int_{-\infty}^{0} x_2(t+\theta)\,d\mu_2(\theta)\right]dt + \sigma_1 x_1(t)\,dB_1(t), \\ dx_2(t) &= x_2(t)\left[b_2 - a_{21}\int_{-\infty}^{0} x_1(t+\theta)\,d\mu_1(\theta) - a_{22}x_2(t)\right]dt + \sigma_2 x_2(t)\,dB_2(t). \end{aligned} \tag{2.4}$$

Note that the coefficients of (2.4) are also locally Lipschitz continuous, then the solution of (2.4) is also unique. Consequently, the solution of the model (1.2) is unique. The proof is complete. □

A lot of authors have paid their attention to the permanence (see e.g., [8,10,31], which is one of the dominant themes in investigating dynamical of population. Now we study the permanence of model (1.2). We first do some preparation work.

Definition 2.4. System (1.2) is said to be permanent in time average a.s. if there are positive constants m_i and M_i $(i = 1, 2)$ such that

$$m_i \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \le M_i \quad \text{a.s., } i = 1, 2,$$

hold for any solution $(x_1(t), x_2(t))$ of system (1.2) with initial condition $\xi \in C_g$.

Theorem 2.5. *For model* (1.2), *we let the assumptions* (A1)–(A2) *hold.*

(I) *If* $b_1 < 0.5\sigma_1^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]$ *and* $b_2 < 0.5\sigma_2^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]$, *then both* x_1 *and* x_2 *are extinct almost surely (a.s.), i.e.,* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.,* $i = 1, 2$.

(II) *If* $b_1 > 0.5\sigma_1^2 - \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]$ *and* $b_2 < 0.5\sigma_2^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]$, *then* x_2 *is extinct a.s. and* x_1 *is permanent in time average a.s., i.e.,*

$$\liminf_{t\to+\infty}\langle x_1(t)\rangle \ge \frac{b_1 - 0.5\sigma_1^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]}{a_{11}} \quad a.s.,$$

$$\limsup_{t\to+\infty}\langle x_1(t)\rangle \le \frac{b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]}{a_{11}} \quad a.s.$$

(III) *If* $b_1 < 0.5\sigma_1^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]$ *and* $b_2 > 0.5\sigma_2^2 - \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]$, *then* x_1 *is extinct a.s. and* x_2 *is permanent in time average a.s., i.e.,*

$$\liminf_{t\to+\infty}\langle x_2(t)\rangle \geq \frac{b_2 - 0.5\sigma_2^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]}{a_{22}} \quad a.s.,$$
$$\limsup_{t\to+\infty}\langle x_2(t)\rangle \leq \frac{b_2 - 0.5\sigma_2^2 + \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]}{a_{22}} \quad a.s.$$

(IV) *If the conditions* $b_1 > 0.5\sigma_1^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]$, $b_2 > 0.5\sigma_2^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]$, $b_1 > 0.5\sigma_1^2 - \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right] + a_{12}M_2$ *and* $b_2 > 0.5\sigma_2^2 - \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right] + a_{21}M_1$ *hold, then for any initial data* $\xi \in C_g$, *the solution* $(x_1(t), x_2(t))$ *of* (1.2) *has the properties that*

$$\limsup_{t\to+\infty}\langle x_1(t)\rangle \leq M_1 \quad a.s., \qquad \limsup_{t\to+\infty}\langle x_2(t)\rangle \leq M_2 \quad a.s.,$$
$$\liminf_{t\to+\infty}\langle x_1(t)\rangle \geq m_1 \quad a.s., \qquad \liminf_{t\to+\infty}\langle x_2(t)\rangle \geq m_2 \quad a.s.,$$

where

$$M_1 = \frac{b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]}{a_{11}},$$
$$M_2 = \frac{b_2 - 0.5\sigma_2^2 + \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]}{a_{22}},$$
$$m_1 = \frac{b_1 - 0.5\sigma_1^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right] - a_{12}M_2}{a_{11}},$$
$$m_2 = \frac{b_2 - 0.5\sigma_2^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right] - a_{21}M_1}{a_{22}}.$$

That is, model (1.2) *will be permanent in time average a.s.*

Proof. Applying Itô's formula to the first equality in (2.2), we obtain

$$\begin{aligned}
&\ln y_1(t) - \ln y_1(0) \\
&= (b_1 - 0.5\sigma_1^2)t - a_{11}\int_0^t \sum_{0<t_k<s}\ln(1+I_k)y_1(s)\,ds \\
&\quad - a_{12}\int_0^t\int_{-\infty}^0 \prod_{0<t_k<s+\theta}(1+H_k)y_2(s+\theta)\,d\mu_2(\theta)ds + \sigma_1 B_1(t) \\
&= (b_1 - 0.5\sigma_1^2)t - a_{11}\int_0^t x_1(s)\,ds - a_{12}\int_0^t\int_{-\infty}^0 x_2(s+\theta)\,d\mu_2(\theta)ds + \sigma_1 B_1(t).
\end{aligned} \tag{2.5}$$

On the other hand, it follows from (2.5) that

$$\begin{aligned}
&\sum_{0<t_k<t} \ln(1+I_k) + \ln y_1(t) - \ln y_1(0) \\
&= \sum_{0<t_k<t} \ln(1+I_k) + (b_1 - 0.5\sigma_1^2)t - a_{11}\int_0^t \sum_{0<t_k<s} \ln(1+I_k) y_1(s)\,ds \\
&\quad - a_{12}\int_0^t\int_{-\infty}^0 \prod_{0<t_k<s+\theta}(1+H_k)y_2(s+\theta)\,d\mu_2(\theta)ds + \sigma_1 B_1(t) \\
&= \sum_{0<t_k<t} \ln(1+I_k) + (b_1 - 0.5\sigma_1^2)t - a_{11}\int_0^t x_1(s)\,ds \\
&\quad - a_{12}\int_0^t\int_{-\infty}^0 x_2(s+\theta)\,d\mu_2(\theta)ds + \sigma_1 B_1(t).
\end{aligned}$$

In other words, we have

$$\begin{aligned}
&\ln x_1(t) - \ln x_1(0) \\
&= \sum_{0<t_k<t} \ln(1+I_k) + (b_1 - 0.5\sigma_1^2)t - a_{11}\int_0^t \sum_{0<t_k<s} \ln(1+I_k) y_1(s)\,ds \\
&\quad - a_{12}\int_0^t\int_{-\infty}^0 \prod_{0<t_k<s+\theta}(1+H_k)y_2(s+\theta)\,d\mu_2(\theta)ds + \sigma_1 B_1(t) \\
&= \sum_{0<t_k<t} \ln(1+I_k) + (b_1 - 0.5\sigma_1^2)t - a_{11}\int_0^t x_1(s)\,ds \\
&\quad - a_{12}\int_0^t\int_{-\infty}^0 x_2(s+\theta)\,d\mu_2(\theta)ds + \sigma_1 B_1(t).
\end{aligned} \tag{2.6}$$

For $i = 1, 2$, we compute

$$\begin{aligned}
&\int_0^t\int_{-\infty}^0 x_i(s+\theta)\,d\mu_i(\theta)ds \\
&= \int_0^t \left[\int_{-\infty}^{-s} x_i(s+\theta)\,d\mu(\theta)ds + \int_{-s}^0 x_i(s+\theta)\,d\mu_i(\theta)\right] ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\,d\mu_i(\theta) + \int_{-t}^0 d\mu_i(\theta)\int_{-\theta}^t x_i(s+\theta)\,ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\,d\mu_i(\theta) + \int_{-t}^0 d\mu_i(\theta)\int_0^{t+\theta} x_i(s)\,ds.
\end{aligned} \tag{2.7}$$

By the assumption (A1), for $i = 1, 2$, we get that

$$\begin{aligned}
&\int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\,d\mu_i(\theta) \\
&\le \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} \left|x(s+\theta)\right| e^{-\mathbf{r}(s+\theta)}\,d\mu_i(\theta) \le \|\xi\|_{c_g} \int_0^t e^{-\mathbf{r}s}\,ds \int_{-\infty}^0 e^{-\mathbf{r}\theta}\,d\mu_i(\theta) \\
&\le \|\xi\|_{c_g} \int_0^t e^{-\mathbf{r}s}\,ds \int_{-\infty}^0 e^{-2\mathbf{r}\theta}\,d\mu_i(\theta) \le \frac{1}{\mathbf{r}}\|\xi\|_{c_g}\,\mu_{\mathbf{r}}(1-e^{-\mathbf{r}t}).
\end{aligned} \tag{2.8}$$

Substituting (2.7) into (2.6), we derive that

$$
\begin{aligned}
\ln x_1(t) - \ln x_1(0) = & \left(t^{-1} \sum_{0<t_k<t} \ln(1+I_k) + (b_1 - 0.5\sigma_1^2) \right) t \\
& - a_{11} \int_0^t x_1(s)\,ds - a_{12} \int_{-t}^0 d\mu_2(\theta) \int_0^{t+\theta} x_2(s)\,ds \\
& - a_{12} \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_2(s+\theta) e^{-\mathbf{r}(s+\theta)}\,d\mu_2(\theta) + \sigma_1 B_1(t).
\end{aligned} \tag{2.9}
$$

Similarly,

$$
\begin{aligned}
\ln x_2(t) - \ln x_2(0) = & \left(t^{-1} \sum_{0<t_k<t} \ln(1+H_k) + (b_2 - 0.5\sigma_2^2) \right) t \\
& - a_{22} \int_0^t x_2(s)\,ds - a_{21} \int_{-t}^0 d\mu_1(\theta) \int_0^{t+\theta} x_1(s)\,ds \\
& - a_{21} \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_1(s+\theta) e^{-\mathbf{r}(s+\theta)}\,d\mu_1(\theta) + \sigma_2 B_2(t).
\end{aligned} \tag{2.10}
$$

(I) Assume that $b_1 < 0.5\sigma_1^2 - \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k)\right]$ and $b_2 < 0.5\sigma_2^2 - \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+H_k)\right]$. By (2.9), we get

$$
t^{-1} \ln \frac{x_1(t)}{x_1(0)} \le b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k) \right] + t^{-1}\sigma_1 B_1(t).
$$

In view of the strong law of large numbers for martingles, we obtain $\lim_{t\to+\infty} B_1(t)/t = 0$ a.s. Therefore, we have

$$
\limsup_{t\to+\infty} t^{-1} \ln x_1(t) \le b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k) \right] < 0.
$$

Consequently, $\lim_{t\to+\infty} x_1(t) = 0$ a.s. Likewise, by (2.10), we can show that if $b_2 < 0.5\sigma_2^2 - t^{-1} \sum_{0<t_k<t} \ln(1+H_k)$, then $\lim_{t\to+\infty} x_2(t) = 0$ a.s.

(II) Suppose that $b_1 > 0.5\sigma_1^2 - \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k)\right]$ and $b_2 < 0.5\sigma_2^2 - \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+H_k)\right]$. Since $b_2 < 0.5\sigma_2^2 - \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+H_k)\right]$, then by (I), we have $\lim_{t\to+\infty} x_2(t) = 0$ a.s. Therefore, in virtue of (2.8), for arbitrary $\varepsilon > 0$, there is $T > 0$ such that for $t \ge T$,

$$
t^{-1} \int_{-t}^0 d\mu(\theta) \int_0^{t+\theta} x_2(s)\,ds \le t^{-1} \int_0^t x_2(s)\,ds \le \frac{\varepsilon}{4}, \quad \frac{1}{\mathbf{r}} \|\xi\|_{c_g} \mu_{\mathbf{r}} (1 - e^{-\mathbf{r}t}) \le \frac{\varepsilon}{4}.
$$

This implies that

$$
-\varepsilon/2 \le a_{12} t^{-1} \int_0^t \int_{-\infty}^0 x_2(s+\theta) d\mu_2(\theta) ds \le \frac{\varepsilon}{2}.
$$

Since $x_1(0)$ is bounded, we can see that

$$-\frac{\varepsilon}{2} \le t^{-1} \ln x_1(0) \le \frac{\varepsilon}{2}.$$

Substituting the two inequalities above into (2.9), we can derive that for $t \ge T$,

$$\begin{aligned}\ln x_1(t) \le & \left[b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty} \left(t^{-1} \sum_{0<t_k<t} \ln(1+I_k) \right) + \varepsilon \right] t \\ & - a_{11} \int_0^t x_1(s)\, ds + \sigma_1 B_1(t),\end{aligned} \tag{2.11}$$

$$\begin{aligned}\ln x_1(t) \ge & \left[b_1 - 0.5\sigma_1^2 + \liminf_{t\to+\infty} \left(t^{-1} \sum_{0<t_k<t} \ln(1+I_k) \right) - \varepsilon \right] t \\ & - a_{11} \int_0^t x_1(s)\, ds + \sigma_1 B_1(t).\end{aligned} \tag{2.12}$$

Owing to $b_1 > 0.5\sigma_1^2$, we can choose ε sufficiently small such that $b_1 - 0.5\sigma_1^2 - \varepsilon > 0$. Applying (i) and (ii) in Lemma 2.2 to (2.11), (2.12) and the arbitrariness of ε respectively, we derive

$$\liminf_{t\to+\infty} \langle x_1(t) \rangle \ge \frac{b_1 - 0.5\sigma_1^2 + \liminf_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k) \right]}{a_{11}} \quad a.s.$$

and

$$\limsup_{t\to+\infty} \langle x_1(t) \rangle \le \frac{b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k) \right]}{a_{11}} \quad a.s.$$

The proof of (III) is similar to (II), so we omit it here.

(IV) By assumption (A1), for $i = 1, 2$, we may compute that

$$\begin{aligned}
&\int_0^t \int_{-\infty}^0 x_i(s+\theta)\, d\mu_i(\theta) ds \\
&= \int_0^t \left[\int_{-\infty}^{-s} x(s+\theta)\, d\mu_i(\theta) ds + \int_{-s}^0 x_i(s+\theta)\, d\mu_i(\theta) \right] ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_i(\theta) + \int_{-t}^0 d\mu_i(\theta) \int_{-\theta}^t x_i(s+\theta)\, ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_i(\theta) + \int_{-t}^0 d\mu_i(\theta) \int_0^{t+\theta} x_i(s)\, ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_i(\theta) \\
&\quad + \int_{-t}^0 d\mu_i(\theta) \int_0^t x_i(s)\, ds + \int_{-t}^0 d\mu_i(\theta) \int_t^{t+\theta} x_i(s)\, ds
\end{aligned} \tag{2.13}$$

$$
\begin{aligned}
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_i(\theta) + \int_0^t x_i(s)\, ds \\
&\quad - \int_{-\infty}^{-t} d\mu_i(\theta) \int_0^t x_i(s)\, ds + \int_{-t}^{0} d\mu_i(\theta) \int_t^{t+\theta} x_i(s)\, ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_i(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_i(\theta) + \int_0^t x_i(s)\, ds \\
&\quad - \int_{-\infty}^{-t} d\mu_i(\theta) \int_0^t x_i(s)\, ds - \int_{-t}^{0} d\mu_i(\theta) \int_{t+\theta}^{t} x_i(s)\, ds.
\end{aligned}
$$

Substituting (2.13) into (2.6), we have

$$
\begin{aligned}
&\ln x_1(t) - \ln x_1(0) \\
&= \left(t^{-1} \sum_{0<t_k<t} \ln(1+I_k) + b_1 - 0.5\sigma_1^2 \right) t - a_{11} \int_0^t x_1(s)\, ds \\
&\quad - a_{12} \int_0^t x_2(s)\, ds - a_{12} \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_2(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_2(\theta) \\
&\quad + a_{12} \int_{-t}^{0} d\mu_2(\theta) \int_{t+\theta}^{t} x_2(s) ds + a_{12} \int_{-\infty}^{-t} d\mu_2(\theta) \int_0^t x_2(s)\, ds + \sigma_1 B_1(t).
\end{aligned}
\tag{2.14}
$$

Similarly, by (2.10) and (2.13), we obtain

$$
\begin{aligned}
&\ln x_2(t) - \ln x_2(0) \\
&= \left(t^{-1} \sum_{0<t_k<t} \ln(1+H_k) + b_2 - 0.5\sigma_2^2 \right) t - a_{22} \int_0^t x_2(s)\, ds \\
&\quad - a_{21} \int_0^t x_1(s)\, ds - a_{21} \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)} x_1(s+\theta) e^{-\mathbf{r}(s+\theta)}\, d\mu_1(\theta) \\
&\quad + a_{21} \int_{-t}^{0} d\mu_1(\theta) \int_{t+\theta}^{t} x_1(s)\, ds + a_{21} \int_{-\infty}^{-t} d\mu_1(\theta) \int_0^t x_1(s)\, ds + \sigma_2 B_2(t).
\end{aligned}
\tag{2.15}
$$

Making use of the conditions $b_1 > 0.5\sigma_1^2 - \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k)\right]$ and (2.11), we obtain

$$
\limsup_{t\to+\infty} \langle x_1(t) \rangle \le \frac{b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+I_k)\right]}{a_{11}} = M_1 \quad a.s. \tag{2.16}
$$

In the same way, we can get that if $b_2 > 0.5\sigma_2^2$, then

$$
\limsup_{t\to+\infty} \langle x_2(t) \rangle \le \frac{b_2 - 0.5\sigma_2^2 + \limsup_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+H_k)\right]}{a_{22}} = M_2 \quad a.s. \tag{2.17}
$$

Let ε be sufficiently small such that $\liminf_{t\to+\infty} \left[t^{-1} \sum_{0<t_k<t} \ln(1+H_k)\right] + b_2 - 0.5\sigma_2^2 -$

$a_{21}M_1 > a_{21}\varepsilon$. When (2.16) and (2.8) are used in (2.15), we have

$$\begin{aligned}
&t^{-1}\ln x_2(t) - t^{-1}\ln x_2(0)\\
&= \left(t^{-1}\sum_{0<t_k<t}\ln(1+H_k) + b_2 - 0.5\sigma_2^2\right) - a_{22}\langle x_2(t)\rangle\\
&\quad - a_{21}\langle x_1(t)\rangle - t^{-1}a_{21}\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{r}(s+\theta)}x_1(s+\theta)e^{-\mathbf{r}(s+\theta)}\,d\mu_1(\theta)\\
&\quad + t^{-1}a_{21}\int_{-t}^{0}d\mu_1(\theta)\int_{t+\theta}^{t}x_1(s)\,ds + t^{-1}a_{21}\int_{-\infty}^{-t}d\mu_1(\theta)\int_0^t x_1(s)\,ds + t^{-1}\sigma_2 B_2(t)\\
&\geq \left(t^{-1}\sum_{0<t_k<t}\ln(1+H_k) + b_2 - 0.5\sigma_2^2\right) - a_{22}\langle x_2(t)\rangle - a_{21}\limsup_{t\to+\infty}\langle x_1(t)\rangle - a_{21}\varepsilon\\
&\geq \left(\liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right] + b_2 - 0.5\sigma_2^2\right) - a_{22}\langle x_2(t)\rangle - a_{21}M_1 - a_{21}\varepsilon
\end{aligned}$$

for sufficiently large t. In view of (ii) in Lemma 2.2 and the arbitrariness of ε, we get

$$\begin{aligned}
\liminf_{t\to+\infty}\langle x_2(t)\rangle &\geq \frac{b_2 - 0.5\sigma_2^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right] - a_{21}M_1}{a_{22}}\\
&= m_2 \quad a.s.
\end{aligned}$$

Similarly, by (2.17) and (2.8) are used in (2.14), we derive

$$\begin{aligned}
\liminf_{t\to+\infty}\langle x_1(t)\rangle &\geq \frac{b_1 - 0.5\sigma_1^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right] - a_{12}M_2}{a_{11}}\\
&= m_1 \quad a.s.
\end{aligned}$$

The whole proof is completed. □

Remark 2.6. By (II) in Theorem 2.5, we can see that when

$$\limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right] = \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right],$$

then

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1 - 0.5\sigma_1^2 + \lim_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]}{a_{11}} \quad a.s.$$

This means x_1 is stable in time average a.s. Similarly, in virtue of (III) in Theorem 2.5, we can obtain that x_2 is stable in time average a.s. when

$$\limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right] = \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right].$$

Remark 2.7. In view of (I)–(IV) in Theorem 2.5, we can find that the impulsive perturbations does not affect extinction and permanence in time average if the impulsive perturbations are bounded. When impulsive perturbations are unbounded, it can maintain some important and significant properties of population models as much as possible. For example, (I) in Theorem 2.5 shows that all the species modelled by stochastic model (1.2) are extinct when suffering environmental noise; whereas (II) and (III) in Theorem 2.5 show that one population of stochastic model (1.2) is permanent in time average when enlarge the intensity of impulsive perturbations. In addition, (IV) in Theorem 2.5 points out the important fact that all the species described by stochastic model (1.2) can survive for a long time under the control of impulsive perturbations which represent human behavior.

Remark 2.8. From the condition $b_1 < 0.5\sigma_1^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]$ and $b_2 < 0.5\sigma_2^2 - \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]$ in (I) of Theorem 2.5, it is easy to see that the environmental noise is disadvantageous of permanence in time average of model (1.2) if we enhance the intensity of the noise σ_1^2 and σ_2^2, simultaneously.

Remark 2.9. From the conditions of (IV) in Theorem 2.5, it is easy to see that the interaction rate a_{ij} $(i = 1, 2,\ i \neq j)$ is unfavorable for permanence in time average of model (1.2).

3. Global attractivity

In this section, we turn to establish sufficient criteria for the global attractivity of (1.2). To end this, we prepare some useful definition and lemmas.

Definition 3.1. Let $x(t) = (x_1(t), x_2(t))$ and $x^*(t) = (x_1^*(t), x_2^*(t)$ be two arbitrary solutions of model (1.2) with initial values $\xi \in C_g$ and $\xi^* \in C_g$, respectively. If, for $i = 1, 2$, $\lim_{t\to+\infty} |x_i(t) - x_i^*(t)| = 0$ a.s., then we say model (1.2) is globally attractive (globally asymptotically stable).

(A3) There are two positive constants l_i and L_i such that $l_1 \leq \prod_{0<t_k<t}(1+I_k) \leq L_1$ and $l_2 \leq \prod_{0<t_k<t}(1+H_k) \leq L_2$ for all $t > 0$, respectively.

Remark 3.2. Assumption (A3) is easy to be satisfied. For example, if $I_k = e^{(-1)^{k+1}/k^2} - 1$, then $e^{0.75} < \prod_{0<t_k<t}(1+I_k) < e$ for all $t > 0$. Thus $1 \leq \prod_{0<t_k<t}(1+I_k) \leq e$ for all $t > 0$.

Lemma 3.3. *Let the assumptions* (A1)–(A3) *hold. For any initial value* $\xi \in C_g$ *and* $p > 0$*, there is a constant* $K = K(p) > 0$ *such that the solution* $y(t) = (y_1(t), y_2(t))$ *of model* (2.2) *satisfies*

$$E(y_i^p(t)) \leq K_i(p), \quad t \geq 0,\ p > 0,\ i = 1, 2.$$

Proof. The proof is rather standard and hence is omitted (see e.g., [23, 26, 28]). ☐

Lemma 3.4. [12, 33] *Suppose that an n-dimensional stochastic process $X(t)$ on $t \geq 0$ satisfies the condition*

$$E\left|X(t) - X(s)\right|^{\alpha} \leq c\left|t - s\right|^{1+\beta}, \quad 0 \leq s, t < \infty$$

for some positive constants α, β and c. Then there exists a continuous modification $\widetilde{X}(t)$ of $X(t)$, which has the property that for every $\gamma \in (0, \beta/\alpha)$, there is a positive random variable $h(w)$ such that

$$p\left\{\omega : \sup_{\substack{0<|t-s|<h(\omega)\\ 0\leq s,t<\infty}} \frac{\left|\widetilde{X}(t,\omega) - \widetilde{X}(s,\omega)\right|}{|t-s|^{\gamma}} \leq \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every sample path of $\widetilde{X}(t)$ is locally but uniformly Hölder continuous with exponent γ.

Lemma 3.5. *Let the assumptions* (A1)–(A3) *hold. If $y(t) = (y_1(t), y_2(t))$ is a solution of model* (2.2) *with initial value $\xi \in C_g$, then almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$.*

Proof. We shall consider the following stochastic integral equation instead of (2.2):

$$\begin{aligned}
y_1(t) = y_1(0) + \int_0^t y_1(s)\Big(b_1 - a_{11} \prod_{0<t_k<t} (1+I_k) y_1(s) \\
&- a_{12} \int_{-\infty}^0 \prod_{0<t_k<t+\theta} (1+H_k) y_2(s+\theta)\, d\mu_2(\theta)\Big) ds + \int_0^t \sigma_1 y_1(s)\, dB_i(s), \\
y_2(t) = y_2(0) + \int_0^t y_2(s)\Big(b_2 - a_{21} \int_{-\infty}^0 \prod_{0<t_k<t+\theta} (1+I_k) y_1(s+\theta)\, d\mu_1(\theta) \\
&- a_{22} \prod_{0<t_k<t} (1+H_k) y_2(s)\Big) ds + \int_0^t \sigma_2 y_2(s)\, dB_2(s).
\end{aligned}$$

Note that

$$\begin{aligned}
&E\left|y_1(s)\left(b_1 - a_{11}\prod_{0<t_k<t}(1+I_k)y_1(s) - a_{12}\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+H_k)y_2(s+\theta)\,d\mu_2(\theta)\right)\right| \\
&= E\left[|y_1(s)|^p \left|b_1 - a_{11}\prod_{0<t_k<t+\theta}(1+I_k)y_1(s) - a_{12}\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+H_k)y_2(s+\theta)\,d\mu_2(\theta)\right|^p\right] \\
&\leq 0.5E\left|y_1(s)\right|^{2p} + 0.5E\bigg|b_1 - a_{11}\prod_{0<t_k<t+\theta}(1+I_k)y_1(s) \\
&\qquad - a_{12}\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+H_k)y_2(s+\theta)\,d\mu_2(\theta)\bigg|^{2p} \\
&\leq 0.5\left\{E\left|y_1(s)\right|^{2p} + 3^{2p-1}E\left[b_1^{2p} + a_{11}^{2p}L_1^{2p}E\left|y_1(s)\right|^{2p} + a_{12}^{2p}L_2^{2p}E\left|\int_{-\infty}^0 y_2(s+\theta)\,d\mu_2(\theta)\right|^{2p}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le 0.5\left\{E\left|y_1(s)\right|^{2p}+3^{2p-1}\left[b_1^{2p}+a_{11}^{2p}L_1^{2p}E\left|y_1(s)\right|^{2p}+a_{12}^{2p}L_2^{2p}E\left(\int_{-\infty}^{0}\left|y_2(s+\theta)\right|d\mu_2(\theta)\right)^{2p}\right]\right\}\\
&\le 0.5\left\{E\left|y_1(s)\right|^{2p}+3^{2p-1}\left[b_1^{2p}+a_{11}^{2p}L_1^{2p}E\left|y_1(s)\right|^{2p}+a_{12}^{2p}L_2^{2p}E\int_{-\infty}^{0}\left|y_2(s+\theta)\right|^{2p}d\mu_2(\theta)\right]\right\}\\
&\le 0.5\left\{E\left|y_1(s)\right|^{2p}+3^{2p-1}\left[b_1^{2p}+a_{11}^{2p}L_1^{2p}E\left|y_1(s)\right|^{2p}+a_{12}^{2p}L_2^{2p}\int_{-\infty}^{0}E\left|y_2(s+\theta)\right|^{2p}d\mu_2(\theta)\right]\right\}\\
&\le 0.5\left\{L_1^{2p}+3^{2p-1}E\left[b_1^{2p}+a_{11}^{2p}L_1^{2p}K_1(2p)+a_{12}^{2p}L_2^{2p}\int_{-\infty}^{0}K_2(2p)\,d\mu_2(\theta)\right]\right\}\\
&=0.5\left\{L_1^{2p}+3^{2p-1}E\left[b_1^{2p}+a_{11}^{2p}L_1^{2p}K_1(2p)+a_{12}^{2p}L_2^{2p}K_2(2p)\right]\right\}=K.
\end{aligned}
$$

Here the second inequality follows from the discrete Hölder inequality. In the same way, we have

$$
E\left|y_2(s)\left[b_2-a_{22}\prod_{0<t_k<t}(1+H_k)y_2(s)-a_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y_1(s+\theta)\,d\mu_1(\theta)\right]\right|\le K.
$$

The rest of proof is similar to Lemma 9 in [23], we hence omit it. □

Lemma 3.6. [3] *Let $f(t)$ be a nonnegative function defined on $[0,\infty)$ such that $f(t)$ is integrable on $[0,\infty)$ and is uniformly continuous on $[0,\infty)$. Then $\lim_{t\to\infty} f(t)=0$.*

Now, we are in the position to give our main result of this section.

Theorem 3.7. *Let the assumptions* (A1)–(A3) *hold. If $l_1a_{11}>L_1a_{21}$, $l_2a_{22}>L_2a_{12}$, then model* (1.2) *is global attractivity.*

Proof. Let $(x_1(t),x_2(t))$ and $(x_1^*(t),x_2^*(t))$ be two arbitrary solutions of (1.2) with initial values $\xi\in C_g$, $\xi^*\in C_g$, respectively. Suppose that the solution of

$$
\begin{aligned}
dy_1(t)&=y_1(t)\left[b_1-\prod_{0<t_k<t}(1+I_k)a_{11}y_1(t)-a_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+H_k)y_2(t+\theta)\,d\mu_2(\theta)\right]dt\\
&\quad+\sigma_1y_1(t)\,dB_1(t),\\
dy_2(t)&=y_2(t)\left[b_2-a_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y_1(t+\theta)\,d\mu_1(\theta)-\prod_{0<t_k<t}(1+H_k)a_{22}y_2(t)\right]dt\\
&\quad+\sigma_2y_2(t)\,dB_2(t)
\end{aligned}
$$

is $(y_1(t),y_2(t))$ and the same initial values $\xi\in C_g$ as (1.2). On the other hand, the solution of

$$
\begin{aligned}
dy_1(t)&=y(t)\left[b_1-\prod_{0<t_k<t}(1+I_k)a_{11}y_1(t)-a_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y_2(t+\theta)\,d\mu(\theta)\right]dt\\
&\quad+\sigma_1y_1(t)\,dB_1(t),\\
dy_2(t)&=y(t)\left[b_2-a_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y_1(t+\theta)\,d\mu_1(\theta)-\prod_{0<t_k<t}(1+H_k)a_{22}y_2(t)\right]dt\\
&\quad+\sigma_2y_2(t)\,dB_2(t)
\end{aligned}
$$

is $(y_1^*(t), y_2^*(t))$ and the same initial values $\xi^* \in C_g$ as (1.2). Then we have

$$x_1(t) = \prod_{0<t_k<t}(1+I_k)y_1(t), \qquad x_2(t) = \prod_{0<t_k<t}(1+H_k)y_2(t),$$
$$x_1^*(t) = \prod_{0<t_k<t}(1+I_k)y_1^*(t), \qquad x_2^*(t) = \prod_{0<t_k<t}(1+H_k)y_2^*(t).$$

Define

$$\begin{aligned}V(t) = &\sum_{i=1}^{2}|\ln(y_i(t)) - \ln(y_i^*(t))| + a_{21}L_1\int_{-\infty}^{0}\int_{t+\theta}^{t}|y_1(s) - y_1^*(s)|\,ds d\mu_1(\theta)\\ &+ a_{12}L_2\int_{-\infty}^{0}\int_{t+\theta}^{t}|y_2(s) - y_2^*(s)|\,ds d\mu_2(\theta).\end{aligned}$$

A calculation of the right differential $D^+V(t)$, and then making use of the generalized Itô's formula, we can observe

$$\begin{aligned}
&D^+V(t)\\
&= \sum_{i=1}^{2}\mathrm{sgn}(y_1(t) - y_1^*(t))\,d(\ln(y_i(t)) - \ln(y_i^*(t)))\\
&\quad + a_{21}L_1\int_{-\infty}^{0}|y_1(t) - y_1^*(t)|\,d\mu_1(\theta)dt - a_{21}L_1\int_{-\infty}^{0}|y_1(t+\theta) - y_1^*(t+\theta)|\,d\mu_1(\theta)dt\\
&\quad + a_{12}L_1\int_{-\infty}^{0}|y_2(t) - y_2^*(t)|\,d\mu_2(\theta)dt - a_{12}L_1\int_{-\infty}^{0}|y_2(t+\theta) - y_2^*(t+\theta)|\,d\mu_2(\theta)dt\\
&= \mathrm{sgn}(y_1(t) - y_1^*(t))\Big(- a_{11}\prod_{0<t_k<t}(1+I_k)(y_1(t) - y_1^*(t))\\
&\qquad - a_{12}\Big(\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+H_k)(y_2(t+\theta) - y_2^*(t+\theta))\,d\mu_2(\theta)\Big)\Big)dt\\
&\quad + \mathrm{sgn}(y_2(t) - y_2^*(t))\Big(- a_{22}\prod_{0<t_k<t}(1+H_k)(y_2(t) - x_2(t))\\
&\qquad - a_{21}\Big(\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)(y_1(t+\theta) - y_1^*(t+\theta))\,d\mu_2(\theta)\Big)\Big)dt\\
&\quad + a_{21}L_1\int_{-\infty}^{0}|y_1(t) - y_1^*(t)|\,d\mu_1(\theta)dt - a_{21}L_1\int_{-\infty}^{0}|y_1(t+\theta) - y_1^*(t+\theta)|\,d\mu_1(\theta)dt\\
&\quad + a_{12}L_1\int_{-\infty}^{0}|y_2(t) - y_2^*(t)|\,d\mu_2(\theta)dt - a_{12}L_1\int_{-\infty}^{0}|y_2(t+\theta) - y_2^*(t+\theta)|\,d\mu_2(\theta)dt\\
&\le -a_{11}l_1\,|y_1(t) - y_1^*(t)|\,dt + a_{12}L_2\int_{-\infty}^{0}|y_2(t+\theta) - y_2^*(t+\theta)|\,d\mu_2(\theta)dt\\
&\quad - a_{22}l_2\,|y_1(t) - y_1^*(t)|\,dt + a_{21}L_1\int_{-\infty}^{0}|y_1(t+\theta) - y_1^*(s+\theta)|\,d\mu_2(\theta)dt
\end{aligned}$$

$$
\begin{aligned}
&+ a_{21}L_1 \int_{-\infty}^{0} |y_1(t) - y_1^*(t)|\, d\mu_1(\theta)dt - a_{21}L_1 \int_{-\infty}^{0} |y_1(t+\theta) - y_1^*(t+\theta)|\, d\mu_1(\theta)dt \\
&+ a_{12}L_2 \int_{-\infty}^{0} |y_2(t) - y_2^*(t)|\, d\mu_2(\theta)dt - a_{12}L_2 \int_{-\infty}^{0} |y_2(t+\theta) - y_2^*(t+\theta)|\, d\mu_2(\theta)dt \\
= &-(a_{11}l_1 - a_{21}L_1)\,|y_1(t) - y_1^*(t)|\, dt - (a_{22}l_2 - a_{12}L_2)\,|y_2(t) - y_2^*(t)|\, dt.
\end{aligned}
$$

Integrating both sides and then taking the expectation yield

$$
V(t) \le V(0) - \int_0^t (a_{11}l_1 - a_{21}L_1)\,|y_1(s) - y_1^*(s)|\, ds - \int_0^t (a_{22}l_2 - a_{12}L_2)\,|y_2(s) - y_2^*(s)|\, ds.
$$

In other words, we have already shown that

$$
\begin{aligned}
&V(t) + \int_0^t (a_{11}l_1 - a_{21}L_1)\,|y_1(s) - y_1^*(s)|\, ds + \int_0^t (a_{22}l_2 - a_{12}L_2)\,|y_2(s) - y_2^*(s)|\, ds \\
&\le V(0) < \infty.
\end{aligned}
$$

From $l_1a_{11} > L_1a_{21}$ and $l_2a_{22} > L_2a_{12}$, we derive, for $i = 1, 2$,

$$
|y_i(t) - y_i^*(t)| \in L^1[0, +\infty).
$$

Then it follows from Lemmas 3.5 and 3.6 that, for $i = 1, 2$,

$$
\lim_{t\to+\infty} |y_i(t) - y_i^*(t)| = 0,
$$

from almost all $\omega \in \Omega$. Therefore,

$$
\lim_{t\to+\infty} |x_1(t) - x_1^*(t)| = \lim_{t\to+\infty} \prod_{0<t_k<t} (1 + I_k)\,|y_1(t) - y_1^*(t)| = 0,
$$

from almost all $\omega \in \Omega$;

$$
\lim_{t\to+\infty} |x_2(t) - x_2^*(t)| = \lim_{t\to+\infty} \prod_{0<t_k<t} (1 + H_k)\,|y_2(t) - y_2^*(t)| = 0,
$$

from almost all $\omega \in \Omega$. This completes the proof of Theorem 3.7. □

Remark 3.8. From Theorems 2.5 and 3.7, it is easy to see that infinite delay does not effect the extinction, permanence in time average and global attractivity of model (1.2).

4. Examples and numerical simulations

In this section, we shall use the Euler scheme (see e.g., [39]) to illustrate the analytical findings.

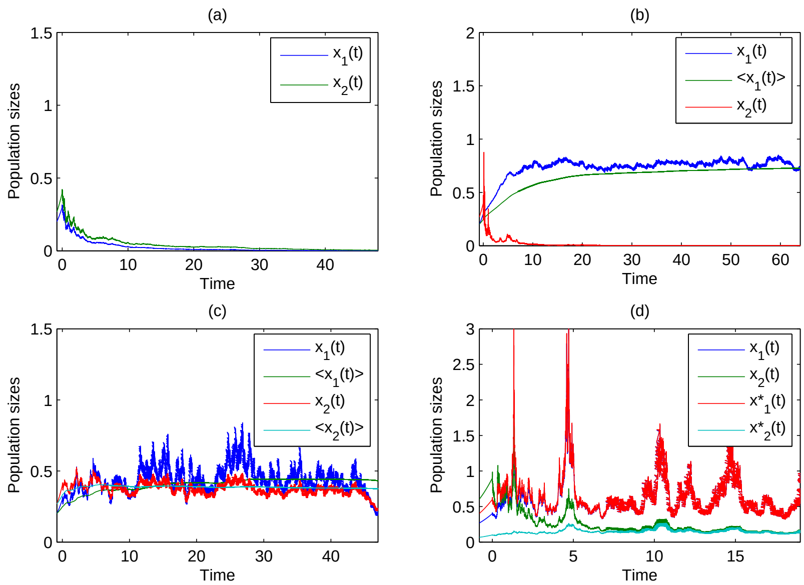

Figure 4.1: The horizontal axis and the vertical axis in this and following figures represent the time t and the populations size (step size $\Delta t = 0.001$).

In Figure 4.1, we choose the initial data $\xi_1(\theta) = 0.3e^{\theta}$, $\xi_2(\theta) = 0.4e^{\theta}$, $b_1 = 0.58$, $b_2 = 0.52$, $a_{11} = 0.78$, $a_{22} = 0.7$, $a_{12} = a_{21} = 0.4$, $\sigma_1^2 = 1.3$, $\sigma_2^2 = 1.2$, $t_k = 10k$. Then $\Psi = a_{11}a_{22} - a_{12}a_{21} = 0.25$, $\Psi_1 = b_1 a_{22} - b_2 a_{12} = 0.046$, $\Psi_2 = b_2 a_{11} - b_1 a_{21} = 0.16$. We let $\sigma_1 = \sigma_2 = 0$. Then by virtue of Kuang's work [13], we have that the positive equilibrium $(\Psi_1/\Psi, \Psi_2/\Psi) = (0.184, 0.64)$ is globally asymptotically stable.

In addition, the only difference in Figure 4.1(a)–(c) is that the representations of I_k and H_k are different. In Figure 4.1(a), we choose $I_k = H_k = 0$. From (I) of Theorem 2.5, the population $x_1(t)$ and $x_2(t)$ are extinct a.s. In Figure 4.1(b), we choose $I_k = e^{0.9} - 1$, $H_k = 0$. In view of (II) of Theorem 2.5, population $x_1(t)$ will be permanent in time average a.s., $x_2(t)$ is extinct a.s. In Figure 4.1(c), we consider $I_k = e^4 - 1$, $H_k = e^5 - 1$, then

$$\begin{aligned}
\limsup_{t\to+\infty} \langle x_1(t)\rangle \le M_1 &= \frac{b_1 - 0.5\sigma_1^2 + \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right]}{a_{11}} \\
&= 0.538 \quad \text{a.s.}, \\
\limsup_{t\to+\infty} \langle x_2(t)\rangle \le M_2 &= \frac{b_2 - 0.5\sigma_2^2 + \limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right]}{a_{22}} \\
&= 0.471 \quad \text{a.s.}, \\
\liminf_{t\to+\infty} \langle x_1(t)\rangle &\ge \frac{b_1 - 0.5\sigma_1^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+I_k)\right] - a_{12}M_2}{a_{11}} \\
&= m_1 = 0.296 \quad \text{a.s.},
\end{aligned}$$

$$\liminf_{t\to+\infty}\langle x_2(t)\rangle \geq \frac{b_2 - 0.5\sigma_2^2 + \liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+H_k)\right] - a_{21}M_1}{a_{22}}$$
$$= m_2 = 0.164 \quad \text{a.s.}$$

By virtue of Theorem 2.5, population $x_i(t)$ will be permanent in time average a.s., $i = 1,2$. By comparing Figure 4.1(a)–(c), we can see that the impulsive perturbation can change the properties of the population system significantly.

In Figure 4.1(d), we choose the initial data $\xi_1(\theta) = 0.3e^{\theta}$, $\xi_2(\theta) = 0.4e^{\theta}$, $b_1 = 0.62$, $b_2 = 0.54$, $a_{11} = 0.78$, $a_{22} = 0.7$, $a_{12} = 0.3$, $a_{21} = 0.2$, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.2$. Let $I_k = H_k = e^{(-1)^{k+1}/k^2} - 1$, then $e^{0.75} < \prod_{0<t_k<t}(1+I_k) < e$ for all $t > 0$. Thus we set $l_i = e^{0.75}$, $L_i = e$ for $i = 1,2$. Making use of Theorem 3.7, the model (1.2) will be global attractivity.

5. Conclusions and remarks

This paper incorporates two important factors, environmental noise and impulsive perturbations, into the classic delay competitive model (1.1) for the first time and obtain the nice dynamical properties including permanence in time average, extinction, stability in time average and global attractivity of model (1.2) by choosing C_g space as phase space. From the conclusions we know that the impulsive perturbations can have important impact on the population. In particular, when the population will be extinct, we should take measure, i.e., positive impulsive perturbation, to avert the case as far as possible. Furthermore, the results also implies that, firstly, environmental noise have an influence on permanence in time average, extinction and stability in time average; secondly, infinite delay has not affect permanence in time average, extinction, stability in time average and global attractivity of model (1.2); thirdly, permanence in time average of model (1.2) has close relationships with the interaction rates.

Some interesting and significant topics deserve our further engagement. One may find a more realistic and sophisticated model to introduce the Lévy jumps [2] into the model. Recently, Liu and Chen [21] investigated the general stochastic non-autonomous logistic model with delays and Lévy jumps:

$$\begin{aligned} dx(t) &= x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau(t)) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\right]dt \\ &\quad + \sigma_1(t)x(t)\,d\omega_1(t) + \sigma_2(t)x^{1+\alpha}(t)\,d\omega_2(t) + \sigma_3(t)x^2(t)x^{\beta}(t-\tau(t))\,d\omega_3(t) \\ &\quad + \sigma_4(t)x(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)d\omega_4(t) + \int_{\mathbb{Y}} \gamma(t,u)x(t^{-1})\widetilde{N}(dt,du), \end{aligned} \tag{5.1}$$

where, $x(t^-) = \lim_{s\uparrow t} x(s)$, $\omega_i(t)$ $(i = 1,2,3,4)$ are the white noises, $N(dt,du)$ is a real-valued Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Y}$

of $\overline{\mathbb{R}}_+$ with $\lambda(\mathbb{Y}) < +\infty$, $\widetilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$. By choosing space C_g as phase space, they discussed the persistence and extinction of model (5.1). Here

$$C_g = \left\{ \varphi \in C((-\infty, 0]; R) : \|\varphi\|_{c_g} = \sup_{-\infty < s \leq 0} e^{\mathbf{r}s} |\varphi(s)| < +\infty \right\},$$

where $g(s) = e^{-\mathbf{r}s}$, $\mathbf{r} > 0$. However, we need point out the fact that unlike the Brown process whose almost all sample paths are continuous, the Poisson random measure $\widetilde{N}(dt, du)$ is a jump process and has the sample paths which are right-continuous and have left limits (see e.g., [35, 38]). Because all sample paths are continuous in C_g above (see [40, 41]), the statement from [21] is incorrect with space C_g as phase space. Therefore, we will find another space as phase space when studying stochastic model with infinite delay and impulsive perturbation, and such investigations are to be done in future.

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