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A Counterexample for a Problem on Quasi Baer Modules

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Abstract. In this note we provide a counterexample to two questions on quasi-Baer modules raised recently by Lee and Rizvi in [5].

1. Introduction

Baer rings have been introduced by Kaplansky in [3]. In representing finite dimensional algebras as twisted matrix unit semigroup algebras, W. E. Clark introduced quasi-Baer rings in [1]. Later, Rizvi and Roman generalized the setting of quasi-Baer rings to modules in [7]. In [5], Lee and Rizvi asked whether a quasi-Baer module is always quasi-retractable and whether a q-local-retractable module is local-retractable. For a unital right R-module M over an associative unital ring R and a subset I of the the endomorphism ring $S = \mathrm{End}(M_R)$ we set $\mathrm{Ann}_M(I) = \bigcap_{f \in I} \mathrm{Ker}(f)$. Following [5] M is called a *quasi-Baer* module if $\mathrm{Ann}_M(I)$ is direct summand of M, for any 2-sided ideal I of S. By [5, Theorem 2.15], a module M is quasi-Baer if and only if S is quasi-Baer and M is *q-local-retractable*, where the latter means that $\mathrm{Ann}_M(I) = \mathrm{r.ann}_S(I)M$ for any 2-sided ideal I of S and $\mathrm{r.ann}_S(I)$ is the right annihilator of I in S. Related to the notion of q-local-retractability, Lee and Rizvi defined a module M to be *local-retractable* if $\mathrm{Ann}_M(L) = \mathrm{r.ann}_S(L)M$ for any left ideal L of S. Following [8, Definition 2.3], a module M is called *quasi-retractable* if $\mathrm{r.ann}_S(L) = 0$ implies $\mathrm{Ann}_M(L) = 0$ for any left ideal L of S. The following questions have been raised at the end of [5]:

(1) Is a quasi-Baer module always quasi-retractable?

(2) Is a q-local-retractable module local-retractable?

We will answer these questions in the negative. Note that any quasi-Baer module is q-local-retractable by [5, Theorem 2.15] and any local-retractable module is quasi-retractable by definition. Hence a negative answer to the first question will also answer the second question in the negative. We also note, that the notion of being quasi-Baer only involves

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2-sided ideals of the endomorphism ring. Therefore, any module M whose endomorphism ring S is a simple ring is a quasi-Baer module. Moreover, if S is a domain, then $\mathrm{r.ann}_S(I) = 0$ for any non-zero subset I of S. Hence a module M with S being a domain is local-retractable if and only if it is quasi-retractable if and only if any non-zero endomorphism of S is injective, because $\mathrm{Ann}_M(Sf) = \mathrm{Ker}(f)$ for any $f \in S$. This means that any module M whose endomorphism ring S is a simple domain and that admits an endomorphism which is not injective yields a counter example to both of the questions.

For any ring S and S-S-bimodule ${}_SN_S$, consider the generalized matrix ring $R = \left(\begin{smallmatrix} S & N \\ N & S \end{smallmatrix}\right)$ with multiplication given by

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} \cdot \begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' & ax' + xb' \\ ya' + by' & bb' \end{pmatrix}$$

for all $a, a', b, b' \in S$ and $x, x', y, y' \in N$. The ring R can be seen as the matrix ring of the Morita context (S, N, N, S) with zero multiplication $N \times N \to S$. Let $M = N \oplus S$. Then M is a right R-module by the following action:

$$(n, s) \cdot \begin{pmatrix} a & x \\ y & b \end{pmatrix} = (na + sy, sb).$$

Furthermore, $\mathrm{End}_R(M) \simeq S$, because for any right R-linear endomorphism $f\colon M \to M$ with $f(0,1) = (n', t)$, we have that

$$(0,0) = f(0,0) = f\left((0,1) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = (n', t) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (n', 0).$$

Hence $n' = 0$, i.e., $f(0,1) = (0,t)$ and therefore for all $(n,s) \in M$:

$$f(n,s) = f\left((0,1) \cdot \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix} \right) = (0,t) \cdot \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix} = (tn, ts).$$

Thus f is determined by the left multiplication of $t \in S$ on $M = N \oplus S$. Now it is easy to check that the map $S \to \mathrm{End}_R(M)$ sending t to $\lambda_t \in \mathrm{End}_R(M)$ with $\lambda_t(n,s) = (tn, ts)$, for any $(n,s) \in M$, is an isomorphism of rings. This means that the structure of M as left $\mathrm{End}_R(M)$-module is the same as the left S-module structure of M.

Lemma 1.1. *Let N be an S-bimodule, $R = \left(\begin{smallmatrix} S & N \\ N & S \end{smallmatrix}\right)$ and $M = N \oplus S$ be as above.*

(1) *If S is a simple ring, then M is a quasi-Baer right R-module.*

(2) *If S is a domain, then M is quasi-retractable if and only if M is local-retractable if and only if N is a torsionfree left S-module.*

(3) *If S is a simple domain and N is not torsionfree as left S-module, then M is a quasi-Baer right R-module that is neither quasi-retractable nor local-retractable.*

Proof. (1) Clearly if S is a simple ring, then the only 2-sided ideals of $S \simeq \mathrm{End}_R(M)$ are 0 and $\mathrm{End}_R(M)$, whose left annihilators are direct summands of M.

(2) Any local-retractable module is quasi-retractable. Suppose S is a domain and M is quasi-retractable. Let $0 \neq s \in S$. Since S is a domain, $\mathrm{r.ann}_S(L) = 0$ for $L = Ss$. Hence, as M is quasi-retractable, $0 = \mathrm{Ann}_M(L) = \mathrm{l.ann}_N(L) \oplus \mathrm{l.ann}_S(L)$, i.e., $sn \neq 0$ for all $0 \neq n \in N$. This shows that N is torsionfree as left S-module. If S is a domain and N is a torsionfree left S-module, then for any non-zero left ideal L of S, $s \in S$ and $n \in N$: $L \cdot (n, s) = (Ln, Ls) \neq 0$, i.e., $\mathrm{Ann}_M(L) = 0$ if $L \neq 0$, which shows that M is local-retractable.

(3) follows from (1) and (2). □

Theorem 1.2. *For any simple Noetherian domain D with ring of fraction Q, such that $Q \neq D$, there exist a ring R and a right R-module M such that* $\mathrm{End}(M_R) \simeq D$ *and M is a quasi-Baer right R-module, hence q-local-retractable, but neither quasi-retractable nor local-retractable.*

Proof. The bimodule $N = Q/D$ and ring $S = D$ satisfy the conditions of Lemma 1.1(3), i.e., S is a simple domain and N is not torsionfree. □

In order to obtain a concrete example, one might choose $D = A_n(k)$, the n-th Weyl algebra over a field k of characteristic zero, or more generally $D = K[x; d]$ with d a non-inner derivation of K and K a d-simple right Noetherian domain (see [4, Theorem 3.15]).

Remark 1.3. The construction in Lemma 1.1 is a special case of a more general construction. Let G be a finite group with neutral element e. A unital associative G-graded algebra is an algebra A with decomposition $A = \bigoplus_{g\in G} A_g$ (as $\mathbb{Z}$-module), such that $A_g A_h \subseteq A_{gh}$. The dual $\mathbb{Z}[G]^*$ of the integral group ring of G acts on A as follows. For all $f \in \mathbb{Z}[G]^*$ and $a = \sum_{g\in G} a_g$ one sets $f \cdot a = \sum_{g\in G} f(g) a_g$. Let $\{p_g \mid g \in G\}$ denote the dual basis of $\mathbb{Z}[G]^*$. The smash product of A and $\mathbb{Z}[G]^*$ is defined to be the algebra $A\#\mathbb{Z}[G]^* = A \otimes_{\mathbb{Z}} \mathbb{Z}[G]^*$ with product defined as $(a\#p_g)(b\#p_h) = ab_{gh^{-1}}\#p_h$ for all $a\#p_g, b\#p_h \in A\#\mathbb{Z}[G]^*$ (see [2, p. 241]). The identity element of $A\#\mathbb{Z}[G]^*$ is $1\#\epsilon$, where ϵ is the function defined by $\epsilon(g) = 1$ for all $g \in G$. The algebra A is a left $A\#\mathbb{Z}[G]^*$-module via $(a\#f) \cdot b = a(f \cdot b)$ for all $a\#f \in A\#\mathbb{Z}[G]^*$ and $b \in A$. Writing endomorphisms opposite to scalars we note that for any $z \in A_e$, the right multiplication $\rho_z \colon A \to A$ defined as $(b)\rho_z = bz$, for all $b \in A$ is left $A\#\mathbb{Z}[G]^*$-linear, because for any $f \in \mathbb{Z}[G]^*$, homogeneous $b \in A_g$ and $a \in A$ we have $((a\#f) \cdot b)\rho_z = af(g)bz = af(g)(b)\rho_z = (a\#f) \cdot (b)\rho_z$, since $(b)\rho_z = bz \in A_g$. On the other hand let $\psi \colon A \to A$ be any left $A\#\mathbb{Z}[G]^*$-linear map and set $z = (1)\psi$, the image of 1 under ψ. Then $z \in A_e$, because for any $g \in G \setminus \{e\}$ we have $p_g \cdot z = (1\#p_g) \cdot (1)\psi = (p_g \cdot 1)\psi = 0$, as $1 \in A_e$. Thus $z \in A_e$ and for any $b \in A$: $(b)\psi = ((b\#p_e) \cdot 1)\psi = (b\#p_e) \cdot z = bz = (b)\rho_z$. Hence we have a bijective correspondence between elements of A_e and left $A\#\mathbb{Z}[G]^*$-linear endomorphisms of A, i.e., $\rho\colon A_e \to \mathrm{End}({}_{A\#\mathbb{Z}[G]^*}A)$ with $\rho(z) = \rho_z$ is an isomorphism of $\mathbb{Z}$-modules. Since $(b)\rho_{zz'} = bzz' = ((b)\rho_z)\rho_{z'} = (b)(\rho_z \circ \rho_{z'})$ and $\rho_1 = \mathrm{id}_A$ we see that ρ is an isomorphism of rings (see also [6]). To summarize: any associative unital G-graded algebra A is a left module over $R = A\#\mathbb{Z}[G]^*$, such that $\mathrm{End}({}_RA)$ is isomorphic to the zero component A_e.

In the case of Lemma 1.1 consider $A = S \oplus N$ as the trivial extension of S by the S-bimodule N, i.e., $(a,x)(b,y) = (ab, ay + xb)$. Then A is graded by the group $G = \{0,1\}$ of order 2 with $A_0 = S$ and $A_1 = N$. It is not difficult to show that the smash product $A\#(\mathbb{Z}G)^*$ is anti-isomorphic to the ring $R = \left(\begin{smallmatrix} S & N \\ N & S \end{smallmatrix}\right)$ and that the left action of $A\#(\mathbb{Z}G)^*$ on A corresponds to the right action of R on $M = N \oplus S$.

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