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Maximal Multilinear Commutators on Non-homogeneous Metric Measure Spaces

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying the so-called upper doubling condition and the geometrically doubling condition. Let T_* be the maximal Calderón-Zygmund operator and $\vec{b} := (b_1, \ldots, b_m)$ be a finite family of $\widetilde{\mathrm{RBMO}}(\mu)$ functions. In this paper, the authors establish the boundedness of the maximal multilinear commutator $T_{*,\vec{b}}$ generated by T_* and $\vec{b}$ on the Lebesgue space $L^p(\mu)$ with $p \in (1, \infty)$. For $\vec{b} = (b_1, \ldots, b_m)$ being a finite family of Orlicz type functions, the weak type endpoint estimate for the maximal multilinear commutator $T_{*,\vec{b}}$ generated by T_* and $\vec{b}$ is also presented. The main tool to deal with these estimates is the smoothing technique.

1. Introduction

It is well known that the theory of Calderón-Zygmund operators is one of the core research areas in harmonic analysis. During the development of Calderón-Zygmund theory, the space of homogeneous type introduced by Coifman and Weiss [7, 8] is considered to be a natural setting for Calderón-Zygmund operators and function spaces. Recall that a quasi-metric space $(\mathcal{X}, d)$ equipped with a non-negative measure μ is called a *space of homogeneous type* in the sense of Coifman and Weiss [7,8] if $(\mathcal{X}, d, \mu)$ satisfies the *measure doubling condition*: there exists a positive constant $C_{(\mu)}$ such that, for all balls $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \le C_{(\mu)}\mu(B(x, r)). \tag{1.1}$$

This measure doubling condition is one of the most crucial assumptions in the classical harmonic analysis.

On the other hand, in the last two decades, many classical results concerning the Calderón-Zygmund operators and function spaces have been proved still valid for metric

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spaces equipped with *non-doubling measures*; see, for example, [4, 6, 14, 27–30, 34–38]. In particular, let μ be a non-negative Radon measure on $\mathbb{R}^d$ which only satisfies the *polynomial growth condition* that there exist some positive constants C_0 and $n \in (0, d]$ such that, for all $x \in \mathbb{R}^d$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq C_0 r^n, \tag{1.2}$$

where $B(x, r) := \left\{y \in \mathbb{R}^d : |x - y| < r\right\}$. Such a measure does not need to satisfy the doubling condition (1.1). The analysis on such non-doubling context plays a striking role in solving several long-standing problems related to the analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [36, 38]. Tolsa [34] introduced the atomic Hardy space $H^{1,p}_{\mathrm{atb}}(\mu)$, for $q \in (1, \infty]$, and its dual space, $\mathrm{RBMO}(\mu)$, the *space of functions with regularized bounded mean oscillation*, with respect to μ as in (1.2), and established the boundedness on $L^p(\mu)$ with $p \in (1, \infty)$ of commutators generated by Carderón-Zygmund operators and $\mathrm{RBMO}(\mu)$ functions. Chen and Miao [5] proved that the maximal commutator generated by the maximal Calderón-Zygmund operator and the $\mathrm{RBMO}(\mu)$ function is bounded on $L^p(\mu)$ with $p \in (1, \infty)$. The weak type endpoint estimate for the maximal commutator generated by the maximal Calderón-Zygmund operator and the Orlicz type function was obtained by Hu et al. [16]. Li and Jiang [22] established the corresponding results for the maximal multilinear commutators.

However, as was pointed out by Hytönen in [17], the measure μ satisfying the polynomial growth condition is different from, not general than, the doubling measure. Hytönen [17] introduced a new class of metric measure spaces satisfying both the so-called upper doubling condition and the geometrically doubling condition (see, respectively, Definitions 1.1 and 1.3 below), which are also simply called *non-homogeneous metric measure spaces*. These new class of metric measure spaces include both metric measure spaces of homogeneous type and metric measure spaces equipped with non-doubling measures as special cases. We mention that several equivalent characterizations for the upper doubling condition were recently established by Tan and Li [32, 33].

From now on, we always assume that $(\mathcal{X}, d, \mu)$ is a metric measure space of non-homogeneous type in the sense of Hytönen [17]. In this new setting, Hytönen [17] introduced the space $\mathrm{RBMO}(\mu)$ and established the corresponding John-Nirenberg inequality, and Hytönen and Martikainen [19] further established a version of Tb theorem. Later, Hytönen et al. [21] and Bui and Duong [2], independently, introduced the atomic Hardy space $H^{1,p}_{\mathrm{atb}}(\mu)$ and proved that the dual space of $H^{1,p}_{\mathrm{atb}}(\mu)$ is $\mathrm{RBMO}(\mu)$. Recently, Fu et al. [11] established the boundedness of multilinear commutators generated by Calderón-Zygmund operators and $\mathrm{RBMO}(\mu)$ functions. Bui [1] obtained the $L^p(\mu)$-boundedness of the maximal commutator generated by the maximal Calderón-Zygmund operator and the $\mathrm{RBMO}(\mu)$ function under the additional assumption that there exists a positive constant

m such that $\lambda(x, ar) = a^m\lambda(x, r)$ for all $x \in \mathcal{X}$ and $a, r \in (0, \infty)$, where λ is the dominating function of the measure μ (see Definition 1.1 below). The boundedness of commutators of multilinear singular integrals on Lebesgue spaces was obtained by Xie et al. [40]. In addition, Fu et al. [10] introduced a version of the atomic Hardy space $\widetilde{H}^{1,p,\gamma}_{\mathrm{atb},\rho}(\mu)$ ($\subset H^{1,p}_{\mathrm{atb}}(\mu)$ and simply denoted by $\widetilde{H}^1(\mu)$) and its corresponding dual space $\widetilde{\mathrm{RBMO}}(\mu)$ ($\supset \mathrm{RBMO}(\mu)$; see Definition 1.8 below) via the discrete coefficients $\widetilde{K}^{(\rho)}_{B,S}$. Very recently, Lin et al. [23] proved that the commutator of the Calderón-Zygmund operator with $\widetilde{\mathrm{RBMO}}(\mu)$ function is bounded from the atomic Hardy space $\widetilde{H}^1(\mu)$ into the weak Lebesgue space $L^{1,\infty}(\mu)$. More research on function spaces and the boundedness of various operators on metric measure spaces of non-homogeneous type can be found in [18, 20, 24, 26]. We refer the reader to the survey [41] and the monograph [42] for more developments on harmonic analysis in this setting.

The main purpose of this paper is to establish the boundedness of the maximal multilinear commutators in the present setting $(\mathcal{X}, d, \mu)$. Precisely, let T_* be the maximal Calderón-Zygmund operator associated with the truncated operator T_ϵ and $\vec{b} := (b_1, \dots, b_m)$ be a finite family of $\widetilde{\mathrm{RBMO}}(\mu)$ functions. We establish the boundedness of the maximal multilinear commutator $T_{*,\vec{b}}$ generated by T_* and $\vec{b}$ on the Lebesgue space $L^p(\mu)$ with $p \in (1, \infty)$. This generalizes the corresponding result in [1]. For $\vec{b} = (b_1, \dots, b_m)$ being a finite family of Orlicz type functions (see Definition 1.12 below), the $L \log L$ type endpoint estimate for the maximal multilinear commutator $T_{*,\vec{b}}$ generated by T_* and $\vec{b}$ is also presented. The main tool to deal with these estimates is the smoothing technique. We mention that this smoothing technique was used by Segovia and Torrea [31] in the setting of classical Euclidean spaces and by García-Cuerva and Martell [12] in the setting of metric measure spaces equipped with non-doubling measures.

To state our main results, we first recall some necessary notions and notations. We start with the following notion of upper doubling metric measure spaces originally introduced by Hytönen [17] (see also [18, 26]).

Definition 1.1. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda\colon \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_{(\lambda)}$, depending on λ, such that, for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_{(\lambda)}\lambda(x, r/2). \tag{1.3}$$

Remark 1.2. (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. On the other hand, the d-dimensional Euclidean space

$\mathbb{R}^d$ with any Radon measure μ as in (1.2) is also an upper doubling space by taking $\lambda(x,r) := C_0 r^n$ for all $x \in \mathbb{R}^d$ and $r \in (0,\infty)$.

(ii) Let $(\mathcal{X}, d, \mu)$ be upper doubling with λ being the dominating function on $\mathcal{X} \times (0,\infty)$ as in Definition 1.1. It was proved in [21] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C_{(\widetilde{\lambda})} \le C_{(\lambda)}$ and, for all $x, y \in \mathcal{X}$ with $d(x,y) \le r$,

$$\widetilde{\lambda}(x,r) \le C_{(\widetilde{\lambda})} \widetilde{\lambda}(y,r). \tag{1.4}$$

(iii) It was shown in [32] that the upper doubling condition is equivalent to the *weak growth condition*: there exist a dominating function $\lambda\colon \mathcal{X} \times (0,\infty) \to (0,\infty)$, with $r \to \lambda(x,r)$ non-decreasing, positive constants $C_{(\lambda)}$, depending on λ, and σ such that

(iii)$_1$ for all $r \in (0,\infty)$, $t \in [0,r]$, $x, y \in \mathcal{X}$ and $d(x,y) \in [0,r]$,

$$|\lambda(y, r+t) - \lambda(x,r)| \le C_{(\lambda)} \left[\frac{d(x,y)+t}{r}\right]^{\sigma} \lambda(x,r);$$

(iii)$_2$ for all $x \in \mathcal{X}$ and $r \in (0,\infty)$, $\mu(B(x,r)) \le \lambda(x,r)$.

(iv) It was proved in [23] that the dominating function λ satisfying (1.4) has the following property: for any fixed ball $B \subset \mathcal{X}$, if $x_1, x_2 \in B$ and $y \in \mathcal{X} \setminus (kB)$ with $k \in [2,\infty)$, then there exists a positive constant C such that $C^{-1}\lambda(x_1, d(x_1,y)) \le \lambda(x_2, d(x_2,y)) \le C\lambda(x_1, d(x_1,y))$; see [23, Lemma 2.3].

The following definition of geometrically doubling is well known in analysis on metric spaces, which can be found in Coifman and Weiss [7, pp. 66–67], and is also known as *metrically doubling* (see, for example, [13, p. 81]). Moreover, spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [7, pp. 66–68].

Definition 1.3. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N} := \{1, 2, \ldots\}$ such that, for any ball $B(x,r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most N_0.

Remark 1.4. Let $(\mathcal{X}, d)$ be a metric space. In [17], Hytönen showed that the following statements are mutually equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling;

(ii) for any $\epsilon \in (0,1)$ and any ball $B(x,r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most ϵ^{-n_0}, here and hereafter, N_0 is as in Definition 1.3 and $n_0 := \log_2 N_0$;

(iii) for every $\epsilon \in (0,1)$, any ball $B(x,r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$ contains at most ϵ^{-n_0} centers of disjoint balls $\{B(x_i,\epsilon r)\}_i$;

(iv) there exists $M \in \mathbb{N}$ such that any ball $B(x,r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i,r/4)\}_{i=1}^M$.

A metric measure space $(\mathcal{X},d,\mu)$ is called a *non-homogeneous metric measure space* if $(\mathcal{X},d)$ is geometrically doubling and $(\mathcal{X},d,\mu)$ is upper doubling. Based on Remark 1.2(ii), from now on, we *always assume* that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space with the dominating function λ satisfying (1.4).

Although the measure doubling condition is not assumed uniformly for all balls in the non-homogeneous metric measure space $(\mathcal{X},d,\mu)$, it was shown in [17] that there still exist many balls which have the following (α,β)-doubling property. In what follows, for any ball $B \subset \mathcal{X}$, we denote its *center* and *radius*, respectively, by c_B and r_B and, moreover, for any $\rho \in (0,\infty)$, we denote the ball $B(c_B,\rho r_B)$ by ρB.

Definition 1.5. Let $\alpha,\beta \in (1,\infty)$. A ball $B \subset \mathcal{X}$ is said to be (α,β)-*doubling* if $\mu(\alpha B) \leq \beta\mu(B)$.

To be precise, it was proved in [17, Lemma 3.2] that, if a metric measure space $(\mathcal{X},d,\mu)$ is upper doubling and $\alpha,\beta \in (1,\infty)$ with $\beta > [C_{(\lambda)}]^{\log_2 \alpha} =: \alpha^\nu$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ such that $\alpha^j B$ is (α,β)-doubling. Moreover, let $(\mathcal{X},d)$ be geometrically doubling, $\beta > \alpha^{n_0}$ with $n_0 := \log_2 N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [17, Lemma 3.3] also showed that, for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (α,β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j}r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0,\infty)$. Throughout this article, for any $\alpha \in (1,\infty)$ and ball B, the *smallest* (α,β_α)-*doubling ball of the form* $\alpha^j B$ *with* $j \in \mathbb{Z}_+$ is denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha := \alpha^{3(\max\{n_0,\nu\})} + [\max\{5\alpha,30\}]^{n_0} + [\max\{3\alpha,30\}]^\nu.$$

Also, for any ball B of $\mathcal{X}$, we denote by $\widetilde{B}$ the smallest $(6,\beta_6)$-doubling cube of the form $6^j B$ with $j \in \mathbb{Z}_+$, especially, throughout this paper.

The following discrete coefficient $\widetilde{K}_{B,S}^{(\rho)}$ was first introduced by Bui and Duong [2] as analogous of the quantity introduced by Tolsa [34] (see also [35]) in the setting of non-doubling measures; see also [9,10]. Before we recall the definition of $\widetilde{K}_{B,S}^{(\rho)}$, we first give an assumption, when we speak of a ball B in $(\mathcal{X},d,\mu)$, it is understood that it comes with a fixed center and radius, although these in general are not uniquely determined by B as a set; see [13, pp. 1–2]. In other words, for any two balls $B,S \subset \mathcal{X}$, if $B = S$, then $c_B = c_S$ and $r_B = r_S$. From this, we deduce that if $B \subset S$, then $r_B \leq 2r_S$, which plays an essential role in the definition of $\widetilde{K}_{B,S}^{(\rho)}$; see also Remark 1.7(i) and [9, pp. 314–315] for some details.

Definition 1.6. For any $\rho \in (1, \infty)$ and any two balls $B \subset S \subset \mathcal{X}$, let

$$\widetilde{K}_{B,S}^{(\rho)} := 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N_{B,S}^{(\rho)}} \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)},$$

here and hereafter, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the *biggest integer which is not bigger than* a, and $N_{B,S}^{(\rho)}$ is the *smallest integer* satisfying $\rho^{N_{B,S}^{(\rho)}} r_B \geq r_S$.

Remark 1.7. (i) With the definition of $N_{B,S}^{(\rho)}$ and the fact that $r_B \leq 2r_S$, we deduce that $N_{B,S}^{(\rho)} \geq \lceil -\log_\rho 2 \rceil = -\lfloor \log_\rho 2 \rfloor$, which makes the definition of $\widetilde{K}_{B,S}^{(\rho)}$ sense.

(ii) By a change of variables and (1.3), we easily conclude that

$$\widetilde{K}_{B,S}^{(\rho)} \sim 1 + \sum_{k=1}^{N_{B,S}^{(\rho)} + \lfloor \log_\rho 2 \rfloor + 1} \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)},$$

where the implicit equivalent positive constants are independent of balls $B \subset S \subset \mathcal{X}$, but depend on ρ.

(iii) A continuous version, $K_{B,S}$, of the coefficient in Definition 1.6 was introduced in [17, 21] as follows. For any two balls $B \subset S \subset \mathcal{X}$, let

$$K_{B,S} := 1 + \int_{(2S)\setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \, d\mu(x).$$

It was proved in [21] that $K_{B,S}$ has all properties similar to those for $\widetilde{K}_{B,S}^{(\rho)}$ as in Lemma 2.1 below. Unfortunately, $K_{B,S}$ and $\widetilde{K}_{B,S}^{(\rho)}$ are usually not equivalent, but, for $(\mathbb{R}^d, |\cdot|, \mu)$ with μ as in (1.2),

$$K_{B,S} \sim \widetilde{K}_{B,S}^{(\rho)} \tag{1.5}$$

with implicit equivalent positive constants independent of B and S; see [10] for more details on this.

Now we recall the $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ space associated with $\widetilde{K}_{B,S}^{(\rho)}$, which was first introduced by Fu et al. [10].

Definition 1.8. Let $\rho \in (1, \infty)$ and $\gamma \in [1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ if there exists a positive constant $\widetilde{C}$ and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \, d\mu(x) \leq \widetilde{C} \tag{1.6}$$

and, for any two balls B and B_1 such that $B \subset B_1$,

$$|f_B - f_{B_1}| \le \widetilde{C}\left[\widetilde{K}_{B,B_1}^{(\rho)}\right]^{\gamma}. \tag{1.7}$$

The infimum of the positive constant $\widetilde{C}$ satisfying both (1.6) and (1.7) is defined to be the $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ norm of f and denoted by $\|f\|_{\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)}$.

Remark 1.9. (i) It was pointed out by Fu et al. [10] that the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ is independent of $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$. In what follows, we denote $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ simply by $\widetilde{\mathrm{RBMO}}(\mu)$.

(ii) When $(\mathcal{X}, d, \mu) = (\mathbb{R}^d, |\cdot|, \mu)$ with μ as in (1.2), by (1.5), we see that $\widetilde{\mathrm{RBMO}}(\mu)$ becomes the regularized $\mathrm{BMO}(\mu)$ space, $\mathrm{RBMO}(\mu)$, introduced in [34] for $\gamma = 1$ and in [14] for $\gamma \in (1,\infty)$. For general metric measure spaces of non-homogeneous type, if we replace $\widetilde{K}_{B,S}^{(\rho)}$ by $K_{B,S}$ in Definition 1.8, then $\widetilde{\mathrm{RBMO}}(\mu)$ becomes the space $\mathrm{RBMO}(\mu)$ in [17]. Obviously, for $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$, $\mathrm{RBMO}(\mu) \subset \widetilde{\mathrm{RBMO}}(\mu)$. However, it is unclear whether $\mathrm{RBMO}(\mu) = \widetilde{\mathrm{RBMO}}(\mu)$ or not.

Definition 1.10. A function $K \in L^1_{\mathrm{loc}}(\{\mathcal{X} \times \mathcal{X}\} \setminus \{(x,x) : x \in \mathcal{X}\})$ is called a *Calderón-Zygmund kernel* if there exists a positive constant C, such that,

(i) for all $x, y \in \mathcal{X}$ with $x \ne y$,

$$|K(x,y)| \le C\frac{1}{\lambda(x, d(x,y))}; \tag{1.8}$$

(ii) there exist positive constants $\delta \in (0,1]$ and $c_{(K)}$, depending on K, such that, for all $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x,y) \ge c_{(K)} d(x,\widetilde{x})$,

$$|K(x,y) - K(\widetilde{x},y)| + |K(y,x) - K(y,\widetilde{x})| \le C\frac{[d(x,\widetilde{x})]^\delta}{[d(x,y)]^\delta \lambda(x, d(x,y))}. \tag{1.9}$$

A linear operator T is called the *Calderón-Zygmund operator* with kernel K satisfying (1.8) and (1.9) if, for all $f \in L^\infty_b(\mu) := \{f \in L^\infty(\mu) : \mathrm{supp}(f) \text{ is bounded}\}$,

$$Tf(x) := \int_{\mathcal{X}} K(x,y) f(y)\, d\mu(y), \quad x \notin \mathrm{supp}(f). \tag{1.10}$$

A new example of the operator with kernel satisfying (1.8) and (1.9) is the so-called Bergman-type operator appearing in [39]; see also [19] for an explanation. Let $\epsilon \in (0,\infty)$. The *truncated operator* T_ϵ is defined by setting, for suitable f and $x \in \mathcal{X}$,

$$T_\epsilon f(x) := \int_{d(x,y)>\epsilon} K(x,y) f(y)\, d\mu(y).$$

The *maximal operator* T_* associated with the $\{T_\epsilon\}_{\epsilon>0}$ is defined by setting, for suitable f and $x \in \mathcal{X}$,

$$T_* f(x) := \sup_{\epsilon>0} |T_\epsilon f(x)| .$$

As a corollary of [25, Theorem 1.5], we see that T_* is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$; see also [1, 2]. Let $\vec{b} := (b_1, \ldots, b_m)$ be a finite family of $\widetilde{\mathrm{RBMO}}(\mu)$ functions. We simply write $\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} := \|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)} \cdots \|b_m\|_{\widetilde{\mathrm{RBMO}}(\mu)}$. The *maximal multilinear commutator* $T_{*,\vec{b}}$ generated by T_* and $\vec{b}$ is defined by setting, for $x \in \mathcal{X}$,

$$T_{*,\vec{b}} f(x) := \sup_{\epsilon>0} \left|T_{\epsilon,\vec{b}} f(x)\right| = \sup_{\epsilon>0} \left| \int_{d(x,y)>\epsilon} \prod_{i=1}^m [b_i(x) - b_i(y)] K(x,y) f(y)\, d\mu(y) \right| . \tag{1.11}$$

One of the main results of this paper is stated as follows.

Theorem 1.11. *Let $m \in \mathbb{N}$ and $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$ for all $i \in \{1, \ldots, m\}$. Let T and $T_{*,\vec{b}}$ be as in* (1.10) *and* (1.11)*, respectively. Assume that T is bounded on $L^2(\mu)$. Then the maximal multilinear commutator $T_{*,\vec{b}}$ is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$. More precisely, there exists a positive constant C such that, for all $f \in L^p(\mu)$,*

$$\left\|T_{*,\vec{b}} f\right\|_{L^p(\mu)} \le C \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \|f\|_{L^p(\mu)} .$$

To consider the endpoint estimate for $T_{*,\vec{b}}$, we first introduce the following *Orlicz type function space* $\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)$, which is a variant with non-doubling measure of the space $\mathrm{Osc}_{\exp L^r}$ in [15].

Definition 1.12. For $r \in [1,\infty)$, a locally integrable function f is said to belong to the space $\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)$ if there exists a positive constant C such that,

(i) for all balls B,

$$\begin{aligned} &\left\|f - m_{\widetilde{B}}(f)\right\|_{\exp L^r, B, \mu/\mu(2B)} \\ &:= \inf \left\{ \lambda \in (0,\infty) : \frac{1}{\mu(2B)} \int_B \exp\left(\frac{\left|f - m_{\widetilde{B}}(f)\right|}{\lambda} \right)^r d\mu \le 2 \right\} \le C; \end{aligned}$$

(ii) for any doubling balls $B \subset S \subset \mathcal{X}$,

$$|m_B(f) - m_S(f)| \le C \widetilde{K}^{(\rho)}_{B,S},$$

here and hereafter, for all balls B and $f \in L^1_{\mathrm{loc}}(\mu)$, $m_B(f)$ denotes its mean over B, namely, $m_B(f) := \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x)$.

The minimal constant C satisfying (i) and (ii) is the $\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)$ norm of f and defined by $\|f\|_{\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)}$.

Remark 1.13. If we replace $\widetilde{K}^{(\rho)}_{B,S}$ by $K_{B,S}$ in Definition 1.12, then $\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)$ becomes the space $\mathrm{Osc}_{\exp L^r}(\mu)$ in [11]. Obviously, for any $r \in [1,\infty)$, $\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu) \subset \widetilde{\mathrm{RBMO}}(\mu)$ and, for all $f \in \widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)$, $\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \le \|f\|_{\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)}$. Moreover, from John-Nirenberg's inequality in [34], it follows that $\widetilde{\mathrm{Osc}}_{\exp L^1}(\mu) = \widetilde{\mathrm{RBMO}}(\mu)$.

Let $m \in \mathbb{N}$, $r_i \in [1,\infty)$ and $b_i \in \widetilde{\mathrm{Osc}}_{\exp L^{r_i}}(\mu)$ for $i \in \{1,\dots,m\}$. Let $\vec{b} = (b_1,\dots,b_m)$ and $r \in [1,\infty)$ with $1/r = 1/r_1 + \cdots + 1/r_m$, we simply write

$$\|\vec{b}\|_{\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)} := \|b_1\|_{\widetilde{\mathrm{Osc}}_{\exp L^{r_1}}(\mu)} \cdots \|b_m\|_{\widetilde{\mathrm{Osc}}_{\exp L^{r_m}}(\mu)}.$$

Now we state another main result of this paper as follows.

Theorem 1.14. *Let $m \in \mathbb{N}$, $r_i \in [1,\infty)$ and $b_i \in \widetilde{\mathrm{Osc}}_{\exp L^{r_i}}(\mu)$ for $i \in \{1,\dots,m\}$. Let T and $T_{*,\vec{b}}$ be as in* (1.10) *and* (1.11)*, respectively. Assume that T is bounded on $L^2(\mu)$. Then there exists a positive constant C such that, for all $t \in (0,\infty)$ and all $f \in L^\infty_b(\mu)$,*

$$\mu\left(\left\{x \in \mathcal{X} : \left|T_{*,\vec{b}}f(x)\right| > t\right\}\right) \le C\Phi_{1/r}\left(\|\vec{b}\|_{\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)}\right) \int_{\mathcal{X}} \Phi_{1/r}\left(\frac{|f(y)|}{t}\right) d\mu(y),$$

where $1/r = 1/r_1 + \cdots + 1/r_m$ and, for all $t \in (0,\infty)$ and $s \in (0,\infty)$, $\Phi_s(t) = t\log^s(2+t)$.

This paper is organized as follows. Section 2 is devoted to proving Theorem 1.11. We first recall some necessary lemmas, and then introduce some new "smooth" kernels. Moreover, we prove that the smoothing technique is still suitable for the present setting. At the end of this section, by borrowing some ideas from the proofs of [1, Theorem 3.3], [11, Theorem 1.9] and [22, Theorem 1.1], we prove Theorem 1.11. In Section 3, we prove Theorem 1.14 via the generalized Hölder's inequality and the Calderón-Zygmund decomposition.

Finally, we make some conventions on notation. Throughout this paper, we always denote by C, $\widetilde{C}$, c or $\widetilde{c}$ a *positive constant* which is independent of the main parameters, but they may vary from line to line. *Constants with subscripts*, such as C_0 and c_0, do not change in different occurrences. Furthermore, we use $C_{(\alpha)}$ to denote a positive constant depending on the parameter α. The expression $Y \lesssim Z$ means that there exists a positive constant C such that $Y \le CZ$. The expression $A \sim B$ means that $A \lesssim B \lesssim A$. Given any $q \in (0,\infty)$, let $q' := q/(q-1)$ denote its *conjugate index*. Also, for any subset $E \subset \mathcal{X}$, χ_E denotes its *characteristic function*.

2. Proof of Theorem 1.11

We begin with some necessary lemmas. The following useful properties of $\widetilde{K}^{(\rho)}_{B,S}$ were proved in [9].

Lemma 2.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space.*

(i) *For any $\rho \in (1,\infty)$, there exists a positive constant $C_{(\rho)}$, depending on ρ, such that, for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{B,R} \le C_{(\rho)} \widetilde{K}^{(\rho)}_{B,S}$.*

(ii) *For any $\alpha \in [1,\infty)$ and $\rho \in (1,\infty)$, there exists a positive constant $C_{(\alpha,\rho)}$, depending on α and ρ, such that, for all balls $B \subset S$ with $r_S \le \alpha r_B$, $\widetilde{K}^{(\rho)}_{B,S} \le C_{(\alpha,\rho)}$.*

(iii) *For any $\rho \in (1,\infty)$, there exists a positive constant $C_{(\rho,\nu)}$, depending on ρ and ν, such that, for all balls B, $\widetilde{K}^{(\rho)}_{B,\widetilde{B}^{\rho}} \le C_{(\rho,\nu)}$. Moreover, letting $\alpha, \beta \in (1,\infty)$, $B \subset S$ be any two concentric balls such that there exists no (α,β)-doubling ball in the form of $\alpha^k B$ with $k \in \mathbb{N}$, satisfying $B \subset \alpha^k B \subset S$, then there exists a positive constant $C_{(\alpha,\beta,\nu)}$, depending on α, β and ν, such that $\widetilde{K}^{(\rho)}_{B,S} \le C_{(\alpha,\beta,\nu)}$.*

(iv) *For any $\rho \in (1,\infty)$, there exists a positive constant $c_{(\rho,\nu)}$, depending on ρ and ν, such that, for all balls $B \subset R \subset S$,*
$$\widetilde{K}^{(\rho)}_{B,S} \le \widetilde{K}^{(\rho)}_{B,R} + c_{(\rho,\nu)} \widetilde{K}^{(\rho)}_{R,S}.$$

(v) *For any $\rho \in (1,\infty)$, there exists a positive constant $\widetilde{c}_{(\rho,\nu)}$, depending on ρ and ν, such that, for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{R,S} \le \widetilde{c}_{(\rho,\nu)} \widetilde{K}^{(\rho)}_{B,S}$.*

(vi) *For any $\rho_1, \rho_2 \in (1,\infty)$, there exist positive constants $c_{(\rho_1,\rho_2,\nu)}$ and $C_{(\rho_1,\rho_2,\nu)}$, depending on ρ_1, ρ_2 and ν, such that, for all balls $B \subset S$,*
$$c_{(\rho_1,\rho_2,\nu)} \widetilde{K}^{(\rho_1)}_{B,S} \le \widetilde{K}^{(\rho_2)}_{B,S} \le C_{(\rho_1,\rho_2,\nu)} \widetilde{K}^{(\rho_1)}_{B,S}.$$

The following four lemmas are related to the space $\widetilde{\mathrm{RBMO}}(\mu)$. Lemma 2.2 can be proved by an argument similar to that used in the proof of [11, Lemma 3.1]. Lemma 2.3 is an equivalent characterization of the space $\widetilde{\mathrm{RBMO}}(\mu)$ established in [23, Lemma 2.15]. Lemmas 2.4 and 2.5 were proved in [3].

Lemma 2.2. *Let $f \in \widetilde{\mathrm{RBMO}}(\mu)$, $q \in (0,\infty)$ and, for all $x \in \mathcal{X}$,*
$$f_q(x) := \begin{cases} f(x) & \text{if } |f(x)| \le q, \\ q\frac{f(x)}{|f(x)|} & \text{if } |f(x)| > q. \end{cases}$$
Then $f_q \in \widetilde{\mathrm{RBMO}}(\mu)$ and there exists a positive constant C, independent of f, such that $\|f_q\|_{\widetilde{\mathrm{RBMO}}(\mu)} \le C \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}$.

Lemma 2.3. *Let $\eta, \rho \in (1, \infty)$, and β_ρ be as in (1.5). For $f \in L^1_{\mathrm{loc}}(\mu)$, the following statements are equivalent:*

(i) *$f \in \widetilde{\mathrm{RBMO}}(\mu)$;*

(ii) *there exists a positive constant C such that, for all balls B,*

$$\frac{1}{\mu(\eta B)} \int_B \left| f(x) - m_{\widetilde{B}^\rho}(f) \right| d\mu(x) \le C$$

and, for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$|m_B(f) - m_S(f)| \le C \widetilde{K}^{(\rho)}_{B,S}.$$

Moreover, the infimum of the above constant C is equivalent to $\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}$.

Lemma 2.4. *Let $m \in \mathbb{N}$, $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$ for $i \in \{1, \dots, m\}$, $\rho, \eta \in (1, \infty)$ and $q \in [1, \infty)$. Then there exists a positive constant C such that, for any ball B,*

$$\left\{ \frac{1}{\mu(\rho B)} \int_B \prod_{i=1}^m \left| b_i(x) - m_{\widetilde{B}^\eta}(b_i) \right|^q d\mu(x) \right\}^{1/q} \le C \prod_{i=1}^m \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

Lemma 2.5. *Let $f \in \widetilde{\mathrm{RBMO}}(\mu)$ and $\rho \in (1, \infty)$. Then, for all two balls $B \subset S \subset \mathcal{X}$, we have*

$$\left| m_{\widetilde{B}^\rho}(f) - m_{\widetilde{S}^\rho}(f) \right| \lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \widetilde{K}^{(\rho)}_{B,S}.$$

We also need to recall some known conclusion from [2, Sections 4.1 and 7.1] and [17, Corollary 3.6].

Lemma 2.6. *Let $p \in (1, \infty)$.*

(i) *Let $r \in (1, p)$ and $\rho \in [5, \infty)$. The following maximal operators, defined by setting, for all $f \in L^p(\mu)$ and $x \in \mathcal{X}$,*

$$M_{r,(\rho)} f(x) := \sup_{B \ni x} \left[\frac{1}{\mu(\rho B)} \int_B |f(x)|^r d\mu(x) \right]^{1/r},$$

$$Nf(x) := \sup_{\substack{B \ni x \\ B\ (6,\beta_6)\text{-doubling}}} \frac{1}{\mu(B)} \int_B |f(x)| \, d\mu(x)$$

and

$$M_{(\rho)} f(x) := \sup_{B \ni x} \frac{1}{\mu(\rho B)} \int_B |f(x)| \, d\mu(x)$$

are bounded on $L^p(\mu)$. Moreover, for $\rho \in [5, \infty)$, $M_{(\rho)}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

(ii) $|f(x)| \le Nf(x)$ *for almost every* $x \in \mathcal{X}$.

For any $f \in L^1_{\rm loc}(\mu)$, recall that the *sharp maximal function* $M^{\#}f$ in [2] is defined by setting, for all $x \in \mathcal{X}$,

$$M^{\#}f(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B \left|f(y) - m_{\widetilde{B}}(f)\right| d\mu(y) + \sup_{\substack{x \in B \subset S \\ B,S\ (6,\beta_6)\text{-doubling}}} \frac{|m_B(f) - m_S(f)|}{\widetilde{K}^{(6)}_{B,S}}.$$

The following lemma is just [2, Theorem 4.2].

Lemma 2.7. *Let* $f \in L^1_{\rm loc}(\mu)$ *satisfy* $\int_{\mathcal{X}} f(x)\, d\mu(x) = 0$ *when* $\|\mu\| := \mu(\mathcal{X}) < \infty$. *Assume that, for some* $p \in (1,\infty)$, $\inf\{1, Nf\} \in L^p(\mu)$. *Then there exists a positive constant* C, *independent of* f, *such that*

$$\|Nf\|_{L^p(\mu)} \le C\|M^{\#}f\|_{L^p(\mu)}.$$

Notice that the truncated kernel $K_\epsilon(x,y) = K(x,y)\chi_{\{d(x,y)\ge\epsilon\}}(x,y)$ may not be a Calderón-Zygmund kernel, which is a problem in studying the boundedness of the maximal multilinear commutators. To overcome this problem, we use the smoothing technique (see [1, 12, 31]) by replacing $K_\epsilon(x,y)$ with some new "smooth" kernels, and then use the properties of sharp maximal operator $M^{\#}$ to estimate the multilinear commutators associated with the "smooth" kernels and $\widetilde{\rm RBMO}(\mu)$ functions.

Definition 2.8. Let K be the Calderón-Zygmund kernel and $\phi, \psi \in C^\infty([0,\infty))$ such that $\chi_{[2,\infty)} \le \phi \le \chi_{[1,\infty)}$, $\chi_{[0,1/2)} \le \psi \le \chi_{[0,3)}$ and, for all $t \in (0,\infty)$, $|\phi'(t)| \le C/t$, $|\psi'(t)| \le C/t$, where C is a positive constant. Let $\epsilon \in (0,\infty)$. Define the kernel $K^\phi_\epsilon(x,y)$ associated with K and ϕ, and the kernel $K^\psi_\epsilon(x,y)$ associated with ψ, respectively, by setting

$$K^\phi_\epsilon(x,y) := K(x,y)\,\phi\left(\frac{d(x,y)}{\epsilon}\right) \quad \text{and} \quad K^\psi_\epsilon(x,y) := \frac{1}{\lambda(x,\epsilon)}\,\psi\left(\frac{d(x,y)}{\epsilon}\right).$$

Lemma 2.9. *Let* K^ϕ_ϵ *and* K^ψ_ϵ *be as in Definition* 2.8. *Then* K^ϕ_ϵ *and* K^ψ_ϵ *are Calderón-Zygmund kernels satisfying conditions* (1.8) *and* (1.9), *where the positive constants are independent of* ϵ.

Proof. We first deal with the kernel K^ϕ_ϵ. By the properties of ϕ, it is easy to see that the kernel $K^\phi_\epsilon(x,y)$ satisfies condition (1.8). To prove (1.9), let $x, x', y \in \mathcal{X}$ with $d(x,y) \ge c_{(K)}d(x,x')$ and $\delta \in (0,1]$ be as in Definition 1.10(ii). Here, we may assume that $c_{(K)} > 1$. In fact, if $c_{(K)} \in (0,1]$, then we can choose $\widetilde{c}_{(K)} > 1$ such that, for all $d(x,y) \ge \widetilde{c}_{(K)}d(x,x')$, (1.9) holds true. By $c_{(K)} > 1$, we see that

$$d(x',y) \ge d(x,y) - d(x,x') \ge \frac{c_{(K)} - 1}{c_{(K)}} d(x,y)$$

and

$$d(x',y) \le d(x,x') + d(x,y) \le \frac{c_{(K)}+1}{c_{(K)}} d(x,y).$$

Hence, we have

$$d(x,y) \sim d(x',y). \tag{2.1}$$

We consider the following four cases of $d(x,y)$ and $d(x',y)$.

Case (I): $d(x,y) \le \epsilon$ and $d(x',y) \le \epsilon$. In this case, $d(x,y)/\epsilon \le 1$ and $d(x',y)/\epsilon \le 1$. Notice that $\phi \in C^\infty$ and $\chi_{[2,\infty)} \le \phi \le \chi_{[1,\infty)}$. From this, it follows that $\phi(t) = 0$ for all $t \in [0,1]$, which leads to

$$\left| K_\epsilon^\phi(x,y) - K_\epsilon^\phi(x',y) \right| = 0.$$

Case (II): $d(x,y) > \epsilon$ and $d(x',y) \le \epsilon$. In this case, by Case (I), we have $K_\epsilon^\phi(x',y) = 0$. By the mean value theorem and $|\phi'(t)| \le C/t$, we conclude that, for all $t_1, t_2 \in (0,\infty)$,

$$|\phi(t_1) - \phi(t_2)| \lesssim |t_1 - t_2| \frac{1}{\min\{t_1, t_2\}}, \tag{2.2}$$

which, together with the fact that $\phi(1) = 0$, $d(x,y)/\epsilon > 1$, (2.1) and (1.8), implies that

$$\begin{aligned}
\left| K_\epsilon^\phi(x,y) - K_\epsilon^\phi(x',y) \right| &= |K(x,y)| \left| \phi\left(\frac{d(x,y)}{\epsilon} \right) - \phi(1) \right| \\
&\lesssim \frac{1}{\lambda(x, d(x,y))} \left| \frac{d(x,y)}{\epsilon} - 1 \right| \frac{1}{\min\{d(x,y)/\epsilon, 1\}} \\
&\lesssim \frac{1}{\lambda(x, d(x,y))} \left| \frac{d(x,y) - d(x',y)}{\epsilon} \right| \\
&\lesssim \frac{1}{\lambda(x, d(x,y))} \frac{d(x,x')}{d(x',y)} \lesssim \frac{[d(x,x')]^\delta}{[d(x,y)]^\delta \lambda(x, d(x,y))}.
\end{aligned}$$

Case (III): $d(x,y) \le \epsilon$ and $d(x',y) > \epsilon$. In this case, $d(x,y) \le \epsilon < d(x',y)$. This, together with the fact that $\phi(1) = 0$, (2.2), (1.8) and (1.4), shows that

$$\begin{aligned}
\left| K_\epsilon^\phi(x,y) - K_\epsilon^\phi(x',y) \right| &= \left| K(x',y) \right| \left| \phi\left(\frac{d(x',y)}{\epsilon} \right) - \phi(1) \right| \\
&\lesssim \frac{1}{\lambda(x', d(x',y))} \left(\frac{d(x',y)}{\epsilon} - 1 \right) \frac{1}{\min\{d(x',y)/\epsilon, 1\}} \\
&\lesssim \frac{1}{\lambda(y, d(x',y))} \frac{d(x',y) - d(x,y)}{\epsilon} \\
&\lesssim \frac{1}{\lambda(y, d(x,y))} \frac{d(x',y) - d(x,y)}{d(x,y)} \\
&\lesssim \frac{1}{\lambda(x, d(x,y))} \frac{d(x,x')}{d(x,y)} \lesssim \frac{[d(x,x')]^\delta}{[d(x,y)]^\delta \lambda(x, d(x,y))}.
\end{aligned}$$

Case (IV): $d(x,y) > \epsilon$ and $d(x',y) > \epsilon$. In this case, from the fact that $0 \le \phi \le 1$, (1.9), (2.1) and (2.2), we deduce that

$$\begin{aligned}
&\left|K_\epsilon^\phi(x,y) - K_\epsilon^\phi(x',y)\right| \\
&\le \left|K(x,y) - K(x',y)\right| \phi\left(\frac{d(x',y)}{\epsilon}\right) + |K(x,y)| \left|\phi\left(\frac{d(x,y)}{\epsilon}\right) - \phi\left(\frac{d(x',y)}{\epsilon}\right)\right| \\
&\lesssim \frac{[d(x,x')]^\delta}{[d(x,y)]^\delta \lambda(x,d(x,y))} + \frac{1}{\lambda(x,d(x,y))} \frac{d(x,x')}{\min\{d(x,y), d(x',y)\}} \\
&\lesssim \frac{[d(x,x')]^\delta}{[d(x,y)]^\delta \lambda(x,d(x,y))} + \frac{1}{\lambda(x,d(x,y))} \frac{d(x,x')}{d(x,y)} \\
&\lesssim \frac{[d(x,x')]^\delta}{[d(x,y)]^\delta \lambda(x,d(x,y))},
\end{aligned}$$

which implies that K_ϵ^ϕ satisfies (1.9).

Now we turn to estimate K_ϵ^ψ. We first prove that K_ϵ^ψ satisfies (1.8). Indeed, if $d(x,y) \ge 3\epsilon$, by the properties of ψ, we have $K_\epsilon^\psi = 0$, and if $d(x,y) < 3\epsilon$, by the fact ψ is bounded and (1.3), we see that

$$K_\epsilon^\psi(x,y) \le \frac{1}{\lambda(x,\epsilon)} \lesssim \frac{1}{\lambda(x,d(x,y))}. \tag{2.3}$$

Finally, we show that K_ϵ^ψ satisfies (1.9) with $\delta = \min\{\sigma, 1\}$, where σ is as in Remark 1.4(iii). We consider the following four cases:

Case (I) $d(x,y) \ge 3\epsilon$ and $d(x',y) \ge 3\epsilon$;

Case (II) $d(x,y) < 3\epsilon$ and $d(x',y) \ge 3\epsilon$;

Case (III) $d(x,y) \ge 3\epsilon$ and $d(x',y) < 3\epsilon$;

Case (IV) $d(x,y) < 3\epsilon$ and $d(x',y) < 3\epsilon$.

By (2.3) and an argument similar to the estimate of K_ϵ^ϕ, we can prove that our desired results hold in Cases (I), (II) and (III). It remains to deal with the Case (IV). Let $x, x', y \in \mathcal{X}$ with $d(x,y) \ge 2d(x,x')$.

In this case, from the fact that $0 \le \psi \le 1$, Remark 1.2(iii), (2.2), (2.1) and (2.3), we deduce that

$$\begin{aligned}
\left|K_\epsilon^\psi(x,y) - K_\epsilon^\psi(x',y)\right| &= \left|\frac{1}{\lambda(x,\epsilon)} \psi\left(\frac{d(x,y)}{\epsilon}\right) - \frac{1}{\lambda(x',\epsilon)} \psi\left(\frac{d(x',y)}{\epsilon}\right)\right| \\
&\le \left|\frac{1}{\lambda(x,\epsilon)} - \frac{1}{\lambda(x',\epsilon)}\right| + \frac{1}{\lambda(x,\epsilon)} \left|\psi\left(\frac{d(x,y)}{\epsilon}\right) - \psi\left(\frac{d(x',y)}{\epsilon}\right)\right| \\
&\lesssim \frac{|\lambda(x',\epsilon) - \lambda(x,\epsilon)|}{\lambda(x,\epsilon)\lambda(x',\epsilon)} + \frac{1}{\lambda(x,\epsilon)} \frac{d(x,x')}{\min\{d(x,y), d(x',y)\}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{\lambda(x,\epsilon)}\left(\frac{d(x,x')}{\epsilon}\right)^{\sigma}+\frac{1}{\lambda(x,\epsilon)}\frac{d(x,x')}{d(x,y)}\\
&\lesssim \frac{1}{\lambda(x,\epsilon)}\left(\frac{d(x,x')}{d(x,y)}\right)^{\sigma}+\frac{1}{\lambda(x,\epsilon)}\frac{d(x,x')}{d(x,y)}\\
&\lesssim \frac{[d(x,x')]^{\delta}}{\lambda(x,d(x,y))[d(x,y)]^{\delta}},
\end{aligned}$$

and finish the proof of K_ϵ^{ψ}.

Combining the estimate for K_ϵ^{ϕ} and K_ϵ^{ψ}, we complete the proof of Lemma 2.9. $\square$

We introduced the operators T_*^{ϕ} and T_*^{ψ}, respectively, associated with K_ϵ^{ϕ} and K_ϵ^{ψ} by setting, for all $x\in\mathcal{X}$,

$$T_*^{\phi}f(x) := \sup_{\epsilon>0}|T_\epsilon^{\phi}f(x)| = \sup_{\epsilon>0}\left|\int_{\mathcal{X}} K_\epsilon^{\phi}(x,y)f(y)\,d\mu(y)\right|$$

and

$$T_*^{\psi}f(x) := \sup_{\epsilon>0}|T_\epsilon^{\psi}f(x)| = \sup_{\epsilon>0}\int_{\mathcal{X}} K_\epsilon^{\psi}(x,y)\,|f(y)|\,d\mu(y).$$

It is easy to see that, for all $x\in\mathcal{X}$,

$$\max\left\{T_*^{\phi}f(x), T_*^{\psi}f(x)\right\} \le T_*f(x) + CM_{(5)}f(x).$$

Therefore, T_*^{ϕ} and T_*^{ψ} are bounded on $L^p(\mu)$ for $p\in(1,\infty)$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

Define the multilinear commutators $T^{\phi}_{*,\vec{b}}$ and $T^{\psi}_{*,\vec{b}}$, respectively, associated with T_ϵ^{ϕ} and T_ϵ^{ψ} by setting, for all $x\in\mathcal{X}$,

$$\begin{aligned}
T^{\phi}_{*,\vec{b}}f(x) &:= \sup_{\epsilon>0}\left|T^{\phi}_{\epsilon,\vec{b}}f(x)\right|\\
&= \sup_{\epsilon>0}\left|\int_{\mathcal{X}}\prod_{i=1}^{m}[b_i(x)-b_i(y)]K_\epsilon^{\phi}(x,y)f(y)\,d\mu(y)\right|
\end{aligned}$$

and

$$\begin{aligned}
T^{\psi}_{*,\vec{b}}f(x) &:= \sup_{\epsilon>0}\left|T^{\psi}_{\epsilon,\vec{b}}f(x)\right|\\
&= \sup_{\epsilon>0}\int_{\mathcal{X}}\left|\prod_{i=1}^{m}[b_i(x)-b_i(y)]\right|K_\epsilon^{\psi}(x,y)\,|f(y)|\,d\mu(y).
\end{aligned}$$

Lemma 2.10. *Let $T_{*,\vec{b}}$ be the maximal multilinear commutator as in* (1.11). *Then there exist ϕ and ψ as in Definition* 2.8 *and a positive constant C such that, for the multilinear commutators $T^{\phi}_{*,\vec{b}}$ and $T^{\psi}_{*,\vec{b}}$, respectively, associated with ϕ and ψ,*

$$T_{*,\vec{b}}f \le T^{\phi}_{*,\vec{b}}f + CT^{\psi}_{*,\vec{b}}f. \tag{2.4}$$

Proof. Let ϕ be as in Definition 1.10, that is $\phi \in C^\infty([0,\infty))$, $\chi_{[2,\infty)} \le \phi \le \chi_{[1,\infty)}$ and $\phi'(t) \le C/t$ for $t \in (0,\infty)$. From (1.11) and the definition of $T^\phi_{*,\vec{b}}$, we deduce that

$$\begin{aligned}
&\left|T_{\epsilon,\vec{b}}f\right| - \left|T^\phi_{\epsilon,\vec{b}}f\right| \\
&= \left|\int_{d(x,y)>\epsilon} \prod_{i=1}^m (b(x)-b(y))K(x,y)f(y)\,d\mu(y)\right| \\
&\quad - \left|\int_{\mathcal{X}} \prod_{i=1}^m (b(x)-b(y))K^\phi_\epsilon(x,y)f(y)\,d\mu(y)\right| \\
&\le \left|\int_{\mathcal{X}} \prod_{i=1}^m (b(x)-b(y))K(x,y)\left[\chi_{(1,\infty)}\left(\frac{d(x,y)}{\epsilon}\right) - \phi\left(\frac{d(x,y)}{\epsilon}\right)\right] f(y)\,d\mu(y)\right| \\
&\lesssim \int_{\mathcal{X}} \frac{1}{\lambda(x,d(x,y))}\left|\chi_{(1,\infty)}\left(\frac{d(x,y)}{\epsilon}\right) - \phi\left(\frac{d(x,y)}{\epsilon}\right)\right| \left|\prod_{i=1}^m [b(x)-b(y)]\right| |f(y)|\,d\mu(y).
\end{aligned}$$

Let $\psi(t) = \chi_{[0,\infty)}(t) - \phi(t)$. Then it is not hard to show that $\psi \in C^\infty([0,\infty))$, $\chi_{[0,1/2)} \le \psi \le \chi_{[0,3)}$ and $|\psi'(t)| \le C/t$ for $t \in (0,\infty)$, which implies that ψ satisfies the conditions of Definition 1.10. Now we consider the following two cases of $d(x,y)$.

Case (I): $d(x,y) \le \epsilon$ or $d(x,y) > 2\epsilon$. In this case, $\chi_{(1,\infty)}\left(\frac{d(x,y)}{\epsilon}\right) - \phi\left(\frac{d(x,y)}{\epsilon}\right) = 0$. Thus,

$$\left|T_{\epsilon,\vec{b}}f\right| - \left|T^\phi_{\epsilon,\vec{b}}f\right| \lesssim \int_{\mathcal{X}} \frac{1}{\lambda(x,\epsilon)}\psi\left(\frac{d(x,y)}{\epsilon}\right)\left|\prod_{i=1}^m [b(x)-b(y)]\right| |f(y)|\,d\mu(y) = \left|T^\psi_{\epsilon,\vec{b}}\right|.$$

Case (II): $\epsilon < d(x,y) \le 2\epsilon$. In this case, by (1.3), we see that $\lambda(x,\epsilon) \sim \lambda(x,d(x,y))$. It then follows that

$$\left|T_{\epsilon,\vec{b}}f\right| - \left|T^\phi_{\epsilon,\vec{b}}f\right| \lesssim \frac{1}{\lambda(x,\epsilon)}\int_{\mathcal{X}} \psi\left(\frac{d(x,y)}{\epsilon}\right)\left|\prod_{i=1}^m [b(x)-b(y)]\right| |f(y)|\,d\mu(y) = \left|T^\psi_{\epsilon,\vec{b}}\right|,$$

which, together with the estimate for Case (I), completes the proof of Lemma 2.10. □

In the sequel, for $i \in \{1,\ldots,m\}$, we denote by C^m_i the family of all finite subsets $\sigma := \{\sigma(1),\ldots,\sigma(i)\}$ of $\{1,\ldots,m\}$ with i different elements. For any $\sigma \in C^m_i$, the complementary sequence σ' is given by $\sigma' := \{1,\ldots,m\}\backslash\sigma$. Let $\vec{b} = (b_1,\ldots,b_m)$ be a finite family of locally integrable functions. For all $i \in \{1,\ldots,m\}$ and $\sigma = \{\sigma(1),\ldots,\sigma(i)\} \in C^m_i$, we define

$$[b(x)-b(y)]_\sigma := [b_{\sigma(1)}(x) - b_{\sigma(1)}(y)]\cdots[b_{\sigma(i)}(x) - b_{\sigma(i)}(y)],$$
$$[b(x)-m_B(b)]_\sigma := [b_{\sigma(1)}(x) - m_B(b_{\sigma(1)})]\cdots[b_{\sigma(i)}(x) - m_B(b_{\sigma(i)})]$$

and

$$[m_S(b)-m_B(b)]_\sigma := [m_S(b_{\sigma(1)}) - m_B(b_{\sigma(1)})]\cdots[m_S(b_{\sigma(i)}) - m_B(b_{\sigma(i)})],$$

where B and S are balls in $\mathcal{X}$ and $x, y \in \mathcal{X}$. With this notation, we write

$$\|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)} := \left\|b_{\sigma(1)}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)} \cdots \left\|b_{\sigma(i)}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

We are now in position to prove the L^p-boundedness of maximal commutator $T_{*,\vec{b}}$.

Proof of Theorem 1.11. By (2.4), we only need to show that, for all $p \in (1, \infty)$,

$$\left\|T^{\phi}_{*,\vec{b}}f\right\|_{L^p(\mu)} \lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \|f\|_{L^p(\mu)} \tag{2.5}$$

and

$$\left\|T^{\psi}_{*,\vec{b}}f\right\|_{L^p(\mu)} \lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \|f\|_{L^p(\mu)}. \tag{2.6}$$

The proof of (2.5) and (2.6) are completely analogous. So, we only deal with (2.5). We show this by induction on $m \in \mathbb{N}$.

By an argument similar to that used in the proof of [1, Theorem 3.3], we deduce that (2.5) is valid for $m = 1$. Now assume that $m \geq 2$ and, for any $i = \{1, \dots, m-1\}$ and any subset $\sigma = \{\sigma(1), \dots, \sigma(i)\}$ of $\{1, \dots, m\}$, $T^{\phi}_{*,\vec{b}_\sigma}$ is bounded on $L^p(\mu)$ for any $p \in (1, \infty)$. By Lemma 2.2 and a standard limit argument, without loss of generality, we may assume that b_i is a bounded function for any $i \in \{1, \dots, m\}$. Let $p \in (1, \infty)$. We first claim that, for all $r \in (1, p)$, $f \in L^p(\mu)$, and $x \in \mathcal{X}$,

$$\begin{aligned} M^{\#}\left(T^{\phi}_{*,\vec{b}}f\right)(x) \lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} &\left[M_{r,(6)}(T^{\phi}_{*}f)(x) + M_{r,(5)}f(x)\right] \\ &+ \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{r,(6)}\left(T^{\phi}_{*,\vec{b}_{\sigma'}}f\right)(x). \end{aligned} \tag{2.7}$$

To prove (2.7), for all $B \subset \mathcal{X}$, we denote

$$h_B := m_B\left(T^{\phi}_{*}\left(\prod_{i=1}^{m}\left[b_i - m_{\widetilde{B}}(b_i)\right] f\chi_{\mathcal{X}\setminus\frac{6}{5}B}\right)\right).$$

As in the proof of [34, Theorem 9.1], it suffices to show that, for all $x \in \mathcal{X}$ and B with $B \ni x$,

$$\begin{aligned} \frac{1}{\mu(6B)}\int_B \left|T^{\phi}_{*,\vec{b}}f - h_B\right| d\mu \leq C\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} &\left[M_{r,(5)}f(x) + M_{r,(6)}(Tf)(x)\right] \\ &+ \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{r,(6)}\left(T^{\phi}_{*,\vec{b}_{\sigma'}}f\right)(x), \end{aligned} \tag{2.8}$$

and, for all $x \in \mathcal{X}$, ball B and doubling ball S with $x \in B \subset S$,

$$\begin{aligned} |h_B - h_S| \leq C\left[\widetilde{K}^{(6)}_{B,S}\right]^{m+1} &\left\{\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \left[M_{r,(5)}f(x) + M_{r,(6)}(Tf)(x)\right]\right. \\ &\left.+ \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{r,(6)}\left(T^{\phi}_{*,\vec{b}_{\sigma'}}f\right)(x)\right\}. \end{aligned} \tag{2.9}$$

We first prove (2.8). Notice that, for all $x, y \in \mathcal{X}$,

$$\prod_{i=1}^{m}\left[b_i(x)-m_{\widetilde{B}}(b_i)\right]=\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}[b(x)-b(y)]_{\sigma'}\left[b(y)-m_{\widetilde{B}}(b)\right]_{\sigma}, \tag{2.10}$$

where, if $i=0$, we set $\sigma'=\{1,\ldots,m\}$, $\sigma=\emptyset$ and $[b(y)-m_{\widetilde{B}}(b)]_{\emptyset}=1$. It then follows that, for all $y\in\mathcal{X}$,

$$T^{\phi}_{\epsilon,\vec{b}}f(y)=T^{\phi}_{\epsilon}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\right)(y)-\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\left[b(y)-m_{\widetilde{B}}(b)\right]_{\sigma}T^{\phi}_{\epsilon,\vec{b}_{\sigma'}}f(y), \tag{2.11}$$

where, if $i=m$, $T^{\phi}_{\epsilon,\vec{b}_{\sigma'}}f=T^{\phi}_{\epsilon}f$. Therefore, for all balls $B\ni x$,

$$\begin{aligned}
&\frac{1}{\mu(6B)}\int_B\left|T^{\phi}_{*,\vec{b}}f(y)-h_B\right|d\mu(y)\\
&\lesssim\frac{1}{\mu(6B)}\frac{1}{\mu(B)}\int_B\int_B\left|\sup_{\epsilon>0}\left|T^{\phi}_{\epsilon,\vec{b}}f(y)\right|-\sup_{\epsilon>0}\left|T^{\phi}_{\epsilon}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\chi_{\mathcal{X}\setminus\frac{6}{5}B}\right)(z)\right|\right|d\mu(z)\,d\mu(y)\\
&\lesssim\frac{1}{\mu(6B)}\frac{1}{\mu(B)}\int_B\int_B\sup_{\epsilon>0}\left|T^{\phi}_{\epsilon}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\right)(y)-\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\left[b-m_{\widetilde{B}}(b)\right]_{\sigma}T^{\phi}_{\epsilon,\vec{b}_{\sigma'}}f(y)\right.\\
&\qquad\left.-T^{\phi}_{\epsilon}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\chi_{\mathcal{X}\setminus\frac{6}{5}B}\right)(z)\right|d\mu(z)\,d\mu(y)\\
&\lesssim\frac{1}{\mu(6B)}\int_B T^{\phi}_{*}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\chi_{\frac{6}{5}B}\right)(y)\,d\mu(y)\\
&\quad+\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\frac{1}{\mu(6B)}\int_B\left|\left[b(y)-m_{\widetilde{B}}(b)\right]_{\sigma}\right|T^{\phi}_{*,\vec{b}_{\sigma'}}f(y)\,d\mu(y)\\
&\quad+\frac{1}{\mu(6B)}\frac{1}{\mu(B)}\int_B\int_B\sup_{\epsilon>0}\left|T^{\phi}_{\epsilon}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\chi_{\mathcal{X}\setminus\frac{6}{5}B}\right)(y)\right.\\
&\qquad\left.-T^{\phi}_{\epsilon}\left(\prod_{i=1}^{m}\left[b_i-m_{\widetilde{B}}(b_i)\right]f\chi_{\mathcal{X}\setminus\frac{6}{5}B}\right)(z)\right|d\mu(z)\,d\mu(y)\\
&=:I_1+I_2+I_3.
\end{aligned}$$

By a slight modified argument similar to that used in the proof of (3.11) of [11, Theorem 1.9], we conclude that, for all $x\in B$,

$$I_1+I_3\lesssim\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}M_{r,(5)}f(x)$$

and

$$I_2\lesssim\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\|\vec{b}_{\sigma}\|_{\widetilde{\mathrm{RBMO}}(\mu)}M_{r,(6)}\left(T^{\phi}_{*,\vec{b}_{\sigma'}}f\right)(x),$$

which completes the proof of (2.8).

The proof of (2.9) is similar to that of (3.10) of [11, Theorem 1.9]. The details are omitted here. Then we complete the proof of (2.7).

We now prove that (2.5) holds true for any $m \geq 2$ by considering the following two cases.

Case (I): $\mu(\mathcal{X}) = \infty$. In this case, notice we assume that b_i is bounded for $i \in \{1, \ldots, m\}$. It follows from the L^p-boundedness of T_*^ϕ that $T_{*,\vec{b}}^\phi f \in L^p(\mu)$, which, together with Lemma 2.6(i), implies that $\inf\{1, N(T_{*,\vec{b}}^\phi f)\} \in L^p(\mu)$. By Lemma 2.6(ii), Lemma 2.7, the L^p-boundedness of T_*^ϕ and the assumption that $T_{*,\vec{b}_{\sigma'}}^\phi$ is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$, we conclude that, for all $p \in (1,\infty)$ and $f \in L^p(\mu)$,

$$\begin{aligned}
\left\|T_{*,\vec{b}}^\phi f\right\|_{L^p(\mu)} &\leq \left\|N\left(T_{*,\vec{b}}^\phi f\right)\right\|_{L^p(\mu)} \lesssim \left\|M^\#\left(T_{*,\vec{b}}^\phi f\right)\right\|_{L^p(\mu)} \\
&\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\left[\left\|M_{r,(6)}(T_*^\phi f)\right\|_{L^p(\mu)} + \left\|M_{r,(5)}(f)\right\|_{L^p(\mu)}\right] \\
&\quad + \sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)}\left\|M_{r,(6)}\left(T_{*,\vec{b}_{\sigma'}}^\phi f\right)\right\| \\
&\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\left[\left\|T_*^\phi f\right\|_{L^p(\mu)} + \|f\|_{L^p(\mu)}\right] \\
&\quad + \sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)}\left\|T_{*,\vec{b}_{\sigma'}}^\phi f\right\|_{L^p(\mu)} \\
&\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f\|_{L^p(\mu)}.
\end{aligned}$$

Case (II): $\mu(\mathcal{X}) < \infty$. In this case, by Lemma 2.4 and the Lebesgue dominated convergence theorem, we see that, for all $r \in (1,\infty)$,

$$\left[\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}}\prod_{i=1}^m|b_i(x) - b_{\mathcal{X}}|^r\,d\mu(x)\right]^{1/r} \lesssim \prod_{i=1}^m\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}, \tag{2.12}$$

where $b_{\mathcal{X}} := \frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}} b(y)\,d\mu(y)$. Write

$$N\left(T_{*,\vec{b}}^\phi f\right) \leq N\left(T_{*,\vec{b}}^\phi f - m_{\mathcal{X}}\left(T_{*,\vec{b}}^\phi f\right)\right) + \left|m_{\mathcal{X}}\left(T_{*,\vec{b}}^\phi f\right)\right|.$$

Notice that $\int_{\mathcal{X}}[T_{*,\vec{b}}^\phi f(x) - m_{\mathcal{X}}(T_{*,\vec{b}}^\phi f)]\,d\mu(x) = 0$, and, for all $p \in (1,\infty)$,

$$\int_{\mathcal{X}}\left[\min\left\{1, N\left(T_{*,\vec{b}}^\phi f - m_{\mathcal{X}}\left(T_{*,\vec{b}}^\phi f\right)\right)(x)\right\}\right]^p d\mu(x) \leq \mu(\mathcal{X}) < \infty.$$

Then by Lemma 2.7 and the fact that $M^\#(T_{*,\vec{b}}^\phi f - m_{\mathcal{X}}(T_{*,\vec{b}}^\phi f)) = M^\#(T_{*,\vec{b}}^\phi f)$, we see that

$$\begin{aligned}
\left\|N\left(T_{*,\vec{b}}^\phi f - m_{\mathcal{X}}\left(T_{*,\vec{b}}^\phi f\right)\right)\right\|_{L^p(\mu)} &\lesssim \left\|M^\#\left(T_{*,\vec{b}}^\phi f - m_{\mathcal{X}}\left(T_{*,\vec{b}}^\phi f\right)\right)\right\|_{L^p(\mu)} \\
&\sim \left\|M^\#\left(T_{*,\vec{b}}^\phi f\right)\right\|_{L^p(\mu)} \lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f\|_{L^p(\mu)}.
\end{aligned}$$

For the term $|m_{\mathcal{X}}(T^{\phi}_{*,\vec{b}}f)|$, by (2.11), we further write

$$\left|T^{\phi}_{*,\vec{b}}f\right| \le \left|T^{\phi}_{*}\left(\prod_{i=1}^{m}[b_i - m_{\mathcal{X}}(b_i)]f\right)\right| + \sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\left|[b-m_{\mathcal{X}}(b)]_{\sigma}T^{\phi}_{*,\vec{b}_{\sigma'}}f\right|.$$

By the Hölder inequality, (2.12) and the $L^q(\mu)$-boundedness of T^{ϕ}_{*} for all $q \in (1,p]$, we have

$$\begin{aligned}
&\left\|m_{\mathcal{X}}\left(T^{\phi}_{*}\left(\prod_{i=1}^{m}[b_i - m_{\mathcal{X}}(b_i)]f\right)\right)\right\|_{L^p(\mu)} \\
&= \left\{\int_{\mathcal{X}}\left|\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}}T^{\phi}_{*}\left(\prod_{i=1}^{m}[b_i - m_{\mathcal{X}}(b_i)]f\right)(y)\,d\mu(y)\right|^p d\mu(x)\right\}^{1/p} \\
&= [\mu(\mathcal{X})]^{1/p-1}\int_{\mathcal{X}}T^{\phi}_{*}\left(\prod_{i=1}^{m}[b_i - m_{\mathcal{X}}(b_i)]f\right)(y)\,d\mu(y) \\
&\lesssim [\mu(\mathcal{X})]^{1/p+1/q-1}\left\|T^{\phi}_{*}\left(\prod_{i=1}^{m}[b_i - m_{\mathcal{X}}(b_i)]f\right)\right\|_{L^q(\mu)} \\
&\lesssim \left\|\prod_{i=1}^{m}[b_i - m_{\mathcal{X}}(b_i)]f\right\|_{L^q(\mu)} \\
&\lesssim \left\{\int_{\mathcal{X}}\left|\prod_{i=1}^{m}[b_i(y) - m_{\mathcal{X}}(b_i)]\right|^q |f(y)|^q\,d\mu(y)\right\}^{1/q} \\
&\lesssim \left\{\int_{\mathcal{X}}\left|\prod_{i=1}^{m}[b_i(y) - m_{\mathcal{X}}(b_i)]\right|^{pq/(p-q)} d\mu(y)\right\}^{(p-q)/(pq)} \left\{\int_{\mathcal{X}}|f(y)|^p\,d\mu(y)\right\}^{1/p} \\
&\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f\|_{L^p(\mu)}.
\end{aligned}$$

From the Hölder inequality, the assumption that $T^{\phi}_{*,\vec{b}_{\sigma'}}$ is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$ and Lemma 2.4, we deduce that

$$\begin{aligned}
&\left\|m_{\mathcal{X}}\left([b-m_{\mathcal{X}}(b)]_{\sigma}T^{\phi}_{*,\vec{b}_{\sigma'}}f\right)\right\|_{L^p(\mu)} \\
&\lesssim [\mu(\mathcal{X})]^{1/p-1}\int_{\mathcal{X}}|[b(y)-m_{\mathcal{X}}(b)]_{\sigma}|\,T^{\phi}_{*,\vec{b}_{\sigma'}}f(y)\,d\mu(y) \\
&\lesssim \left\|T^{\phi}_{*,\vec{b}_{\sigma'}}f\right\|_{L^p(\mu)}\left\{\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}}|[b(y)-m_{\mathcal{X}}(b)]_{\sigma}|^{p'}\,d\mu(y)\right\}^{1/p'} \\
&\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f\|_{L^p(\mu)},
\end{aligned}$$

which implies that $\|m_{\mathcal{X}}(T^{\phi}_{*,\vec{b}}f)\|_{L^p(\mu)} \lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f\|_{L^p(\mu)}$. Combining the above two estimates, we also obtain the desired conclusion in this case, which completes the proof of (2.5). Then we finish the proof of Theorem 1.11. □

3. Proof of Theorem 1.14

To prove Theorem 1.14, we need the following generalized Hölder's inequality presented in [11].

Lemma 3.1. *There exists a positive constant C, depending only on m, such that, for all locally integrable functions f and b_i with $i \in \{1, \dots, m\}$, all balls B and $1/r = 1/r_1 + \cdots + 1/r_m$ with $r_i \in [1, \infty)$ for $i \in \{1, \dots, m\}$,*

$$\begin{aligned}&\frac{1}{\mu(2B)} \int_B |f(x) b_1(x) \cdots b_m(x)| \, d\mu(x) \\ &\quad \le C \|b_1\|_{\exp L^{r_1}, B, \mu/\mu(2B)} \cdots \|b_m\|_{\exp L^{r_m}, B, \mu/\mu(2B)} \|f\|_{L(\log L)^{1/r}, B, \mu/\mu(2B)},\end{aligned}$$

where, for $\alpha \in (0, \infty)$,

$$\|f\|_{L(\log L)^\alpha, B, \mu/\mu(2B)} := \inf \left\{ \lambda \in (0, \infty) : \frac{1}{2B} \int_B \frac{|f(x)|}{\lambda} \log^\alpha \left(2 + \frac{|f(x)|}{\lambda} \right) d\mu(x) \le 1 \right\}$$

and

$$\|f\|_{\exp L^\alpha, B, \mu/\mu(2B)} := \inf \left\{ \lambda \in (0, \infty) : \frac{1}{\mu(2B)} \int_B \exp \left(\frac{|f(x)|}{\lambda} \right)^\alpha d\mu(x) \le 2 \right\}.$$

The following Calderón-Zygmund decomposition is analogous to [2, Theorem 6.3] and its proof is also analogous to that of [2, Theorem 6.3]. The details are omitted. Let γ_0 be a fixed positive constant satisfying that $\gamma_0 > \max\{C_{(\lambda)}^{3 \log_2 6}, 6^{3n_0}\}$, where $C_{(\lambda)}$ is as in (1.3) and n_0 is as in Remark 1.4(ii).

Lemma 3.2. *Let $p \in [1, \infty)$, $f \in L^p(\mu)$ and $t \in (0, \infty)$ ($t > (\gamma_0)^{1/p} \|f\|_{L^p(\mu)} / [\mu(\mathcal{X})]^{1/p}$ when $\mu(\mathcal{X}) < \infty$). Then the following hold true.*

(i) *There exists an almost disjoint family $\{6B_j\}_j$ of balls such that $\{B_j\}_j$ is pairwise disjoint,*

$$\begin{aligned}\frac{1}{\mu(6^2 B_j)} \int_{B_j} |f(x)|^p \, d\mu(x) &> \frac{t^p}{\gamma_0} \quad \textit{for all } j, \\ \frac{1}{\mu(6^2 \eta B_j)} \int_{\eta B_j} |f(x)|^p \, d\mu(x) &\le \frac{t^p}{\gamma_0} \quad \textit{for all } j \textit{ and all } \eta \in (2, \infty)\end{aligned}$$

and

$$|f(x)| \le t \quad \textit{for } \mu\textit{-almost every } x \in \mathcal{X} \setminus \bigcup_j 6B_j.$$

(ii) *For each j, let S_j be a $(3\times 6^2, C_\lambda^{\log_2(3\times 6^2)+1})$-doubling ball of the family $\{(3\times 6^2)^k B_j\}_{k\in\mathbb{N}}$ and $\omega_j := \chi_{6B_j}/\big(\sum_k \chi_{6B_k}\big)$. Then there exists a family $\{\varphi_j\}_j$ of functions such that, for each j, $\operatorname{supp}(\varphi_j)\subset S_j$, φ_j has a constant sign on S_j,*

$$\int_{\mathcal{X}} \varphi_j(x)\,d\mu(x) = \int_{6B_j} f(x)\omega_j(x)\,d\mu(x),$$
$$\sum_j |\varphi_j(x)| \le \gamma t \quad \text{for } \mu\text{-almost every } x\in\mathcal{X},$$

where γ is some positive constant, depending only on $(\mathcal{X},\mu)$, and there exists a positive constant C, independent of f, t and j, such that, when $p=1$, it holds true that

$$\|\varphi_j\|_{L^\infty(\mu)}\,\mu(S_j) \le C\int_{\mathcal{X}} |f(x)\omega_j(x)|\,d\mu(x)$$

and, when $p\in(1,\infty)$, it holds true that

$$\left[\int_{S_j} |\varphi_j(x)|^p\,d\mu(x)\right]^{1/p} [\mu(S_j)]^{1/p'} \le \frac{C}{t^{p-1}}\int_{\mathcal{X}} |f(x)\omega_j(x)|^p\,d\mu(x).$$

By (2.4), to prove Theorem 1.14, it suffices to prove the operators $T^{\phi}_{*,\vec{b}}$ and $T^{\psi}_{*,\vec{b}}$ satisfy the same type estimate. Precisely, we have the following result.

Lemma 3.3. *Under the same assumption as Theorem 1.14, there exists a positive constant C such that, for all $t\in(0,\infty)$ and all $f\in L^\infty_b$ with bounded support,*

$$\mu\left(\left\{x\in\mathcal{X} : \left|T^{\phi}_{*,\vec{b}}f(x)\right| > t\right\}\right) \le C\Phi_{1/r}\left(\|\vec{b}\|_{\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)}\right)\int_{\mathcal{X}} \Phi_{1/r}\left(\frac{|f(y)|}{t}\right)d\mu(y) \tag{3.1}$$

and

$$\mu\left(\left\{x\in\mathcal{X} : \left|T^{\psi}_{*,\vec{b}}f(x)\right| > t\right\}\right) \le C\Phi_{1/r}\left(\|\vec{b}\|_{\widetilde{\mathrm{Osc}}_{\exp L^r}(\mu)}\right)\int_{\mathcal{X}} \Phi_{1/r}\left(\frac{|f(y)|}{t}\right)d\mu(y). \tag{3.2}$$

Proof. Similar to the proof of [15, Theorem 4], without loss of generality, we may assume that, for all $m\in\mathbb{N}$ and $i\in\{1,\dots,m\}$, $\|b_i\|_{\widetilde{\mathrm{Osc}}_{\exp L^{r_i}}(\mu)} = 1$.

The proof of (3.1) is parallel to that of [14, Theorem 4] with slight modifications. The details are omitted.

We now prove (3.2). For each fixed $|f|$ and $t\in(0,\infty)$, by applying Lemma 3.2 and its notation, we see that $|f(x)| = g(x)+h(x)$, where

$$h(x) := \sum_j [|f(x)|\,\omega_j(x) - \varphi_j(x)] =: \sum_j h_j(x).$$

Write

$$\begin{aligned}
T^{\psi}_{*,\vec{b}}f(x) &= \sup_{\epsilon>0}\left|\int_{\mathcal{X}} K^{\psi}_{\epsilon}(x,y)\left|\prod_{i=1}^{m}[b_i(x)-b_i(y)]\right| |f(y)|\,d\mu(y)\right| \\
&\lesssim \sup_{\epsilon>0}\left|\int_{\mathcal{X}} K^{\psi}_{\epsilon}(x,y)\left|\prod_{i=1}^{m}[b_i(x)-b_i(y)]\right| h(y)\,d\mu(y)\right| \\
&\quad + \sup_{\epsilon>0}\left|\int_{\mathcal{X}} K^{\psi}_{\epsilon}(x,y)\left|\prod_{i=1}^{m}[b_i(x)-b_i(y)]\right| g(y)\,d\mu(y)\right| \\
&\lesssim \sup_{\epsilon>0}\left|\int_{\mathcal{X}} K^{\psi}_{\epsilon}(x,y)\left|\prod_{i=1}^{m}[b_i(x)-b_i(y)]\right| h(y)\,d\mu(y)\right| + T^{\psi}_{*,\vec{b}}g(x) =: M(x)+N(x).
\end{aligned}$$

By Lemma 3.2, it is easy to see that $|g(x)| \lesssim t$ for μ-almost every $x \in \mathcal{X}$ and $\|g\|^2_{L^2(\mu)} \lesssim t\,\|f\|_{L^1(\mu)}$, which, together with the L^p-boundedness of $T^{\psi}_{*,\vec{b}}$, further implies that

$$\mu\left(\{x\in\mathcal{X} : N(x)>t\}\right) \lesssim t^{-2}\left\|T^{\psi}_{*,\vec{b}}g\right\|^2_{L^2(\mu)} \lesssim t^{-2}\,\|g\|^2_{L^2(\mu)} \lesssim t^{-1}\int_{\mathcal{X}}|f(y)|\,d\mu(y).$$

From Lemma 3.2(i), we have

$$\mu\left(\bigcup_j 6^2 B_j\right) \lesssim \frac{1}{t}\int_{\mathcal{X}}|f(y)|\,d\mu(y).$$

Therefore, the proof of (3.2) can be reduced to proving that

$$\mu\left(\left\{x\in\mathcal{X}\setminus\cup_j 6^2B_j : |M(x)|>t\right\}\right) \lesssim \int_{\mathcal{X}}\frac{|f(y)|}{t}\log^{1/r}\left(2+\frac{|f(y)|}{t}\right)d\mu(y). \tag{3.3}$$

We prove (3.3) by induction on $m\in\mathbb{N}$.

For $m=1$, the proof is similar to that of [16, Lemma 2.1]. The details are omitted here. Now we assume that $m\ge 2$ and, for any $i\in\{1,\dots,m-1\}$ and any subset $\sigma = \{\sigma(1),\dots,\sigma(i)\}$ of $\{1,\dots,m-1\}$, (3.2) holds true. The fact that $K^{\psi}_{\epsilon}(x,y)\ge 0$ and $\int_{\mathcal{X}} h_j(y)\,d\mu(y)=0$, together with (2.10), implies that

$$\begin{aligned}
M(x) &\lesssim \sup_{\epsilon>0}\left|\sum_j\int_{\mathcal{X}}\left[K^{\psi}_{\epsilon}(x,y)\left|\prod_{i=1}^{m}[b_i(x)-b_i(y)]\right|\right.\right. \\
&\qquad \left.\left. -K^{\psi}_{\epsilon}(x,x_j)\left|\prod_{i=1}^{m}\left[b(x)-m_{\widetilde{B_j}}(b)\right]\right|\right] h_j(y)\,d\mu(y)\right| \\
&\lesssim \sup_{\epsilon>0}\sum_j\int_{\mathcal{X}}\left|K^{\psi}_{\epsilon}(x,y)\left[\prod_{i=1}^{m}\left[b_i(x)-m_{\widetilde{B_j}}(b_i)\right]-\prod_{i=1}^{m}\left[b_i(y)-m_{\widetilde{B_j}}(b_i)\right]\right.\right. \\
&\qquad \left.\left. -\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}[b_i(x)-b_i(y)]_{\sigma}\left[b_i(y)-m_{\widetilde{B_j}}(b_i)\right]_{\sigma'}\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad -K_\epsilon^\psi(x,x_j)\prod_{i=1}^m\left[b_i(x)-m_{\widetilde{B_j}}(b_i)\right]\Bigg|\,|h_j(y)|\,d\mu(y)\\
&\lesssim \sum_j\left|\prod_{i=1}^m\left[b(x)-m_{\widetilde{B_j}}(b)\right]\right|\sup_{\epsilon>0}\int_{\mathcal{X}}\left|K_\epsilon^\psi(x,y)-K_\epsilon^\psi(x,x_j)\right||h_j(y)|\,d\mu(y)\\
&\quad + T_*^\psi\left(\sum_j\prod_{i=1}^m\left[b_i-m_{\widetilde{B_j}}(b_i)\right]h_j\right)(x)\\
&\quad + \sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}T^\psi_{*,\vec{b}_\sigma}\left(\sum_j\left[b_i-m_{\widetilde{B_j}}(b_i)\right]_{\sigma'}h_j\right)(x)\\
&=: M_1(x)+M_2(x)+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}M_3(x).
\end{aligned}
$$

Similar to the estimate for $V(x)$ in the proof of [16, Lemma 2.1], we have

$$\int_{\mathcal{X}\setminus\bigcup_j 6^2B_j}M_1(x)\,d\mu(x)\lesssim\int_{\mathcal{X}}|f(y)|\,d\mu(y).$$

The generalized Hölder's inequality via an argument similar to that used in the estimates for $T_{\vec{b}}^{\mathrm{II}}h(x)$ and $T_{\vec{b}_{\sigma'}}^{\mathrm{III}}h(x)$ in [14, pp. 252–254] shows that

$$\mu\left(\left\{x\in\mathcal{X}\setminus\cup_j6^2B_j : M_2(x)>t\right\}\right)\lesssim\int_{\mathcal{X}}\frac{|f(y)|}{t}\log^{1/r}\left(2+\frac{|f(y)|}{t}\right)d\mu(y)$$

and

$$\mu\left(\{x\in\mathcal{X} : M_3(x)>t\}\right)\lesssim\int_{\mathcal{X}}\frac{|f(y)|}{t}\log^{1/r}\left(2+\frac{|f(y)|}{t}\right)d\mu(y),$$

which implies (3.3) and hence completes the proof of Lemma 3.3. □

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References

[1] T. A. Bui, *Boundedness of maximal operators and maximal commutators on non-homogeneous spaces*, in *AMSI International Conference on Harmonic Analysis and Applications*, 22–36, Proc. Centre Math. Appl. Austral. Nat. Univ. **45**, Austral. Nat. Univ., Canberra, 2013.

[2] T. A. Bui and X. T. Duong, *Hardy spaces, regularized BMO spaces and the boundedness of Calderón-Zygmund operators on non-homogeneous spaces*, J. Geom. Anal. **23** (2013), no. 2, 895–932. `https://doi.org/10.1007/s12220-011-9268-y`

[3] J. Chen and H. Lin, *Hardy-type space estimates for multilinear commutators of Calderón-Zygmund operators on nonhomogeneous metric measure spaces*, Banach J. Math. Anal. (to appear). `https://doi.org/10.1215/17358787-2017-0002`

[4] W. Chen, Y. Meng and D. Yang, *Calderón-Zygmund operators on Hardy spaces without the doubling condition*, Proc. Amer. Math. Soc. **133** (2005), no. 9, 2671–2680. `https://doi.org/10.1090/S0002-9939-05-07781-6`

[5] W. Chen and C. Miao, *Vector valued commutators on non-homogeneous spaces*, Taiwanese J. Math. **11** (2007), no. 4, 1127–1141.

[6] W. Chen and E. Sawyer, *A note on commutators of fractional integrals with* RBMO(μ) *functions*, Illinois J. Math. **46** (2002), no. 4, 1287–1298.

[7] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogènes*, Étude de certaines intégrales singulières, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin-New York, 1971. `https://doi.org/10.1007/bfb0058946`

[8] ———, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645. `https://doi.org/10.1090/s0002-9904-1977-14325-5`

[9] X. Fu, H. Lin, D. Yang and D. Yang, *Hardy spaces H^p over non-homogeneous metric measure spaces and their applications*, Sci. China Math. **58** (2015), no. 2, 309–388. `https://doi.org/10.1007/s11425-014-4956-2`

[10] X. Fu, D. Yang and D. Yang, *The molecular characterization of the Hardy space H^1 on non-homogeneous metric measure spaces and its application*, J. Math. Anal. Appl. **410** (2014), no. 2, 1028–1042. `https://doi.org/10.1016/j.jmaa.2013.09.021`

[11] X. Fu, D. Yang and W. Yuan, *Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces over non-homogeneous spaces*, Taiwanese J. Math. **16** (2012), no. 6, 2203–2238.

[12] J. García-Cuerva and J. M. Martell, *On the existence of principal values for the Cauchy integral on weighted Lebesgue spaces for non-doubling measures*, J. Fourier Anal. Appl. **7** (2001), no. 5, 469–487. `https://doi.org/10.1007/bf02511221`

[13] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001. `https://doi.org/10.1007/978-1-4613-0131-8`

[14] G. Hu, Y. Meng and D. Yang, *New atomic characterization of H^1 space with non-doubling measures and its applications*, Math. Proc. Cambridge Philos. Soc. **138** (2005), no. 1, 151–171. `https://doi.org/10.1017/s030500410400800x`

[15] ______, *Multilinear commutators of singular integrals with non doubling measures*, Integral Equations Operator Theory **51** (2005), no. 2, 235–255. https://doi.org/10.1007/s00020-003-1251-y

[16] ______, *Endpoint estimate for maximal commutators with non-doubling measures*, Acta Math. Sci. Ser. B Engl. Ed. **26** (2006), no. 2, 271–280. https://doi.org/10.1016/s0252-9602(06)60049-0

[17] T. Hytönen, *A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa*, Publ. Mat. **54** (2010), no. 2, 485–504. https://doi.org/10.5565/publmat_54210_10

[18] T. Hytönen, S. Liu, D. Yang and D. Yang, *Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces*, Canad. J. Math. **64** (2012), no. 4, 892–923. https://doi.org/10.4153/cjm-2011-065-2

[19] T. Hytönen and H. Martikainen, *Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces*, J. Geom. Anal. **22** (2012), no. 4, 1071–1107. https://doi.org/10.1007/s12220-011-9230-z

[20] ______, *Non-homogeneous $T1$ theorem for bi-parameter singular integrals*, Adv. Math. **261** (2014), 220–273. https://doi.org/10.1016/j.aim.2014.02.011

[21] T. Hytönen, D. Yang and D. Yang, *The Hardy space H^1 on non-homogeneous metric spaces*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 1, 9–31. https://doi.org/10.1017/s0305004111000776

[22] L. Li and Y.-S. Jiang, *Estimates for maximal multilinear commutators on non-homogeneous spaces*, J. Math. Anal. Appl. **355** (2009), no. 1, 243–257. https://doi.org/10.1016/j.jmaa.2009.01.022

[23] H. Lin, S. Wu and D. Yang, *Boundedness of certain commutators over non-homogeneous metric measure spaces*, Anal. Math. Phys. **7** (2017), no. 2, 187–218. https://doi.org/10.1007/s13324-016-0136-6

[24] H. Lin and D. Yang, *Equivalent boundedness of Marcinkiewicz integrals on non-homogeneous metric measure spaces*, Sci. China Math. **57** (2014), no. 1, 123–144. https://doi.org/10.1007/s11425-013-4754-2

[25] S. Liu, Y. Meng and D. Yang, *Boundedness of maximal Calderón-Zygmund operators on non-homogeneous metric measure spaces*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), no. 3, 567–589. https://doi.org/10.1017/s0308210512000054

[26] S. Liu, D. Yang and D. Yang, *Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces: equivalent characterizations*, J. Math. Anal. Appl. **386** (2012), no. 1, 258–272. https://doi.org/10.1016/j.jmaa.2011.07.055

[27] F. Nazarov, S. Treil and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces*, Internat. Math. Res. Notices **1998** (1998), no. 9, 463–487. https://doi.org/10.1155/S1073792898000312

[28] ______, *The Tb-theorem on non-homogeneous spaces*, Acta Math. **190** (2003), no. 2, 151–239. https://doi.org/10.1007/bf02392690

[29] Y. Sawano and H. Tanaka, *Morrey spaces for non-doubling measures*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 6, 1535–1544. https://doi.org/10.1007/s10114-005-0660-z

[30] ______, *Sharp maximal inequalities and commutators on Morrey spaces with non-doubling measures*, Taiwanese J. Math. **11** (2007), no. 4, 1091–1112.

[31] C. Segovia and J. L. Torrea, *Vector-valued commutators and applications*, Indiana Univ. Math. J. **38** (1989), no. 4, 959–971.

[32] C. Tan and J. Li, *Littlewood-Paley theory on metric measure spaces with non doubling measures and its applications*, Sci. China Math. **58** (2015), no. 5, 983–1004. https://doi.org/10.1007/s11425-014-4950-8

[33] ______, *Some characterizations of upper doubling conditions on metric measure spaces*, Math. Nachr. **290** (2017), no. 1, 142–158. https://doi.org/10.1002/mana.201400347

[34] X. Tolsa, *BMO, H^1, and Calderón-Zygmund operators for non doubling measures*, Math. Ann. **319** (2001), no. 1, 89–149. https://doi.org/10.1007/pl00004432

[35] ______, *Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures*, Adv. Math. **164** (2001), no. 1, 57–116. https://doi.org/10.1006/aima.2001.2011

[36] ______, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), no. 1, 105–149. https://doi.org/10.1007/bf02393237

[37] ______, *The space H^1 for nondoubling measures in terms of a grand maximal operator*, Trans. Amer. Math. Soc. **355** (2003), no. 1, 315–348. https://doi.org/10.1090/S0002-9947-02-03131-8

[38] ______, *Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón-Zygmund Theory*, Progress in Mathematics **307**, Birkhäuser/Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-00596-6

[39] A. Volberg and B. D. Wick, *Bergman-type singular integral operators and the characterization of Carleson measures for Besov-Sobolev spaces on the complex ball*, Amer. J. Math. **134** (2012), no. 4, 949–992. https://doi.org/10.1353/ajm.2012.0028

[40] R. Xie, H. Gong and X. Zhou, *Commutators of multilinear singular integral operators on non-homogeneous metric measure spaces*, Taiwanese J. Math. **19** (2015), no. 3, 703–723. https://doi.org/10.11650/tjm.19.2015.4715

[41] D. Yang, D. Yang and X. Fu, *The Hardy space H^1 on non-homogeneous spaces and its applications—a survey*, Eurasian Math. J. **4** (2013), no. 2, 104–139.

[42] D. Yang, D. Yang and G. Hu, *The Hardy Space H^1 with Non-doubling Measures and Their Applications*, Lecture Notes in Mathematics **2084**, Springer, Cham, 2013. https://doi.org/10.1007/978-3-319-00825-7

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