

## Haar Adomian Method for the Solution of Fractional Nonlinear Lane-Emden Type Equations Arising in Astrophysics

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**Abstract.** In this paper, we propose a method for solving some well-known classes of fractional Lane-Emden type equations which are nonlinear ordinary differential equations on the semi-infinite domain. The method is proposed by utilizing Haar wavelets in conjunction with Adomian's decomposition method. The operational matrices for the Haar wavelets are derived and constructed. Procedure of implementation and convergence analysis of the method are presented. The method is tested on the fractional standard Lane-Emden equation and the fractional isothermal gas spheres equation. We compare the results produce by present method with some well-known results to show the accuracy and applicability of the method.

### 1. Introduction

In recent years, numerous applications of fractional order ordinary and partial differential equations have appeared in many areas of physics and engineering. There have found a number of works, especially in hereditary solid mechanics and in viscoelasticity theory, where fractional order derivatives are used for a better description of material properties. This is the main advantage of fractional derivatives in comparison with classical integer order models in which such effects are neglected. The mathematical modeling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally lead to differential equations of fractional order and to the necessity of solving such equations. For most of fractional order differential equations, exact solutions are not known. Therefore different numerical methods have been applied for providing approximate solutions. Some of these techniques include, the Adomian decomposition method (ADM) [8,34,37], the homotopy perturbation method (HPM) [21,24], the variational iteration method (VIM) [15,23], the generalized differential transform method (DTM) [14,26], and collocation methods [4,30,35].

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Many problems arising in the field of mathematical physics and astrophysics can be modelled by Lane-Emden type initial value problems. The Caputo fractional order Lane-Emden equation studied in this paper is

$$D^\alpha y + \frac{2}{x} D^\beta y + f(y) = 0, \quad x > 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1,$$

subject to the initial conditions:

$$y(0) = A, \quad y'(0) = B,$$

where  $A$  and  $B$  are constants,  $f(y)$  is the nonlinear function of  $y$ ,  $x$  and  $y$  are the independent and dependent variables respectively. For  $\alpha = 2$  and  $\beta = 1$ , we have classical Lane-Emden type equations which are nonlinear ordinary differential equations on semi-infinite domain and are categorized as singular initial value problems. These equations are used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deals with the issue of energy transport, through the transfer of material between different levels of the star. The equations are used to model the gravitational potential of a degenerate white dwarf star. These equations are one of the basic equations in the theory of stellar structure and has been the focus of many researchers [5, 6, 10, 17, 18, 22, 25].

Wavelet analysis is a new development in the area of applied mathematics. Wavelets are a special kind of functions which exhibits oscillatory behavior for a short period of time and then die out. In wavelets, we use a single function and its dilations and translations to generate a set of orthonormal basis functions to represent a signal. We define wavelet (mother wavelet) by [28]:

$$(1.1) \quad \psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right), \quad a, b \in \mathbb{R}, \quad a \neq 0,$$

where  $a$  and  $b$  are called scaling and translation parameter respectively. If  $|a| < 1$ , the wavelet (1.1) is the compressed version (smaller support in time-domain) of the mother wavelet and corresponds to mainly higher frequencies. On the other hand, when  $|a| > 1$ , the wavelet (1.1) has larger support in time-domain and corresponds to lower frequencies.

A wavelet is a function  $\psi$  which satisfies the condition, known as the wavelet admissibility condition:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

where  $\widehat{\psi}(\omega)$  is the Fourier transform of  $\psi(x)$ . This condition ensures that  $\widehat{\psi}(\omega)$  goes to zero quickly as  $\omega \rightarrow 0$ , it is required that

$$\widehat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x) dx = 0.$$

Another condition impose on wavelet function is finite energy, that is

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty.$$

Discretizing the parameters via  $a = 2^{-k}$  and  $b = n2^{-k}$ , we get the discrete wavelets transform as

$$\psi_{k,n}(x) = 2^{k/2}\psi(2^kx - n).$$

These wavelets for all integers  $k$  and  $n$  produce an orthogonal basis of  $L_2(\mathbb{R})$ .

The Haar wavelet [20, 31] is the most practical orthonormal wavelets with compact support and was constructed by Haar in 1909. Haar wavelet operational matrix of integration was first derived by Chen et al. [7] to solve the differential equations. There are several other wavelets which can be used to solve the differential equations. Some of these include, Daubechies [11, 33], B-spline [12], Legendre [29, 36], Hermite [1, 32] and Chebyshev [2, 16]. Legendre, Hermite and Chebyshev wavelets use Legendre, Hermite and Chebyshev polynomials as their basis functions, respectively.

In [35], author combined tau collocation method with linearization technique or Adomian's decomposition method for solving nonlinear partial differential equation of integer order. The main purpose of this article is to propose a numerical method for solving the fractional nonlinear Lane-Emden type equations by using Haar wavelets in conjunction with Adomian's decomposition method. First, the Adomian's polynomial is used to expand the nonlinear term of Lane-Emden type equation into a set of polynomials and then utilize the properties of Haar wavelets method to convert the obtained Lane-Emden equation into a system of algebraic equations. The solution of the obtained system provides the values of Haar wavelets coefficients which lead to the solution of nonlinear Lane-Emden type equation. No linearization process is required for this approach. This approach is a new idea in the field of wavelets method and it is first time introduce for wavelets method and implemented on nonlinear Lane-Emden equations.

The paper is arranged as follows: in Section 2 we describe the basic definitions of fractional integration and differentiation. Function approximations, Haar wavelets and construction of operational matrices are describe in Section 3. We present the procedure of implementation for nonlinear Lane-Emden equation in Section 4. In Section 5 we discuss the convergence analysis of the method and in Section 6 we apply the proposed method to standard Lane-Emden equation and the isothermal gas spheres equation. Finally in Section 7 we conclude our work.

## 2. Preliminaries

In this section we introduce some necessary definitions and mathematical preliminaries of fractional calculus [13, 19].

**Definition 2.1.** Let  $\alpha \in \mathbb{R}^+$ . The operator  $I_a^\alpha$ , defined on  $L_1[a, b]$  by

$$I_a^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t) dt,$$

is called the Reimann-Liouville fractional integral operator of order  $\alpha$ .

**Definition 2.2.** Let  $\alpha \in \mathbb{R}^+$ ,  $n = \lceil \alpha \rceil$  and  $u \in AC^n[a, b]$ . The operator  ${}^{RL}D_a^\alpha$ , defined by

$${}^{RL}D_a^\alpha u(x) = \frac{d^n}{dx^n} \int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} u(t) dt,$$

is called the Reimann-Liouville fractional derivative of order  $\alpha$ . The notation  $AC[a, b]$  is used to denote the space of functions  $f$  which are absolutely continuous on  $[a, b]$  and  $AC^n[a, b]$  is the space of functions  $f(x)$  which have continuous derivatives up to  $n - 1$  on  $[a, b]$  such that  $f^{(n-1)}(x) \in AC[a, b]$ .

**Definition 2.3.** Let  $\alpha \in \mathbb{R}^+$ ,  $n = \lceil \alpha \rceil$  and  $u \in AC^n[a, b]$ . Then the Caputo fractional derivative of  $u(x)$  is defined by

$$D_a^\alpha u(x) = \int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} u^{(n)}(t) dt.$$

**Lemma 2.4.** Let  $\alpha > 0$ ,  $\beta > -1$  and  $f(x) = (x-a)^\beta$ , then

$$I_a^\alpha f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-a)^{\alpha+\beta}.$$

**Lemma 2.5.** Let  $\alpha, \beta \in \mathbb{R}^+$  and  $u \in L_1[a, b]$  then  $I_a^\alpha I_a^\beta u(x) = I_a^{\alpha+\beta} u(x)$  holds almost everywhere on  $[a, b]$ .

**Lemma 2.6.** If  $\alpha \in \mathbb{R}^+$  and  $u \in C[a, b]$ , then  $D_a^\alpha I_a^\alpha u(x) = u(x)$ .

**Lemma 2.7.** Let  $\alpha > 0$ ,  $n = \lceil \alpha \rceil$  and  $u \in AC^n[a, b]$  then

$$I_a^\alpha D_a^\alpha u = u(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} u^{(k)}(a).$$

### 3. Function approximations and Haar matrices

The Haar functions contains just one wavelet during some subinterval of time, remains zero elsewhere and are orthogonal. The uniform Haar wavelets are useful for the treatment of solution of differential equations which has no abrupt behavior. The  $l$ th uniform Haar wavelet  $h_l(x)$ ,  $x \in [0, 1]$  is defined as [7]:

$$(3.1) \quad h_l(x) = \begin{cases} 1 & a(l) \leq x < b(l), \\ -1 & b(l) \leq x < c(l); \\ 0 & \text{otherwise,} \end{cases}$$

where  $a(l) = k/m$ ,  $b(l) = (k+0.5)/m$ ,  $c(l) = (k+1)/m$  and  $l = 2^j + k + 1$ ,  $j = 0, 1, 2, \dots, J$  is dilation parameter, where  $m = 2^j$  and  $k = 0, 1, 2, \dots, 2^j - 1$  is translation parameter.  $J$  is maximal level of resolution and the maximal value of  $i$  is  $2M$  where  $M = 2^J$ . In particular  $h_1(x) := \chi_{[0,1]}(x)$ , where  $\chi_{[0,1]}(x)$  is the characteristic function on the interval  $[0, 1]$ , is the Haar scaling function. For the uniform Haar wavelet, the wavelet-collocation method is applied. The collocation points for the uniform Haar wavelets are usually taken as  $x_j = (j - 0.5)/(2M)$ , where  $j = 1, 2, \dots, 2M$ .

Any function  $y \in L_2[0, 1]$  can be represented in term of the Haar series

$$(3.2) \quad y(x) = \sum_{l=1}^{\infty} b_l h_l(x),$$

where  $b_l$  are the Haar wavelet coefficients given as  $b_l = \int_0^1 y(x) h_l(x) dx$ . This function (3.2) can be represented by the truncated Haar wavelet series

$$(3.3) \quad y(x) \approx y_M(x) = \sum_{l=1}^{2M} b_l h_l(x), \quad l = 2^j + k + 1, \quad j = 0, 1, 2, \dots, J, \quad k = 0, 1, 2, \dots, 2^j - 1.$$

The wavelets coefficients  $b_l$  are determined in such away that the integral square error  $E$  given by

$$(3.4) \quad E = \int_0^1 \left[ y(x) - \sum_{l=1}^{2M} b_l h_l(x) \right]^2 dx$$

is minimized.

In order to find the numerical approximations of function, we put the Haar wavelets into a discrete form. For this purpose, we utilized the collocation method. The collocation points for the Haar wavelets are taken as  $x_c(i) = (i + 0.5)/(2M)$ , where  $i = 1, 2, \dots, 2M$ . In discrete form, equation (3.4) is written as

$$E_M = \Delta x \sum_{i=1}^{2M} \left[ y(x_c(i)) - \sum_{l=1}^{2M} b_l h_l(x_c(i)) \right]^2.$$

The discrete form of (3.3) is

$$(3.5) \quad y_M(x_c(i)) = \sum_{l=1}^{2M} b_l h_l(x_c(i)).$$

We can represent the equation (3.5) in vector form as

$$\mathbf{y} = \mathbf{bH},$$

where  $\mathbf{b} = [b_1 b_2 \cdots b_{2M}]$  and  $\mathbf{y} = [y_1 y_2 \cdots y_{2M}]$  are  $2M$  dimensional row vectors,  $y_i = y_M(x_c(i))$ , and

$$\mathbf{H}_{2M \times 2M} = \begin{bmatrix} h_1(x_c(1)) & h_1(x_c(2)) & \cdots & h_1(x_c(2M)) \\ h_2(x_c(1)) & h_2(x_c(2)) & \cdots & h_2(x_c(2M)) \\ \dots & \dots & \dots & \dots \\ h_{2M}(x_c(1)) & h_{2M}(x_c(2)) & \cdots & h_{2M}(x_c(2M)) \end{bmatrix}$$

is a Haar matrix. In particular, for  $J = 2$ , we get  $2M = 8$  and the Haar matrix is given as

$$\mathbf{H}_{8 \times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

The Haar coefficients  $b_l$  can be determined by matrix inversion

$$(3.6) \quad \mathbf{b} = \mathbf{y} \mathbf{H}^{-1},$$

where  $\mathbf{H}^{-1}$  is the inverse of  $\mathbf{H}$ . Equation (3.6) gives the Haar coefficients  $b_l$  which are used in (3.3) to get the solution  $y(x)$ .

### 3.1. Fractional integral of the Haar wavelets

The Riemann-Liouville fractional integral, of order  $\alpha$ , of the uniform Haar wavelets is given as

$$(3.7) \quad I_a^\alpha h_1(x) = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)},$$

and

$$(3.8) \quad p_{\alpha,l}(x) = I_a^\alpha h_l(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} h_l(s) ds \\ = \frac{1}{\Gamma(\alpha+1)} \begin{cases} (x-a(l))^\alpha & a(l) \leq x < b(l), \\ (x-a(l))^\alpha - 2(x-b(l))^\alpha & b(l) \leq x < c(l), \\ (x-a(l))^\alpha - 2(x-b(l))^\alpha + (x-c(l))^\alpha & x \geq c(l). \end{cases}$$

### 3.2. Haar wavelet operational matrix of fractional integration

Haar matrix  $\mathbf{H}$  is obtained by using the collocation points in (3.1),  $H(l, i) = h_l(x_c(i))$ . Similarly, we can obtain the fractional order integration matrix  $\mathbf{P}$  of Haar function by substituting the collocation points in equations (3.7) and (3.8),  $\mathbf{P}(l, i) = p_{\alpha,l}(x_c(i))$ , as

$$\mathbf{P}_{2M \times 2M} = \begin{bmatrix} p_{\alpha,1}(x_c(1)) & p_{\alpha,1}(x_c(2)) & \cdots & p_{\alpha,1}(x_c(2M)) \\ p_{\alpha,2}(x_c(1)) & p_{\alpha,2}(x_c(2)) & \cdots & p_{\alpha,2}(x_c(2M)) \\ \dots & \dots & \dots & \dots \\ p_{\alpha,2M}(x_c(1)) & p_{\alpha,2M}(x_c(2)) & \cdots & p_{\alpha,2M}(x_c(2M)) \end{bmatrix}.$$

In particular, we fix  $J = 2$ ,  $\alpha = 0.75$ , we get  $2M = 8$  and the Haar wavelet operational matrix of fractional integration is

$$\mathbf{P}_{8 \times 8} = \begin{bmatrix} 0.1360 & 0.3100 & 0.4548 & 0.5853 & 0.7067 & 0.8215 & 0.9312 & 1.0367 \\ 0.1360 & 0.3100 & 0.4548 & 0.5853 & 0.4347 & 0.2014 & 0.0216 & -0.1340 \\ 0.1360 & 0.3100 & 0.1828 & -0.0347 & -0.0668 & -0.0391 & -0.0275 & -0.0210 \\ 0 & 0 & 0 & 0 & 0.1360 & 0.3100 & 0.1828 & -0.0347 \\ 0.1360 & 0.0380 & -0.0293 & -0.0142 & -0.0091 & -0.0066 & -0.0051 & -0.0042 \\ 0 & 0 & 0.1360 & 0.0380 & -0.0293 & -0.0142 & -0.0091 & -0.0066 \\ 0 & 0 & 0 & 0 & 0.1360 & 0.0380 & -0.0293 & -0.0142 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1360 & 0.0380 \end{bmatrix}.$$

### 4. Procedure of implementation

In the present method, Adomian's polynomials is used to convert the nonlinear terms of the nonlinear differential equation into a set of polynomials. No linearization process is required for the suggested method. We describe the procedure of implementation in more details, which enable the readers to understand the method more efficiently. Consider the following form of Lane-Emden equation

$$(4.1) \quad D^\alpha y + \frac{2}{x} D^\beta y + f(y) = g(x), \quad x > 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \\ y(0) = A, \quad y'(0) = B,$$

where  $f(y)$  is the nonlinear term of the problem and,  $A$  and  $B$  are some constants. In the Adomian decomposition method, we can express the solution of (4.1) in a series form as

$$(4.2) \quad y(x) = \sum_{i=0}^{\infty} y_i(x).$$

We approximate the solution (4.2) by the truncated Adomian series as

$$(4.3) \quad y(x) \approx \sum_{i=0}^N y_i(x), \quad N \in \mathbb{N}.$$

Moreover, the nonlinear term  $f(y)$  in equation (4.1) is decomposed in term of Adomian polynomials as

$$(4.4) \quad f(y) = \sum_{i=0}^{\infty} A_i(y_0, y_1, \dots, y_i) \quad \text{or} \quad f(y) \approx \sum_{i=0}^{N-1} A_i(y_0, y_1, \dots, y_i),$$

where  $A_i = \frac{1}{i!} \frac{d^i}{d\lambda^i} [f(\sum_{n=0}^i \lambda^n y_n(x))]|_{\lambda=0}$ ,  $i = 0, 1, 2, \dots$ , are the Adomian polynomials.

Using equations (4.3) and (4.4) in (4.1), we get

$$D^\alpha \left( \sum_{i=0}^N y_i(x) \right) + \frac{2}{x} D^\beta \left( \sum_{i=0}^N y_i(x) \right) = g(x) - \sum_{i=0}^{N-1} A_i$$

or

$$\begin{aligned} & D^\alpha (y_0(x) + y_1(x) + \dots + y_N(x)) + \frac{2}{x} D^\beta (y_0(x) + y_1(x) + \dots + y_N(x)) \\ &= g(x) - (A_0 + A_1 + \dots + A_{N-1}). \end{aligned}$$

The problem (4.1) can be decomposed into  $N + 1$  subproblems by the principle of superposition as

$$(4.5) \quad D^\alpha y_0(x) + \frac{2}{x} D^\beta y_0(x) = g(x), \quad y_0(0) = A, \quad y'_0(0) = B,$$

and

$$(4.6) \quad D^\alpha y_i(x) + \frac{2}{x} D^\beta y_i(x) = -A_{i-1}, \quad y_i(0) = 0, \quad y'_i(0) = 0,$$

where  $x > 0$ ,  $1 < \alpha \leq 2$ ,  $0 < \beta \leq 1$  and  $i = 1, 2, \dots, N$ .

Use Haar wavelet method on the  $N + 1$  subproblems (4.5) and (4.6). We approximate each component  $y_i(x)$ ,  $i = 0, 1, \dots, N$ , of solution  $y(x)$ , given in equation (4.3), by the truncated Haar wavelet series (3.3) as

$$y_i(x) \approx y_{M,i}(x) = \sum_{l=1}^{2M} b_l^i h_l(x).$$

Applying the Haar wavelet method on equation (4.5), we approximate the higher order derivative term by Haar wavelet series as

$$(4.7) \quad D^\alpha y_0(x) \approx D^\alpha y_{M,0}(x) = \sum_{l=1}^{2M} b_l^0 h_l(x).$$

Lower order derivatives are obtained by integrating (4.7) and use of initial conditions, we get

$$(4.8) \quad \begin{aligned} y_0(x) &\approx y_{M,0}(x) = \sum_{l=1}^{2M} b_l^0 I_a^\alpha h_l(x) + Bx + A, \\ D^\beta y_0(x) &\approx D^\beta y_{M,0}(x) = \sum_{l=1}^{2M} b_l^0 I_a^{\alpha-\beta} h_l(x) + B \left( \frac{x^{1-\beta}}{\Gamma(2-\beta)} \right). \end{aligned}$$

Using equations (4.7) and (4.8) in equation (4.5) to obtain

$$(4.9) \quad \sum_{l=1}^{2M} b_l^0 \left( h_l(x) + \frac{2}{x} I_a^{\alpha-\beta} h_l(x) \right) = g(x) - \frac{2Bx^{-\beta}}{\Gamma(2-\beta)}.$$

Expand equation (4.9) at the collocation points,  $x_j = (j + 0.5)/(2M)$ ,  $j = 1, 2, \dots, 2M$ , which enable us to represent equation (4.9) in vector form by using Section 3 that is

$$(4.10) \quad (\mathbf{b}^0)^T \left[ \mathbf{H}_{2M \times 2M} + \mathbf{P}_{2M \times 2M}^{\alpha-\beta} \mathbf{D} \right] = \mathbf{R},$$

where

$$\mathbf{D} = \begin{bmatrix} \frac{2}{x_1} & 0 & \cdots & 0 \\ 0 & \frac{2}{x_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{2}{x_{2M}} \end{bmatrix},$$

$\mathbf{R} = [r(x_1), r(x_2), \dots, r(x_{2M})]$  and  $r(x) = g(x) - 2Bx^{-\beta}/\Gamma(2-\beta)$ . Solution of equation (4.10) gives  $(\mathbf{b}^0)^T$ , it is used in equation (4.8) to obtain  $y_{M,0}(x)$  which is an approximation to  $y_0(x)$ . Similarly apply the Haar wavelet method on equation (4.6) by approximating higher order derivative by Haar wavelet series

$$(4.11) \quad D^\alpha y_i(x) \approx D^\alpha y_{M,i}(x) = \sum_{l=1}^{2M} b_l^i h_l(x).$$

Integrating (4.11) and use the initial conditions, we get

$$(4.12) \quad \begin{aligned} y_i(x) &\approx y_{M,i}(x) = \sum_{l=1}^{2M} b_l^i I_a^\alpha h_l(x), \\ D^\beta y_i(x) &\approx D^\beta y_{M,i}(x) = \sum_{l=1}^{2M} b_l^i I_a^{\alpha-\beta} h_l(x). \end{aligned}$$

Using equations (4.11) and (4.12) in equation (4.6) and expanding the obtained equation at the collocation points to obtain

$$(4.13) \quad (\mathbf{b}^i)^T \left[ \mathbf{H}_{2M \times 2M} + \mathbf{P}_{2M \times 2M}^{\alpha-\beta} \mathbf{D} \right] = \mathbf{U},$$

where  $\mathbf{U} = -[A_{i-1}(x_1), A_{i-1}(x_2), \dots, A_{i-1}(x_{2M})]$ .

Fix  $i = 1$  and use the obtained approximation,  $y_{M,0}(x)$ , in the calculation of Adomian's polynomials  $A_0$ . Now solve the equation (4.13) for  $(\mathbf{b}^1)^T$  and use it in (4.12) to get  $y_{M,1}(x)$  which is an approximation of  $y_1(x)$ . Similarly, by considering  $i = 2$ , we obtain  $A_1$  by using  $y_{M,0}(x)$  and  $y_{M,1}(x)$ , and then obtain  $(\mathbf{b}^2)^T$  from equation (4.13) which is used in equation (4.12) to get approximate solution  $y_{M,2}(x)$  of  $y_2(x)$ . This process is repeated by using the approximate solutions  $y_{M,i}(x)$ ,  $i = 0, 1, \dots, k$ , in the calculation of Adomian's polynomials  $A_k$  and use it in equation (4.13) to get  $(\mathbf{b}^{k+1})^T$ , which is used in (4.12) to obtain approximate solution,  $y_{M,k+1}(x)$ , of  $y_{k+1}(x)$ . In this way, we obtain a sequence of approximations  $\{y_{M,i}(x)\}$ ,  $i = 0, 1, \dots, N$ , where  $N \in \mathbb{N}$ . Thus the approximate solution of problem (4.1) is obtained as  $\sum_{i=0}^N y_{M,i}(x)$ .

## 5. Convergence analysis of modified Haar wavelet method

The convergence analysis of proposed method based on the convergence analysis of Adomian decomposition method and Haar wavelet method. For convergence of Haar wavelet method. Let  $y_i(x)$ , which is a component of truncated Adomian series (4.3), be a differentiable function and assume that  $y_i(x)$  have bounded first derivative on  $(0, 1)$ , i.e., there exist  $K > 0$ ; for all  $x \in (0, 1)$

$$|y'_i(x)| \leq K.$$

Haar wavelet approximation for the function  $y_i(x)$  is given by

$$y_{M,i}(x) = \sum_{l=1}^{2M} b_l^i h_l(x).$$

By following Babolian and Shahsavar [3], we get the  $L_2$ -error norm for present method

$$\|y_i(x) - y_{M,i}(x)\|^2 \leq \frac{K^2}{3} \cdot \frac{1}{(2M)^2},$$

or

$$(5.1) \quad \|y_i(x) - y_{M,i}(x)\| \leq O(1/M).$$

As  $M = 2^J$  and  $J$  is the maximal level of resolution. According to (5.1), we conclude that error is inversely proportional to the level of resolution. Equation (5.1) ensures the convergence of Haar wavelet approximation  $y_{M,i}(x)$  for components of Adomian's series  $y_i(x)$  at higher level of resolution i.e., when  $M = 2^J$  approaches to infinity. According to the convergence of Adomian's method [9],  $\sum_{i=0}^N y_i(x)$  converges to  $y(x)$  when  $N \rightarrow \infty$ . According to the above analysis, we conclude that solution by the present method converges to the exact solution of (4.1), when  $N$  and  $J$  approach to infinity.

## 6. Applications

The method is tested on the fractional standard Lane-Emden equation and the fractional isothermal gas spheres equation.

### 6.1. The fractional standard Lane-Emden equation

Consider the fractional standard Lane-Emden equation that is used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics

$$(6.1) \quad D^\alpha y + \frac{2}{x} D^\beta y + y^p(x) = 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1,$$

subject to the initial conditions:  $y(0) = 1$ ,  $y'(0) = 0$ .

The exact solutions, when  $\alpha = 2$ ,  $\beta = 1$  and  $p = 0, 1, 5$ , are given in [27]. For other values of  $\alpha$ ,  $\beta$  and  $p$ , exact solutions are not known. Therefore, we apply the proposed method to solve the standard Lane-Emden equation for different values of  $\alpha$ ,  $\beta$  and  $p$ .

We implement the proposed method on equation (6.1) by considering  $p = 5$ , and different values of  $\alpha$  and  $\beta$  as shown in Figure 6.1.  $y_E(x)$  and  $y_H(x)$  represent exact solution and solution by present method respectively. Figure 6.1 shows that solution by the present method converges to the exact solution, when  $\alpha$  and  $\beta$  approaches to 2 and 1 respectively.

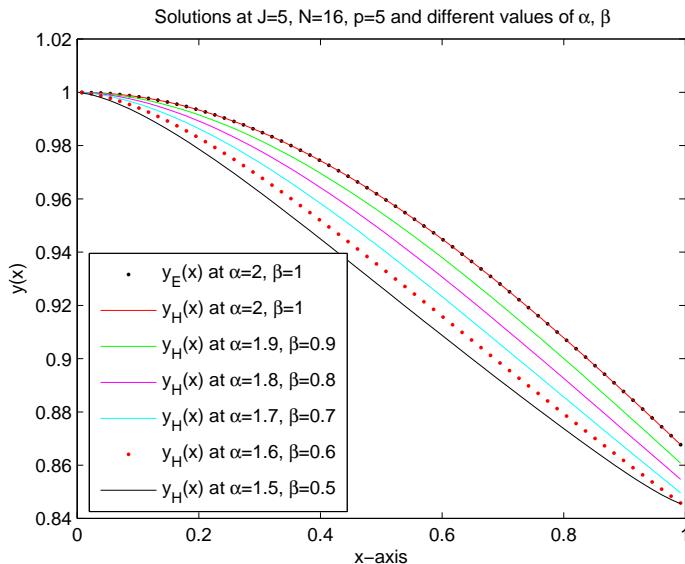


Figure 6.1: Exact solution  $y_E(x)$  at  $\alpha = 2$ ,  $\beta = 1$  and approximate solutions  $y_H(x)$  of the standard Lane-Emden equation at different value of  $\alpha$  and  $\beta$ .

Figure 6.2 is used to plot the obtained solutions,  $y_H(x)$ , of equation (6.1) by present method at  $J = 5$ ,  $N = 16$ ,  $\alpha = 1.6125$ ,  $\beta = 0.7653$ , and different values of  $p$ . We compare our solution with solutions obtained from following methods: Homotopy perturbation method [10], Optimal homotopy asymptotic method [18], Boubaker polynomials expansion scheme [5] and squared remainder minimization method [6] as shown in Table 6.1.  $E_{HPM}$ ,  $E_{OHAM}$ ,  $E_{BPES}$ ,  $E_{SRMM}$  and  $E_{Haar}$  represent the absolute errors by homotopy perturbation method, Optimal homotopy asymptotic method, Boubaker polynomials expansion scheme, squared remainder minimization method and present method respectively. According to the Table 6.1, we conclude that results produced by present method are better than the other methods [5, 6, 10, 18].

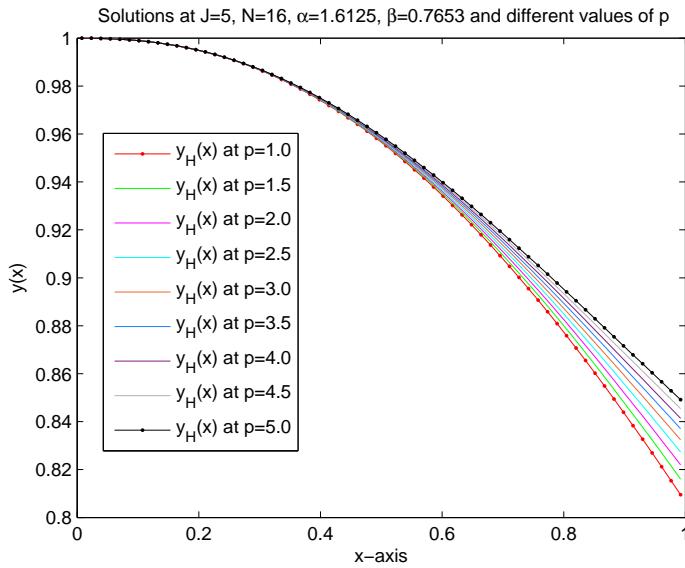


Figure 6.2: Approximate solutions of the standard Lane-Emden equation, at different  $p$  and fix  $\alpha = 1.6125$ ,  $\beta = 0.7653$ .

$J = 9, N = 23$					
$x$	$E_{HPM}$ [10]	$E_{OHAM}$ [18]	$E_{BPES}$ [5]	$E_{SRMM}$ [6]	$E_{Haar}$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	3.36569e-11	4.01135e-5	5.22512e-4	1.46725e-7	6.63284e-11
0.5	1.22677e-5	3.56912e-4	1.30811e-2	3.56494e-6	1.68772e-9
1.0	2.59948e-3	4.49742e-4	8.26346e-2	5.07745e-7	6.22345e-9

Table 6.1: Comparison of absolute errors of the approximate solutions for the standard Lane-Emden equation, when  $p = 5$ ,  $\alpha = 2$  and  $\beta = 1$ .

## 6.2. The fractional isothermal gas spheres equation

The fractional Lane-Emden equation for a self-gravitating isothermal gas sphere can be written as

$$(6.2) \quad D^\alpha y + \frac{2}{x} D^\beta y - e^{-y} = 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1,$$

subject to the initial conditions:  $y(0) = 0$ ,  $y'(0) = 0$ , where  $y(x)$  is the Newtonian gravitational potential function and  $x$  is the dimensionless radius.

Figure 6.3 is used to plot the solutions,  $y_H(x)$ , obtained from present method at different values of  $\alpha$  and  $\beta$ . In [22], B. M. Mirza discussed the fractional approximation technique for the solution of isothermal gas sphere equation (6.2). A power series solution method is used by M. I. Nouh in [25] for the solution of (6.2). Euler-transformed series is used to approximate the solution of isothermal gas sphere equation by C. Hunter [17].

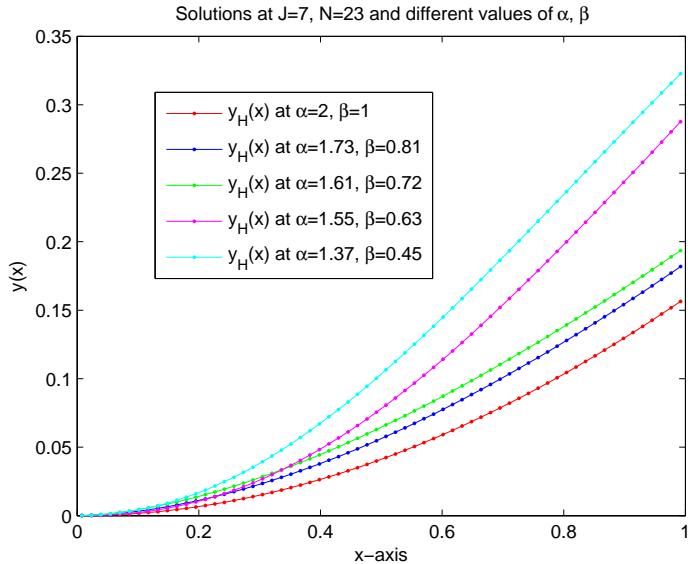


Figure 6.3: Approximate solutions  $y_H(x)$  of the isothermal gas spheres equation at different value of  $\alpha$  and  $\beta$ .

In Table 6.2, exact solution represents the solution obtained from Runge Kutta method of order 4 and Haar represents the obtained solution by present method at  $J = 8$  and  $N = 23$ . Our results for isothermal gas sphere equation (6.2), when  $\alpha = 2$  and  $\beta = 1$ , are better than those obtained in [17, 22, 25].

$J = 8, N = 23$					
$x$	Exact solution	Haar	Mirza [22]	Nouh [25]	Hunter [17]
0.1	0.001666	0.0016658	0.0016	0.0166	0.0016
0.2	0.006653	0.0066534	0.0066	0.0333	0.0065
0.3	0.014933	0.0149329	0.0149	0.0500	0.0145
0.4	0.026455	0.0264555	0.0266	0.0666	0.0253
0.5	0.041154	0.0411540	0.0416	0.0833	0.0385
0.6	0.058944	0.0589441	0.0598	0.1000	0.0536
0.7	0.079726	0.0797260	0.0813	0.1166	0.0700
0.8	0.103386	0.1033861	0.1060	0.1333	0.0870
0.9	0.129799	0.1297985	0.1338	0.1500	0.1038
1.0	0.158828	0.1588277	0.1646	0.1666	0.1198

Table 6.2: Comparison of the approximate solutions for the isothermal gas sphere equation when  $\alpha = 2$  and  $\beta = 1$ .

## 7. Conclusion

We have derived and constructed the Haar wavelets matrix,  $\mathbf{H}_{2M \times 2M}$ , and the Haar wavelets operational matrix of fractional order integration,  $\mathbf{P}_{2M \times 2M}$ . These matrices are successfully utilized to solve the fractional nonlinear Lane-Emden type equations.

According to Tables 6.1 and 6.2, our results are more accurate as compared to Homotopy perturbation method [10], Optimal homotopy asymptotic method [18], Boubaker polynomials expansion scheme [5], fractional approximation technique [22], power series solution method [25], Euler-transformed series method [17] and squared remainder minimization method [6].

Also, fractional order lane emden equation converge to the integer order Lane-Emden equation when  $\alpha$  and  $\beta$  approaches to integer values as shown in Figures 6.1 and 6.3.

It is shown that present method gives excellent results when applied to fractional nonlinear Lane-Emden type equations. The different type of non-linearities in Lane-Emden type equations can easily be handled by the present method.

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