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Multifractal Analysis for Maps with the Gluing Orbit Property

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Abstract. In this paper, we obtain a conditional variational principle for the topological entropy of level sets of Birkhoff averages for maps with the gluing orbit property. Our result can be easily extended to flows.

1. Introduction

This article is devoted to the study of multifractal analysis for maps with gluing orbit property. Before stating our results, we first give some notations and backgrounds. By a topological dynamical system (*TDS* for short) (X, f), we mean that (X, d) is a compact metric space and f is a continuous map from X to itself. For a continuous function $\varphi\colon X \to \mathbb{R}$, X can be divided into the following parts:

$$X = \bigcup_{\alpha \in \mathbb{R}} X(\varphi, \alpha) \cup \widehat{X}(\varphi, f),$$

where for $\alpha \in \mathbb{R}$,

$$X(\varphi, \alpha) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = \alpha \right\}$$

and

$$\widehat{X}(\varphi, f) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \text{ does not exist} \right\}.$$

The level set $X(\varphi, \alpha)$ is called the multifractal decomposition set of ergodic average of φ in multifractal analysis. And the set $\widehat{X}(\varphi, f)$ is called the historic set of ergodic average of φ. The multifractal analysis of dynamical systems is a subfield of dimension theory. Roughly speaking, multifractal analysis studies the complexity of the level sets with invariant local quantities obtained from a dynamical system. There are fruitful results to describe the size of the level sets. The quantities include Hausdorff dimension, topological entropy and topological pressure (see [1, 2, 8–11, 13]).

The concept of the gluing orbit property was introduced by Bofim and Varandas [3] to study large deviations principles for semiflows. It is also called the weak specification

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property, which was introduced by Constantine, Lafont and Thompson [5] independently. In this paper, we prove a variational principle for the topological entropy of the level set $X(\varphi, \alpha)$ for maps with the gluing orbit property.

Let (X, f) be a *TDS*. For a continuous function $\varphi\colon X \to \mathbb{R}$ and $n \geq 1$, let $S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(f^i x)$, and for $c > 0$, we set

$$\mathrm{Var}(\varphi, c) := \sup \{ |\varphi(x) - \varphi(y)| : d(x, y) \leq c \}.$$

For every $\epsilon > 0$, $n \in \mathbb{N}$ and a point $x \in X$, define

$$B_n(x, \epsilon) = \left\{ y \in X : d(f^i x, f^i y) < \epsilon, 0 \leq i \leq n-1 \right\}$$

and

$$\overline{B_n}(x, \epsilon) = \left\{ y \in X : d(f^i x, f^i y) \leq \epsilon, 0 \leq i \leq n-1 \right\}.$$

Let $C(X, \mathbb{R})$ be the Banach algebra of real-valued continuous functions of X equipped with the supremum norm. Denote by $M(X)$, $M(X, f)$ and $E(X, f)$ the set of all Borel probability measures on X, the collection of all f-invariant Borel probability measures and the collection of all ergodic measures, respectively. It is well known that (see [12]) $M(X)$ and $M(X, f)$ equipped with weak* topology are both convex, compact spaces. For measures $m_1, m_2 \in M(X, f)$, we define the metric $D(m_1, m_2)$ compatible with the weak* topology by

$$D(m_1, m_2) = \sum_{n=1}^{\infty} \frac{\left| \int f_n \, dm_1 - \int f_n \, dm_2 \right|}{2^n \left\| f_n \right\|},$$

where $\{f_n\}_{n=1}^{\infty}$ is a dense subset of $C(X, \mathbb{R})$.

Definition 1.1 (Bowen's topological entropy)**.** [4] Given $Z \subset X$, $\epsilon > 0$ and $N \in \mathbb{N}$, let $\Gamma_N(Z, \epsilon)$ be the collection of all finite or countable covers of Z by sets of the form $B_n(x, \epsilon)$ with $n \geq N$. For each $s \in \mathbb{R}$, we set

$$m(Z, s, N, \epsilon) := \inf \left\{ \sum_{B_n(x,\epsilon) \in \mathcal{C}} e^{-ns} : \mathcal{C} \in \Gamma_N(Z, \epsilon) \right\}$$

and

$$m(Z, s, \epsilon) := \lim_{N \to \infty} m(Z, s, N, \epsilon).$$

Define

$$h_{\mathrm{top}}(Z, \epsilon) = \inf \{ s \in \mathbb{R} : m(Z, s, \epsilon) = 0 \} = \sup \{ s \in \mathbb{R} : m(Z, s, \epsilon) = \infty \}$$

and topological entropy of Z as

$$h_{\mathrm{top}}^B(Z) := \lim_{\epsilon \to 0} h_{\mathrm{top}}^B(Z, \epsilon).$$

It is obvious that the following hold:

(1) $h^B_{\rm top}(Z_1) \le h^B_{\rm top}(Z_2)$ for any $Z_1 \subset Z_2 \subset X$;

(2) $h^B_{\rm top}(Z) = \sup_{i=1}^{\infty} h^B_{\rm top}(Z_i)$, where $Z = \bigcup_{i=1}^{\infty} Z_i \subset X$.

Definition 1.2. [3,5] We say a continuous map $f\colon X \to X$ on a compact metric space X satisfying the gluing orbit property if for any $\epsilon > 0$, there exists an integer $N = N(\epsilon) \ge 1$ such that for any points $x_1, \ldots, x_k \in X$ and any positive integers $n_1, \ldots, n_k$, there are $p_1, \ldots, p_k \le N(\epsilon)$ and a point $x \in X$ so that $d(f^j x, f^j x_1) \le \epsilon$ for every $0 \le j < n_1$ and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x), f^j(x_i)) \le \epsilon$$

for every $2 \le i \le k$ and $0 \le j < n_i$.

There are two important classes of examples satisfying the gluing orbit property.

Proposition 1.3. [3,5] *Let Σ be a subshift of finite type. The Σ is transitive if and only if Σ satisfies the gluing orbit property.*

Proposition 1.4. [5] *Let X be a compact, locally* CAT(-1)*, geodesic metric space, with non-elementary fundamental group. Then the geodesic flow on GX satisfies the weak specification property (the gluing orbit property).*

We state our result as follows.

Theorem 1.5. *Suppose (X, f) is a TDS, f satisfies the gluing orbit property, $\alpha \in \mathbb{R}$, and $\varphi \in C(X, \mathbb{R})$. Then*

$$h^B_{\rm top}(X(\varphi,\alpha)) = \sup\left\{ h_\nu(f) : \nu \in M(X,f) \text{ and } \int \varphi \, d\nu = \alpha \right\}.$$

2. Proof of Theorem 1.5

In this section, we will prove our main result. The upper bound on $h^B_{\rm top}(X(\varphi,\alpha))$ is easy to get. In order to obtain the lower bound on $h^B_{\rm top}(X(\varphi,\alpha))$, we will use the gluing orbit property. The proof will be divided into the following two subsections.

2.1. Upper bound on $h^B_{\rm top}(X(\varphi,\alpha))$

Let (X, f) be a *TDS*. For $x \in X$, let δ_x be the unit measure concentrated on x. For $n \in \mathbb{N}$, define

$$\mathcal{E}_n(x) = \frac{\delta_x + \delta_{f(x)} + \cdots + \delta_{f^{n-1}(x)}}{n}.$$

Let $V(x)$ be the set of all limit points in $M(X)$ of the sequence $\mathcal{E}_n(x)$.

The upper bound of $h^B_{\rm top}(X(\varphi,\alpha))$ can be easily obtained by the following lemma.

Lemma 2.1. [4] *For $t \geq 0$, consider the set*

$$B(t) = \{x \in X : \exists\, \nu \in V(x) \text{ satisfying } h_\nu(f) \leq t\}.$$

Then $h^B_{\text{top}}(B(t)) \leq t$.

Let

$$t = \sup\left\{h_\nu(f) : \nu \in M(X,f) \text{ and } \int \varphi\, d\nu = \alpha\right\}.$$

Then $X(\varphi,\alpha) \subset B(t)$. Thus we have

$$h^B_{\text{top}}(X(\varphi,\alpha)) \leq \sup\left\{h_\nu(f) : \nu \in M(X,f) \text{ and } \int \varphi\, d\nu = \alpha\right\}.$$

2.2. Lower bound on $h^B_{\text{top}}(X(\varphi,\alpha))$

To get the lower bound on $h^B_{\text{top}}(X(\varphi,\alpha))$, the dynamical system needs some mild assumption such as gluing orbit property. Our strategy is inspired by the work of Takens and Verbitskiy [10], developed by Chen, Kupper and Shu [6], Pfister and Sullivan [9], Pei and Chen [8] and Thompson [11].

2.2.1. Katok's definition of metric entropy

We use the Katok's definition of metric entropy based on the following lemma.

Lemma 2.2. [7] *Let (X,d) be a compact metric space, $f\colon X \to X$ be a continuous map and ν be an ergodic invariant measure. For $\epsilon > 0$, $\delta \in (0,1)$, denote $N^\nu(n,\epsilon,\delta)$ the minimum number of ϵ-Bowen balls $B_n(x,\epsilon)$, which cover a set of ν-measure larger than $1-\delta$. Then*

$$h_\nu(f) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log N^\nu(n,\epsilon,\delta) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N^\nu(n,\epsilon,\delta).$$

Fix $\delta \in (0,1)$. For $\epsilon > 0$ and $\nu \in E(X,f)$, we define

$$h^{\text{Kat}}_\nu(f,\epsilon) := \liminf_{n\to\infty}\frac{1}{n}\log N^\nu(n,\epsilon,\delta).$$

Then by Lemma 2.2,

$$h_\nu(f) = \lim_{\epsilon\to 0} h^{\text{Kat}}_\nu(f,\epsilon).$$

If ν is non-ergodic, we will define $h^{\text{Kat}}_\nu(f,\epsilon)$ by the ergodic decomposition of ν. The following lemma is necessary.

Lemma 2.3. *Fix $\epsilon,\delta > 0$ and $n \in \mathbb{N}$, the function $s\colon E(X,f) \to \mathbb{R}$ defined by $\nu \mapsto N^\nu(n,\epsilon,\delta)$ is upper semi-continuous.*

Proof. Let $\{\nu_k\}$ be a sequence of ergodic measures satisfying $\nu_k \to \nu$ when $k \to \infty$. Let $a > N^{\nu}(n,\epsilon,\delta)$, then there exists a set S which (n,ϵ) spans some set Z with $\nu(Z) > 1-\delta$ such that $a > \sharp S$, where $\sharp S$ denotes the number of elements in S. If k is large enough, then $\nu_k(\bigcup_{x\in S} B_n(x,\epsilon)) > 1-\delta$, which implies that

$$a > N^{\nu_k}(n,\epsilon,\delta).$$

Thus we obtain

$$N^{\nu}(n,\epsilon,\delta) \geq \limsup_{k\to\infty} N^{\nu_k}(n,\epsilon,\delta),$$

which completes the proof. □

By Lemma 2.3, we know the function $s\colon E(X,f) \to \mathbb{R}$ defined by

$$s(m) = h_m^{\mathrm{Kat}}(f,\epsilon)$$

is measurable.

Assume $\nu \in M(X,f)$, $\nu = \int_{E(X,f)} m\, d\tau(m)$ is the ergodic decomposition of ν. Define

$$h_\nu^{\mathrm{Kat}}(f,\epsilon) := \int_{E(X,f)} h_m^{\mathrm{Kat}}(f,\epsilon)\, d\tau(m).$$

By monotone convergence theorem, we have

$$h_\nu(f) = \int_{E(X,f)} \lim_{\epsilon\to 0} h_m^{\mathrm{Kat}}(f,\epsilon)\, d\tau(m) = \lim_{\epsilon\to 0} h_\nu^{\mathrm{Kat}}(f,\epsilon). \tag{2.1}$$

2.2.2. Some important lemmas

Let

$$C := \sup\left\{ h_\nu(f) : \nu \in M(X,f) \text{ and } \int \varphi\, d\nu = \alpha \right\}.$$

We may assume that C is finite and $C > 0$. The case that C is infinite can be included in our proof. Fix small $0 < \delta, \eta < 1$ and $\eta < C/5$. Choose a measure $\mu \in M(X,f)$ satisfying $\int \varphi\, d\mu = \alpha$ such that

$$h_\mu(f) > C - \frac{\eta}{2}.$$

By (2.1), we can choose $\epsilon > 0$ sufficiently small so that

$$h_\mu^{\mathrm{Kat}}(f,4\epsilon) > C - \eta.$$

Then

$$h_\mu^{\mathrm{Kat}}(f,4\epsilon) - 4\eta > C - 5\eta > 0.$$

Let $\{\delta_k\}_{k\geq 1}$ be a sequence of positive numbers strictly decreasing to 0. We have the following lemma.

Lemma 2.4. *For $\mu \in M(X,f)$, there exists a sequence of measures $\{\mu_k\}_{k\geq 1}$, where*

$$\mu_k = \sum_{i=1}^{s_k} a_{k,i} m_{k,i}, \quad \sum_{i=1}^{s_k} a_{k,i} = 1, \quad a_{k,i} \in \mathbb{Q},\ 1 \leq i \leq s_k,\ m_{k,i} \in E(X,f)$$

such that

$$\left| \int \varphi\, d\mu - \int \varphi\, d\mu_k \right| < \delta_k, \quad h_\mu^{\mathrm{Kat}}(f,4\epsilon) \leq \sum_{i=1}^{s_k} a_{k,i} h_{m_{k,i}}^{\mathrm{Kat}}(f,4\epsilon).$$

Proof. Let

$$\mu = \int_{E(X,f)} m\, d\tau(m)$$

be the ergodic decomposition of μ. Fix k. We make $\zeta_k > 0$ such that if $D(\mu_1,\mu_2) < \zeta_k$, then $\left|\int \varphi\, d\mu_1 - \int \varphi\, d\mu_2\right| < \delta_k/2$. Let $\{A_{k,1},\dots,A_{k,s_k}\}$ be a partition of $E(X,f)$ with $\mathrm{diam}(A_{k,i}) < \zeta_k$, $1 \leq i \leq s_k$. For any $A_{k,i}$, there exists an Ergodic measure $m_{k,i} \in A_{k,i}$ such that

$$\int_{A_{k,i}} h_m^{\mathrm{Kat}}(f,4\epsilon)\, d\tau(m) \leq \tau(A_{k,i}) \cdot h_{m_{k,i}}^{\mathrm{Kat}}(f,4\epsilon), \quad 1 \leq i \leq s_k.$$

So

$$\int_{E(X,f)} h_m^{\mathrm{Kat}}(f,4\epsilon)\, d\tau(m) \leq \sum_{i=1}^{s_k} \tau(A_{k,i}) \cdot h_{m_{k,i}}^{\mathrm{Kat}}(f,4\epsilon).$$

By the density of rational number, we can choose $a_{k,i}$ such that

$$|a_{k,i} - \tau(A_{k,i})| < \frac{1}{s_k \|\varphi\|} \frac{\delta_k}{2}, \quad 1 \leq i \leq s_k$$

and

$$h_\mu^{\mathrm{Kat}}(f,4\epsilon) \leq \sum_{i=1}^{s_k} a_{k,i} h_{m_{k,i}}^{\mathrm{Kat}}(f,4\epsilon).$$

Let $\mu_k = \sum_{i=1}^{s_k} a_{k,i} m_{k,i}$, by Choquet representation theorem, we have

$$\begin{aligned}
&\left| \int \varphi\, d\mu - \int \varphi\, d\mu_k \right| \\
&= \left| \int_{E(X,f)} \left(\int \varphi\, dm \right) d\tau(m) - \sum_{i=1}^{s_k} a_{k,i} \int \varphi\, dm_{k,i} \right| \\
&\leq \left| \sum_{i=1}^{s_k} \int_{A_{k,i}} \left(\int \varphi\, dm \right) d\tau(m) - \sum_{i=1}^{s_k} \int_{A_{k,i}} \left(\int \varphi\, dm_{k,i} \right) d\tau(m) \right| + \sum_{i=1}^{s_k} |a_{k,i} - \tau(A_{k,i})|\, \|\varphi\| \\
&\leq \sum_{i=1}^{s_k} \left| \int_{A_{k,i}} \left(\int \varphi\, dm \right) d\tau(m) - \int_{A_{k,i}} \left(\int \varphi\, dm_{k,i} \right) d\tau(m) \right| + \frac{\delta_k}{2} \\
&\leq \sum_{i=1}^{s_k} \frac{\delta_k}{2} \tau(A_{k,i}) + \frac{\delta_k}{2} = \delta_k.
\end{aligned}$$

□

By Birkhoff ergodic theorem, we can choose a strictly increasing sequence $\{l_k\}_{k\geq 1}$ satisfying $\delta_k \to 0$ and $l_k \to \infty$ as $k \to \infty$ so that the set

$$Y_{k,i} := \left\{ x \in X : \left| \frac{1}{n} S_n\varphi(x) - \int \varphi \, dm_{k,i} \right| < \delta_k \text{ for all } n \geq l_k \right\}$$

satisfies $m_{k,i}(Y_{k,i}) > 1 - \delta$, where $1 \leq i \leq s_k$.

Lemma 2.5. *There exist a sequence of numbers $\{\widehat{n}_k\}_{k\geq 1}$ satisfying $\widehat{n}_k \to \infty$ as $k \to \infty$, $a_{k,i}\widehat{n}_k \in \mathbb{N}$ and $N(\frac{\epsilon}{2^k})/(a_{k,i}\widehat{n}_k) \leq \delta_k$, and a countable collection of finite sets $\mathcal{S}_{k,i}$ such that each $\mathcal{S}_{k,i}$ is an $(a_{k,i}\widehat{n}_k, 4\epsilon)$ separated set for $Y_{k,i}$ and*

$$\sharp\mathcal{S}_{k,i} \geq \exp\left(a_{k,i}\widehat{n}_k(h^{\mathrm{Kat}}_{m_{k,i}}(f, 4\epsilon) - 2\eta)\right),$$

where $1 \leq i \leq s_k$, $\eta > 0$ is a small number.

Proof. For $A \subset X$, we define

$$Q_n(A, 4\epsilon) = \inf\left\{\sharp S : S \text{ is an } (n, 4\epsilon) \text{ spanning set for } A\right\},$$
$$P_n(A, 4\epsilon) = \sup\left\{\sharp S : S \text{ is an } (n, 4\epsilon) \text{ separated set for } A\right\}.$$

Since $m_{k,i}(Y_{k,i}) > 1 - \delta$, we have

$$P_n(Y_{k,i}, 4\epsilon) \geq Q_n(Y_{k,i}, 4\epsilon) \geq N^{m_{k,i}}(n, 4\epsilon, \delta).$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log P_n(Y_{k,i}, 4\epsilon) &\geq \liminf_{n\to\infty} \frac{1}{n} \log N^{m_{k,i}}(n, 4\epsilon, \delta) \\ &= h^{\mathrm{Kat}}_{m_{k,i}}(f, 4\epsilon), \quad k \in \mathbb{N},\ 1 \leq i \leq s_k.\end{aligned}$$

Choose $\widehat{n}_k \to \infty$ such that $a_{k,i}\widehat{n}_k \in \mathbb{N}$ and $N(\frac{\epsilon}{2^k})/(a_{k,i}\widehat{n}_k) \leq \delta_k$. We have

$$\frac{1}{a_{k,i}\widehat{n}_k} \log P_n(Y_{k,i}, 4\epsilon) \geq h^{\mathrm{Kat}}_{m_{k,i}}(f, 4\epsilon) - \eta.$$

Let $\mathcal{S}_{k,i}$ be the $(a_{k,i}\widehat{n}_k, 4\epsilon)$ separated set for $Y_{k,i}$, and satisfying

$$\sharp\mathcal{S}_{k,i} \geq \exp\left\{a_{k,i}\widehat{n}_k(h^{\mathrm{Kat}}_{m_{k,i}}(f, 4\epsilon) - 2\eta)\right\}.$$

The desired result follows. □

We now use the gluing orbit property to define the set $\mathcal{S}_k$ as follows. Let $y_i \in \mathcal{S}_{k,i}$, and define $x = x(y_1, \ldots, y_{s_k})$ to be a choice of point which satisfies

$$d_{a_{k,l}\widehat{n}_k}(y_l, f^{a_l}(x)) \leq \frac{\epsilon}{2^k} \tag{2.2}$$

for all $l \in \{1, 2, \dots, s_k\}$, where $a_1 = 0$, $a_l = \sum_{i=1}^{l-1} a_{k,i}\widehat{n}_k + \sum_{i=1}^{l-1} p_{k,i}$, $p_{k,i} \le N(\frac{\epsilon}{2^k})$ for $l \in \{2, \dots, s_k\}$. Denote $\mathcal{S}_k$ be the set of all points constructed in this way. We set $n_k = \sum_{i=1}^{s_k} a_{k,i}\widehat{n}_k + \sum_{i=1}^{s_k-1} p_{k,i}$. Then n_k is the amount of time for which the orbit of points in $\mathcal{S}_k$ has been prescribed and we have $\frac{n_k}{\widehat{n}_k} \to 1$ as $k \to \infty$. One can verify that $\mathcal{S}_k$ is $(n_k, 3\epsilon)$ separated, so $\sharp\mathcal{S}_k = \sharp\mathcal{S}_{k,1} \cdots \sharp\mathcal{S}_{k,s_k}$.

Lemma 2.6. *Let $\mu \in M(X, f)$, $\alpha = \int \varphi \, d\mu$. For sufficiently large k, we have*

(1) $\sharp\mathcal{S}_{k,1} \cdots \sharp\mathcal{S}_{k,s_k} \ge \exp\{n_k(C - 5\eta)\}$;

(2) *if $x \in \mathcal{S}_k$, then*

$$\left|\frac{1}{n_k} S_{n_k}\varphi(x) - \alpha\right| < \operatorname{Var}(\varphi, \epsilon/2^k) + \frac{n_k + \widehat{n}_k}{n_k}\delta_k + \frac{1}{n_k}\sum_{i=2}^{s_k} p_{k,i} \|\varphi\| + \frac{n_k - \widehat{n}_k}{n_k} \|\varphi\|.$$

Proof. (1) By Lemmas 2.4 and 2.5, for sufficiently large k we have

$$\begin{aligned}
\sharp\mathcal{S}_{k,1} \cdots \sharp\mathcal{S}_{k,s_k} &\ge \exp\left\{\sum_{i=1}^{s_k} a_{k,i}\widehat{n}_k (h^{\mathrm{Kat}}_{m_{k,i}}(f, 4\epsilon) - 2\eta)\right\} \\
&= \exp\left\{\widehat{n}_k \left(\sum_{i=1}^{s_k} a_{k,i}(h^{\mathrm{Kat}}_{m_{k,i}}(f, 4\epsilon) - 2\eta)\right)\right\} \\
&\ge \exp\left\{\widehat{n}_k (h^{\mathrm{Kat}}_{\mu}(f, 4\epsilon) - 2\eta)\right\} \\
&\ge \exp\{\widehat{n}_k (h_\mu(f, \psi) - 3\eta)\} \\
&\ge \exp\{\widehat{n}_k (C - 4\eta)\} \\
&\ge \exp\{n_k (C - 5\eta)\}.
\end{aligned}$$

The last line follows from $\frac{n_k}{\widehat{n}_k} \to 1$ as $k \to \infty$.

(2) Suppose $x = x(y_1, \dots, y_{s_k}) \in \mathcal{S}_k$, then from Lemma 2.4 and (2.2), we have

$$\begin{aligned}
&|S_{n_k}\varphi(x) - n_k\alpha| = \left|S_{n_k}\varphi(x) - n_k\int \varphi \, d\mu\right| \\
&\le \left|S_{n_k}\varphi(x) - n_k\int \varphi \, d\mu_k\right| + \left|n_k\int \varphi \, d\mu_k - n_k\int \varphi \, d\mu\right| \\
&\le \left|S_{n_k}\varphi(x) - \sum_{i=1}^{s_k} a_{k,i} n_k \int \varphi \, dm_{k,i}\right| + n_k\delta_k \\
&\le \left|S_{n_k}\varphi(x) - \sum_{i=1}^{s_k} S_{a_{k,i}\widehat{n}_k}\varphi(y_i)\right| + \left|\sum_{i=1}^{s_k} S_{a_{k,i}\widehat{n}_k}\varphi(y_i) - \sum_{i=1}^{s_k} a_{k,i} n_k \int \varphi \, dm_{k,i}\right| + n_k\delta_k \\
&\le \widehat{n}_k \operatorname{Var}(\varphi, \epsilon/2^k) + \sum_{i=1}^{s_k-1} p_{k,i} \|\varphi\| + \sum_{i=1}^{s_k} a_{k,i}\widehat{n}_k\delta_k + (n_k - \widehat{n}_k)\|\varphi\| + n_k\delta_k \\
&\le \widehat{n}_k \operatorname{Var}(\varphi, \epsilon/2^k) + (n_k + \widehat{n}_k)\delta_k + \sum_{i=1}^{s_k-1} p_{k,i} \|\varphi\| + (n_k - \widehat{n}_k)\|\varphi\|.
\end{aligned}$$

The desired result follows. □

Thus we obtain the following lemma.

Lemma 2.7. *We can choose* $n_k \to \infty$ *such that* $N(\frac{\epsilon}{2^k})/n_k \to 0$, *and* $\gamma_k \to 0$, *and a countable collection of finite sets* $\mathcal{S}_k$ *such that each* $\mathcal{S}_k$ *is an* $(n_k, 3\epsilon)$ *separated set and*

$$\sharp \mathcal{S}_k \geq \exp(n_k(C - 5\eta)),$$

and for any $x \in \mathcal{S}_k$,

$$\left| \frac{1}{n_k} S_{n_k}\varphi(x) - \int \varphi \, d\mu \right| < \gamma_k.$$

2.2.3. Construction of the fractal F

Let us choose a sequence N_k increasing to ∞ sufficiently quickly so that

$$\lim_{k\to\infty} \frac{n_{k+1} + p_{k+1}}{N_k} = 0, \quad \lim_{k\to\infty} \frac{N_1(n_1 + p_1) + \cdots + N_k(n_k + p_k)}{N_{k+1}} = 0, \tag{2.3}$$

where $p_i \leq N(\frac{\epsilon}{2^i})$, $i \geq 1$ satisfying Definition 1.2.

We enumerate the points in the set $\mathcal{S}_k$, i.e., $\mathcal{S}_k = \left\{x_i^k : i = 1, 2, \ldots, \sharp\mathcal{S}_k\right\}$. For $k \geq 1$, we consider the set $\mathcal{S}_k^{N_k}$. For any N_k points $x_1^k, x_2^k, \ldots, x_{N_k}^k$ in $\mathcal{S}_k$, we denote $\underline{x}_k = (x_1^k, x_2^k, \ldots, x_{N_k}^k) \in \mathcal{S}_k^{N_k}$.

Define

$$\begin{aligned} a_1 &= n_1, & b_1 &= a_1 + p_1, \\ a_2 &= b_1 + n_1, & b_2 &= a_2 + p_1, \\ &\vdots & &\vdots \\ a_{N_1-1} &= b_{N_1-2} + n_1, & b_{N_1-1} &= a_{N_1-1} + p_1. \end{aligned}$$

By the gluing orbit property, there exists a $z \in X$ such that

$$\begin{aligned} d(f^j x_1^1, f^j z) &\leq 2^{-1}\epsilon, & 0 \leq j < n_1, \\ d(f^j x_2^1, f^{j+b_1} z) &\leq 2^{-1}\epsilon, & 0 \leq j < n_1, \\ &\vdots & &\vdots \\ d(f^j x_{N_1}^1, f^{j+b_{N_1-1}} z) &\leq 2^{-1}\epsilon, & 0 \leq j < n_1. \end{aligned}$$

Denote $l_1 = N_1 n_1 + (N_1 - 1)p_1$,

$$L_1 = \left\{ z = z(\underline{x}_1) : \underline{x}_1 = (x_1^1, x_2^1, \ldots, x_{N_1}^1) \in \mathcal{S}_1^{N_1} \right\}.$$

We will define recursively L_k and l_k as follows. Suppose we have already defined the set L_k, then

$$L_{k+1} = \left\{ z = z(x, \underline{y}_{k+1}) : x \in L_k, \underline{y}_{k+1} = (y_1^{k+1}, y_2^{k+1}, \ldots, y_{N_{k+1}}^{k+1}) \in \mathcal{S}_{k+1}^{N_{k+1}} \right\},$$

where $z = z(x, \underline{y}_{k+1})$ is a point such that

$$d_{l_k}(x, z) \le \frac{\epsilon}{2^{k+1}} \quad \text{and} \quad d_{n_{k+1}}(y_j^{k+1}, f^{l_k+p_{k+1}+m_j}(z)) \le \frac{\epsilon}{2^{k+1}},$$

and $m_j = (j-1)(p_{k+1} + n_{k+1})$, $j = 1, 2, \dots, N_{k+1}$. Such a point z exists because the map f satisfies the gluing orbit property. Let $l_{k+1} = l_k + N_{k+1}(p_{k+1} + n_{k+1})$, then $l_{k+1} = \sum_{i=1}^{k+1} N_i(p_i + n_i) - p_1$.

Lemma 2.8. *For any $x \in L_k$, $\underline{y} = (y_1^{k+1}, y_2^{k+1}, \dots, y_{N_{k+1}}^{k+1})$, $\underline{y'} = (y_1'^{k+1}, y_2'^{k+1}, \dots, y_{N_{k+1}}'^{k+1}) \in \mathcal{S}_{k+1}^{N_{k+1}}$ with $\underline{y} \ne \underline{y'}$, it follows*

$$d_{l_k}(z(x, \underline{y}), z(x, \underline{y'})) \le \frac{\epsilon}{2^k} \quad \textit{and} \quad d_{l_{k+1}}(z(x, \underline{y}), z(x, \underline{y'})) > 2\epsilon.$$

Proof. First, obviously we have $d_{l_k}(z(x, \underline{y}), z(x, \underline{y'})) \le d_{l_k}(z(x, \underline{y}), x) + d_{l_k}(x, z(x, \underline{y'})) \le \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2^k}$.

Since $\underline{y} \ne \underline{y'}$, there exists $j \in \{1, 2, \dots, N_{k+1}\}$ such that $y_j^{k+1} \ne y_j'^{k+1}$. We may assume $j \ge 2$, then

$$d_{n_{k+1}}(y_j^{k+1}, f^{l_k+p_{k+1}+m_j}(z(x, \underline{y}))) \le \frac{\epsilon}{2^{k+1}},$$
$$d_{n_{k+1}}(y_j'^{k+1}, f^{l_k+p_{k+1}+m_j}(z(x, \underline{y'}))) \le \frac{\epsilon}{2^{k+1}}.$$

Together with $d_{n_{k+1}}(y_j^{k+1}, y_j'^{k+1}) > 3\epsilon$, we have

$$\begin{aligned} d_{l_{k+1}}(z(x, \underline{y}), z(x, \underline{y'})) &\ge d_{n_{k+1}}(f^{l_k+p_{k+1}+m_j}(z(x, \underline{y})), f^{l_k+p_{k+1}+m_j}(z(x, \underline{y'}))) \\ &\ge d_{n_{k+1}}(y_j^{k+1}, y_j'^{k+1}) - d_{n_{k+1}}(y_j^{k+1}, f^{l_k+p_{k+1}+m_j}(z(x, \underline{y}))) \\ &\quad - d_{n_{k+1}}(y_j'^{k+1}, f^{l_k+p_{k+1}+m_j}(z(x, \underline{y'}))) \\ &> 3\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2} = 2\epsilon. \end{aligned}$$ □

By Lemma 2.8, L_k is a $(l_k, 2\epsilon)$ separated set. In particular, if $z, z' \in L_k$, then $\overline{B}_{l_k}(z, \frac{\epsilon}{2^{k-1}}) \cap \overline{B}_{l_k}(z', \frac{\epsilon}{2^{k-1}}) = \emptyset$.

Lemma 2.9. *For $z = z(x, \underline{y}) \in L_{k+1}$, where $x \in L_k$ and $\underline{y} = (y_1^{k+1}, y_2^{k+1}, \dots, y_{N_{k+1}}^{k+1}) \in \mathcal{S}_{k+1}^{N_{k+1}}$, we have*

$$\overline{B}_{l_{k+1}}\left(z, \frac{\epsilon}{2^k}\right) \subset \overline{B}_{l_k}\left(x, \frac{\epsilon}{2^{k-1}}\right).$$

Proof. Let $z' \in \overline{B}_{l_{k+1}}(z, \frac{\epsilon}{2^k})$, then

$$d_{l_k}(z', x) \le d_{l_k}(z', z) + d_{l_k}(x, z) \le \frac{\epsilon}{2^k} + \frac{\epsilon}{2^{k+1}} \le \frac{\epsilon}{2^{k-1}}.$$ □

For every $k \geq 1$, put

$$F_k = \bigcup_{x \in L_k} \overline{B}_{l_k}\left(x, \frac{\epsilon}{2^{k-1}}\right).$$

By Lemma 2.9, $F_{k+1} \subset F_k$, i.e., $\{F_k\}_{k \geq 1}$ is a decreasing sequence of compact sets. Hence the intersection $F = \bigcap_{k=1}^{\infty} F_k$ is non-empty.

For $q \in \mathbb{N}$, we define the sequence M_q as follows:

$$M_q = \begin{cases} n_1 + (q-1)(p_1+n_1), & \text{if } 0 < q \leq N_1, \\ n_1 + (N_1-1)(p_1+n_1) + (q-N_1)(p_2+n_2), & \text{if } N_1 < q \leq N_1+N_2, \\ \cdots \\ -p_1 + \sum_{i=1}^{k} N_i(p_i+n_i) + (q - \sum_{i=1}^{k} N_i)(p_{k+1}+n_{k+1}), & \text{if } \sum_{i=1}^{k} N_i < q \leq \sum_{i=1}^{k+1} N_i, \\ \cdots \end{cases}$$

Lemma 2.10. *For any $y \in F$, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i y) = \alpha$, where $\alpha = \int \varphi \, d\mu$, $\mu \in M(X,f)$.*

Proof. By (2.3), it suffices to show that

$$\lim_{q\to\infty} \frac{1}{M_q} \sum_{i=0}^{M_q-1} \varphi(f^i y) = \alpha.$$

We assume that $M_q = -p_1 + \sum_{k=1}^{j} N_k(p_k+n_k) + t(p_{j+1}+n_{j+1})$, where $0 < t = q - \sum_{k=1}^{j} N_k \leq N_{j+1}$. For $y \in F$, by the gluing orbit property, there exists $(\underline{x}_1, \ldots, \underline{x}_{j+1}) \in \mathcal{S}_1^{N_1} \times \cdots \times \mathcal{S}_{j+1}^{N_{j+1}}$ such that $y \in \overline{B(\underline{x}_1, \ldots, \underline{x}_{j+1})}$, where $\underline{x}_i = (x_1^i, \ldots, x_{N_i}^i) \in \mathcal{S}_i^{N_i}$,

$$B(\underline{x}_1) = B_{n_1}\left(x_1^1, \frac{\epsilon}{2}\right) \cap f^{-n_1-p_1} B_{n_1}\left(x_2^1, \frac{\epsilon}{2}\right) \cap \cdots \cap f^{-(N_1-1)(n_1+p_1)} B_{n_1}\left(x_{N_1}^1, \frac{\epsilon}{2}\right),$$

$l_1 = n_1 + (N_1-1)(p_1+n_1)$,

$$B(\underline{x}_1, \ldots, \underline{x}_k) = B(\underline{x}_1) \cap \left(\bigcap_{i=1}^{N_2} f^{-l_1-(i-1)(p_2+n_2)-p_2} B_{n_2}\left(x_i^2, \frac{\epsilon}{2^2}\right)\right) \cap \cdots$$
$$\cap \left(\bigcap_{i=1}^{N_k} f^{-l_1-\sum_{j=2}^{k-1} N_j(p_j+n_j)-(i-1)(p_k+n_k)-p_k} B_{n_k}\left(x_i^k, \frac{\epsilon}{2^k}\right)\right).$$

For $1 \leq k \leq j$, let $y_k = f^{-p_1+\sum_{i=1}^{k-1} N_i(p_i+n_i)} y$. Let $a_m = p_k + (m-1)(n_k+p_k)$, $c_k = N_k n_k + (N_k-1)p_k$, we have

$$\left| S_{c_k}\varphi(y_k) - c_k \int \varphi \, d\mu \right| \leq \left| \sum_{l=1}^{N_k} S_{n_k}\varphi(f^{a_l} y_k) - c_k \int \varphi \, d\mu \right| + p_k(N_k-1)\|\varphi\|$$
$$\leq \sum_{l=1}^{N_k} \left| S_{n_k}\varphi(f^{a_l} y_k) - S_{n_k}\varphi(x_l^k) \right| + \sum_{l=1}^{N_k} \left| S_{n_k}\varphi(x_l^k) - n_k \int \varphi \, d\mu \right|$$

$$
\begin{aligned}
&+ 2p_k(N_k - 1)\left\|\varphi\right\| \\
&\le N_k n_k \left\{ \mathrm{Var}\left(\varphi, \frac{\epsilon}{2^k}\right) + \gamma_k \right\} + 2p_k(N_k - 1)\left\|\varphi\right\|.
\end{aligned}
$$

Let $y_{j+1} = f^{-p_1 + \sum_{i=1}^{j} N_i(n_i+p_i)} y$. Similarly, we have

$$
\begin{aligned}
&\left| S_{t(p_{j+1}+n_{j+1})}\varphi(y_{j+1}) - t(p_{j+1} + n_{j+1}) \int \varphi \, d\mu \right| \\
&\le t n_{j+1} \left\{ \mathrm{Var}\left(\varphi, \frac{\epsilon}{2^{j+1}}\right) + \gamma_{j+1} \right\} + 2t p_{j+1}\left\|\varphi\right\|.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\left| S_{M_q}\varphi(y) - M_q \int \varphi \, d\mu \right| &\le \sum_{k=1}^{j} \left| S_{c_k}\varphi(y_k) - c_k \int \varphi \, d\mu \right| \\
&\quad + \left| S_{t(p_{j+1}+n_{j+1})}\varphi(y_{j+1}) - t(p_{j+1} + n_{j+1}) \int \varphi \, d\mu \right| + 2\sum_{k=2}^{j} p_k \left\|\varphi\right\| \\
&\le \sum_{k=1}^{j} N_k n_k \left\{ \mathrm{Var}\left(\varphi, \frac{\epsilon}{2^k}\right) + \gamma_k \right\} + t n_{j+1} \left\{ \mathrm{Var}\left(\varphi, \frac{\epsilon}{2^{j+1}}\right) + \gamma_{j+1} \right\} \\
&\quad + 2\left(\sum_{k=1}^{j} p_k N_k + t p_{j+1} \right) \left\|\varphi\right\|.
\end{aligned}
$$

It follows from (2.3) and the fact

$$
\lim_{k\to\infty} \mathrm{Var}\left(\varphi, \frac{\epsilon}{2^k}\right) = 0, \quad \lim_{k\to\infty} \frac{p_k}{n_k} \le \lim_{k\to\infty} \frac{N(\frac{\epsilon}{2^k})}{n_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \gamma_k = 0,
$$

we have

$$
\lim_{q\to\infty} \frac{1}{M_q} \sum_{i=0}^{M_q - 1} \varphi(f^i y) = \alpha.
$$

Thus the desired result follows. □

2.2.4. Computation of the topological entropy of the fractal F

Next we compute the topological entropy of F. Let $z', z'' \in F$. We appoint $L_0 = \emptyset$. Assume that for $k \ge 0$, $z' \in \overline{B}_{l_{k+1}}(z_1, \frac{\epsilon}{2^k})$ and $z'' \in \overline{B}_{l_{k+1}}(z_2, \frac{\epsilon}{2^k})$, where $z_1 = z_1(x, \underline{x}_{k+1}) \in L_{k+1}$ and $z_2 = z_2(y, \underline{y}_{k+1}) \in L_{k+1}$. If $\underline{x}_{k+1} \ne \underline{y}_{k+1} \in \mathcal{S}_{k+1}^{N_{k+1}}$, by Lemma 2.7, we have $d_{l_{k+1}}(z', z'') > \epsilon$. For $k \ge 2$, we define

$$
M_{k,i} = l_{k-1} + i(p_k + n_k), \quad 1 \le i \le N_k.
$$

Since F is compact we can consider finite covers $\mathcal{C}$ of F with the property that if $B_n(x, \epsilon/2) \in \mathcal{C}$, then $B_n(x, \epsilon/2) \cap F \ne \emptyset$. By definition

$$
m(F, s, N, \epsilon/2) = \inf \left\{ \sum_{B_n(x,\epsilon/2)\in\mathcal{C}} e^{-ns} : \mathcal{C} \in \Gamma_N(F, \epsilon/2) \right\}
$$

for each $\mathcal{C} \in \Gamma_N(F, \epsilon/2)$, we can define a new cover $\mathcal{C}'$ in which for $M_{k,i} \le n < M_{k,i+1}$, $B_n(x, \epsilon/2)$ is replaced by $B_{M_{k,i}}(x, \epsilon/2)$. Here we appoint $M_{k,N_k+1} = M_{k+1,0}$. Then

$$m(F, s, N, \epsilon/2) = \inf_{\mathcal{C}\in\Gamma_N(Z,\epsilon/2)} \sum_{B_n(x,\epsilon/2)\in\mathcal{C}} e^{-ns} \ge \inf_{\mathcal{C}\in\Gamma_N(F,\epsilon/2)} \sum_{B_{M_{k,i}}(x,\epsilon/2)\in\mathcal{C}'} e^{-M_{k,i+1}s}.$$

We use the lexicographical order for the set $\{(k, i) : k, i \in \mathbb{N}, 1 \le i \le N_k\}$. Let

$$(k_0, i_0) = \max\left\{(k, i) : B_{M_{k,i}}(x, \epsilon/2) \in \mathcal{C}'\right\}.$$

For $k \ge 2$, define

$$\mathcal{W}_{k,i} := \left(\prod_{j=1}^{k-1} \mathcal{S}_j^{N_j}\right) \times \mathcal{S}_k^i, \quad \overline{\mathcal{W}_{k_0,i_0}} := \bigcup_{(k,i)\le(k_0,i_0)} \mathcal{W}_{k,i}.$$

Each $x \in B_{M_{k,i}} \cap F$ corresponds to a unique point in $\mathcal{W}_{k,i}$. For $(k, i) \le (k_0, i_0)$, each $w \in \mathcal{W}_{k,i}$ is the prefix of exactly $\sharp\mathcal{W}_{k_0,i_0}/\sharp\mathcal{W}_{k,i}$ elements of $\mathcal{W}_{k_0,i_0}$. If $\mathcal{W} \subset \overline{\mathcal{W}_{k_0,i_0}}$ contains a prefix of each element of $\mathcal{W}_{k_0,i_0}$, then

$$\sum_{(k,i)\le(k_0,i_0)} \sharp(\mathcal{W} \cap \mathcal{W}_{k,i}) \frac{\sharp\mathcal{W}_{k_0,i_0}}{\sharp\mathcal{W}_{k,i}} \ge \sharp\mathcal{W}_{k_0,i_0},$$

i.e.,

$$\sum_{(k,i)\le(k_0,i_0)} \frac{\sharp(\mathcal{W} \cap \mathcal{W}_{k,i})}{\sharp\mathcal{W}_{k,i}} \ge 1.$$

It follows from

$$\sharp\mathcal{S}_k \ge \exp(n_k(C - 5\eta)),$$

that

$$\begin{aligned}\sharp\mathcal{W}_{k,i} &\ge (\sharp\mathcal{S}_1)^{N_1}(\sharp\mathcal{S}_2)^{N_2} \cdots (\sharp\mathcal{S}_{k-1})^{N_{k-1}}(\sharp\mathcal{S}_k)^i \\ &\ge \exp\left\{(n_1N_1 + n_2N_2 + \cdots + n_{k-1}N_{k-1} + n_k i)(C - 5\eta)\right\}.\end{aligned}$$

One can readily verify that

$$\lim_{k\to\infty} \frac{n_1N_1 + n_2N_2 + \cdots + n_{k-1}N_{k-1} + n_k i}{M_{k,i+1}} = 1.$$

Since $\mathcal{C}'$ is a cover, each point of $\mathcal{W}_{k_0,i_0}$ has a prefix associated with some $B_{M_{k,i}} \in \mathcal{C}'$. When N is large enough, we have

$$\begin{aligned}m(F, C - 6\eta, N, \epsilon/2) &\ge \sum_{B_{M_{k,i}}(x,\epsilon/2)\in\mathcal{C}'} \exp\left\{-M_{k,i+1}(C - 6\eta)\right\} \\ &\ge \sum_{B_{M_{k,i}}(x,\epsilon/2)\in\mathcal{C}'} \left\{-(n_1N_1 + n_2N_2 + \cdots + n_{k-1}N_{k-1} + n_k i)(C - 5\eta)\right\} \\ &\ge \sum_{B_{M_{k,i}}(x,\epsilon/2)\in\mathcal{C}'} \frac{1}{\sharp\mathcal{W}_{k,i}} \ge 1,\end{aligned}$$

which implies that

$$h_{\text{top}}\left(F, \frac{\epsilon}{2}\right) \geq C - 6\eta.$$

Since ϵ and η is arbitrary, we have $h^B_{\text{top}}(X(\varphi, \alpha)) \geq h^B_{\text{top}}(F) \geq C$, which completes the proof of Theorem 1.5.

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